Solving of time varying quadratic optimal control problems by using Bézier control points

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Abstract. In this paper, linear quadratic optimal control problems are solved by applying least square method based on Bézier control points. We divide the interval which includes t, into k subintervals and approximate the trajectory and control functions by Bézier curves. We have chosen the Bézier curves as piacewise polynomials of degree three, and determined Bézier curves on any subinterval by four control points. By using least square method, we introduce an optimization problem and compute the control points by solving this optimization problem. Numerical experiments are presented to illustrate the proposed method.

Mathematical subject classification: 49N10.

Key words: Bézier Control Points, Quadratic optimal Control Problem, Least Square.

1 Introduction

Optimal control problems arise in a wide variety of disciplines. Apart from traditional areas such as aerospace engineering, robotics and chemical engineering, optimal control theory has also been used with great success in areas as diverse as economics to biomedicine. In particular, finding the analytical solution of optimal control problems are difficult. Thus, numerical methods are needed for the solution of these problems.

The linear quadratic optimal control problems are a class of optimal control and there is an extensive literature on them. There are many papers which their

authors have given methods for solving linear quadric optimal control problems. For example, spectral method [10], time-domain decomposition iterative method [16], and etc.

This paper aims at minimizing quadratic cost functionals over solutions of time varying linear systems of the form

$$\min I = \frac{1}{2} \boldsymbol{x}^T(t_f) H(t_f) \boldsymbol{x}(t_f) + \int_{t_0}^{t_f} (\boldsymbol{x}^T P \boldsymbol{x} + \boldsymbol{u}^T Q \boldsymbol{u} + K \boldsymbol{x} + R \boldsymbol{u}) dt,$$

$$s.t. :$$

$$\dot{\boldsymbol{x}} = A(t) \boldsymbol{x} + B(t) \boldsymbol{u} + F(t),$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0,$$

(1)

where t_0 and t_f are given constants and $\boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t))^T \in R^n$, $\boldsymbol{u}(t) = (u_1(t), \ldots, u_m(t))^T \in R^m$ are unknown vectors functions. Also, we assume $H(t) = [h_{ij}(t)]_{n \times n}$, $P(t) = [p_{ij}(t)]_{n \times n}$, $Q(t) = [q_{ij}(t)]_{n \times m}$, $A(t) = [a_{ij}(t)]_{n \times n}$ and $B(t) = [b_{ij}(t)]_{n \times m}$ are matrices functions and $K(t) = (k_1(t), \ldots, k_n(t))^T$, $R(t) = (r_1(t), \ldots, r_m(t))^T$ and $F(t) = (f_1(t), \ldots, f_n(t))^T$ are vectors functions, which their all elements are polynomial in $[t_0, t_f]$. Also, $\boldsymbol{x}_0$ is finite constant vectors and t_f is fixed constant. If $\boldsymbol{x}(t_f) = \boldsymbol{x}_f$ be given, the first term of the objective function will be converted to a constant and can be omitted.

If A and B independent of t, the problem (1) will be called time-invariant [3], and otherwise it called time varying problem, [7], [11], and [21].

Lots of papers and books exist dealing with the Bézier curves or surfaces techniques that are applied to different contexts. Harada [14] and Nürnberger [6] have used the Bézier control points in approximate data and functions. Zheng [13] proposed the use of control points of the Bernstein-Bézier form for solving differential equations numerically and also Evrenosoglu [15] have used this approach for solving singularity perturbed two points boundary value problems. The Bézier curves are used in partial differential equations. For example, the Wave and Heat equations are solved in Bézier form, [1, 2, 12, 18]. Other applications of the Bézier functions and control points are found in [4, 9, 19, 22], that are used in computer aided geometric design and image compression.

We suggest a technique similar to what is in [13] and [15], for solving of linear quadratic optimal control problems.

2 Least square method

We divide the interval $t_0 \leq t \leq t_f$ into a set of grid points such that

$$t_j = t_0 + jh, \qquad j = 0, 1, \cdots, 2k,$$

where $h = \frac{t_f - t_0}{2k}$, and k is a positive integer. Let $S_j = [t_{2j-2}, t_{2j}]$ for $j = 1, 2, \cdots, k$, the control problem (1) will be defined piecewise as

$$\min I_j = C_j + \int_{t_{2j-2}}^{t_{2j}} (\boldsymbol{x}_j^T P \boldsymbol{x}_j + \boldsymbol{u}_j^T Q \boldsymbol{u}_j + K \boldsymbol{x}_j + R \boldsymbol{u}_j) dt,$$

$$s.t. : \tag{2}$$

$$\dot{\boldsymbol{x}}_j = A(t)\boldsymbol{x}_j + B(t)\boldsymbol{u}_j + F(t), \quad t \in S_j,$$

$$\boldsymbol{x}_j(t_{2j-2}) = \mu_{j-1}, \quad \boldsymbol{x}_j(t_{2j}) = \mu_j,$$

where $\mu_0 = \boldsymbol{x}_0$ is known, $\mu_1, \mu_2, \ldots, \mu_{k-1}, \mu_k$ are unknown, $\mu_k = \boldsymbol{x}_f$ and

$$C_j = \begin{cases} \frac{1}{2}\boldsymbol{x}_k^T(t_f)H(t_f)\boldsymbol{x}_k(t_f), & j = k, \\ 0, & j \neq k. \end{cases}$$

Our strategy is to divide the interval S_j into two subintervals and then using a Bézier spline curve to approximate the solutions $\boldsymbol{x}_j(t)$ and $\boldsymbol{u}_j(t)$ by $\boldsymbol{v}_j(t)$ and $\boldsymbol{w}_j(t)$, respectively where $\boldsymbol{v}_j(t)$ and $\boldsymbol{w}_j(t)$ are given below. Individual Bézier curves that are defined over the subintervals are joined together to form the Bézier spline curve. Let the Bézier segment over $[t_{2j-2+\ell}, t_{2j-1+\ell}]$ be

$$\begin{bmatrix} \boldsymbol{v}_{2j-1+\ell}(t) \\ \boldsymbol{w}_{2j-1+\ell}(t) \end{bmatrix} = \sum_{r=0}^{3} \begin{bmatrix} \boldsymbol{a}_r^{2j-1+\ell} \\ \boldsymbol{b}_r^{2j-1+\ell} \end{bmatrix} B_r^3 \left(\frac{t - t_{2j-2+\ell}}{h} \right), \quad \ell = 0, 1, \tag{3}$$

where

$$B_r^3 \left(\frac{t - t_{2j-2+\ell}}{h} \right) = \begin{pmatrix} 3 \\ r \end{pmatrix} \frac{1}{h^3} (t_{2j-1+\ell} - t)^{3-r} (t - t_{2j-2+\ell})^r,$$

are the Bernstein polynomials of degree 3 over the interval $[t_{2j-2+\ell}, t_{2j-1+\ell}]$ and $\begin{bmatrix} \boldsymbol{a}_r^{2j-1+\ell} \\ \boldsymbol{b}_r^{2j-1+\ell} \end{bmatrix}$ are vectors of the control points. Now, by substituting (3) into

differential equation of problem (2), we define the piecewise residual functions $\boldsymbol{E}_{1,2j-1+\ell}(t)$ and $\boldsymbol{E}_{2,2j-1+\ell}(t)$ for $t \in [t_{2j-2+\ell}, t_{2j-1+\ell}]$ and $\ell = 0, 1$, by

$$\boldsymbol{E}_{1,2j-1+\ell}(t) = \dot{\boldsymbol{v}}_{2j-1+\ell}(t) - A(t)\boldsymbol{v}_{2j-1+\ell}(t) - B(t)\boldsymbol{w}_{2j-1+\ell}(t) - F(t),$$

$$\boldsymbol{E}_{2,2j-1+\ell}(t) = \boldsymbol{v}_{2j-1+\ell}^T P(t)\boldsymbol{v}_{2j-1+\ell} + \boldsymbol{w}_{2j-1+\ell}^T Q(t)\boldsymbol{w}_{2j-1+\ell} + R(t)\boldsymbol{w}_{2j-1+\ell}.$$

The boundary conditions should be applied to the first and last Bézier segments, i.e.,

$$\boldsymbol{v}_{2j-1}(t_{2j-2}) = \mu_{j-1}, \quad \boldsymbol{v}_{2j}(t_{2j}) = \mu_j.$$

Beside the boundary conditions, there are also continuity constraints imposed on each successive pair of Bézier segments. Since the differential equation is of first order, the continuity of the first derivative of $\boldsymbol{x}$ (or $\boldsymbol{v}$) is required and this gives

$$\boldsymbol{v}_{2j-1}^{(s)}(t_{2j-1}) = \boldsymbol{v}_{2j}^{(s)}(t_{2j-1}), \quad s = 0, 1, \quad j = 1, 2, \ldots, k.$$

Thus the control points $\boldsymbol{a}_r^{2j-1+\ell}$ must satisfy

$$\begin{aligned}
\boldsymbol{a}_0^{2j-1} &= \mu_{j-1}, \\
\boldsymbol{a}_3^{2j} &= \mu_j, \\
\boldsymbol{a}_0^{2j} &= \boldsymbol{a}_3^{2j-1}, \\
\boldsymbol{a}_1^{2j} &= 2\boldsymbol{a}_3^{2j-1} - \boldsymbol{a}_2^{2j-1}.
\end{aligned} \tag{4}$$

If we consider the continuity of w, the following constraints will be added to constraints (4),

$$\begin{aligned}
\boldsymbol{b}_0^{2j} &= \boldsymbol{b}_3^{2j-1}, \\
\boldsymbol{b}_1^{2j} &= 2\boldsymbol{b}_3^{2j-1} - \boldsymbol{b}_2^{2j-1}.
\end{aligned}$$

Now, for each $j = 1, 2, \cdots, k$ we define residual function as follows

$$E_j = \|C_j\|^2 + \sum_{\ell=0}^{1} \int_{t_{2j-2+\ell}}^{t_{2j-1+\ell}} (M\|\boldsymbol{E}_{1,2j-1+\ell}(t)\|^2 + \boldsymbol{E}_{2,2j-1+\ell}(t))dt, \tag{5}$$

where $\|.\|$ is L_2 norm and M is a big enough number. Our goal is to minimize the residual function over S_j. Taking the derivative of E_j with respect to component of the unknown control points

$$\boldsymbol{a}_1^{2j-1}, \ \boldsymbol{a}_2^{2j-1}, \ \boldsymbol{a}_3^{2j-1}, \ \boldsymbol{a}_2^{2j}, \ \boldsymbol{b}_0^{2j-1}, \ \boldsymbol{b}_1^{2j-1}, \ \boldsymbol{b}_2^{2j-1}, \ \boldsymbol{b}_3^{2j-1}, \ \boldsymbol{b}_2^{2j} \quad \text{and} \quad \boldsymbol{b}_3^{2j}$$

reduces the problem (5) to solve a linear system include $4n + 6m$ equations and unknowns where n and m are dimensions of x and u, respectively. Solving this system, we have the solution in terms of $\mu_i, i = 1, \ldots, k - 1$. From the continuity of the first derivative of the solutions, we have

$$v'_{2j}(t_{2j}) = v'_{2j+1}(t_{2j}), \quad j = 1, \ldots, k - 1,$$

which gives a linear equation system of $k - 1$ equations with $k - 1$ unknowns $\mu_1, \ldots, \mu_{k-1}$. Substituting the minimum solution back into (4) we derive the approximate solution of the quadratic optimal control problem.

Note 1: For $j = k$, we consider $a_3^{2k} = \mu_k$ as unknown in (5), and we compute derivative of E_k with respect to component of a_3^{2k}.

Note 2: In problem (1), if $x(t_f)$ be known, then we set $C_k = 0$ and $\mu_k = x(t_f)$ will be known.

Note 3: We can use this method for solving a problem of calculus of variation as follows

$$\min I = \int_{t_0}^{t_f} (\dot{x}^T P \dot{x} + x^T Q x + K \dot{x} + R x) dt,$$

with the conditions $x(t_0) = x_0$ and $x(t_f) = x_f$.

3   Numerical examples

In this section, numerical experiments are conducted to validate the proposed method. We have solved two following problems. First, an optimal control problem and then a problem in calculus of variation and in each example, we show the graphs of trajectories and optimal control functions.

Example 1. We consider the problem of minimizing the performance index [8]:

$$I = \frac{1}{2} \int_0^1 u^2 dt,$$

on trajectories of the systems

$$\dot{x}_1 = x_2 + u,$$
$$0 = -x_2 + u,$$

which satisfy the boundary values

$$x_1(0) = 1, \ x_1\left(\frac{1}{2}\right) = x_1(1) = 0.$$

We solve this problem with $k = 1$ and the continuity of x_2 is omitted. We obtain the solution of the considered problem:

$$x_1(t) = \begin{cases} 8\left(\frac{1}{2} - t\right)^3 + 16.02400\left(\frac{1}{2} - t\right)^2 t + 7.97601\left(\frac{1}{2} - t\right)t^2, & 0 \le t \le \frac{1}{2}, \\ 0, & \frac{1}{2} \le t \le 1, \end{cases}$$

$$x_2(t) = \begin{cases} -7.97601\left(\frac{1}{2} - t\right)^3 - 24.07199\left(\frac{1}{2} - t\right)^2 t \\ \qquad\qquad -24.07199\left(\frac{1}{2} - t\right)t^2 - 7.97601t^3, & 0 \le t \le \frac{1}{2}, \\ 0, & \frac{1}{2} \le t \le 1, \end{cases}$$

$$u(t) = \begin{cases} -7.976003\left(\frac{1}{2} - t\right)^3 - 24.07198\left(\frac{1}{2} - t\right)^2 t \\ \qquad\qquad -24.07198\left(\frac{1}{2} - t\right)t^2 - 7.97601t^3, & 0 \le t \le \frac{1}{2}, \\ 0, & \frac{1}{2} \le t \le 1, \end{cases}$$
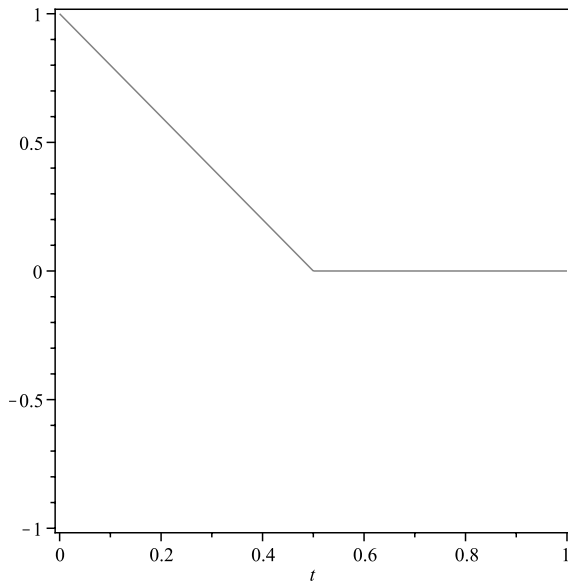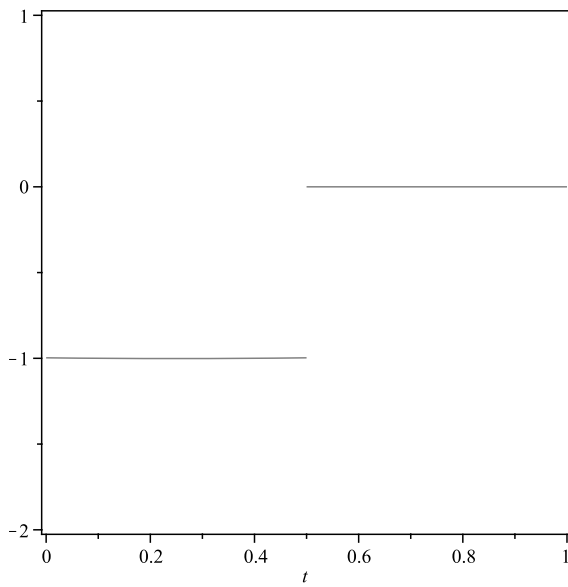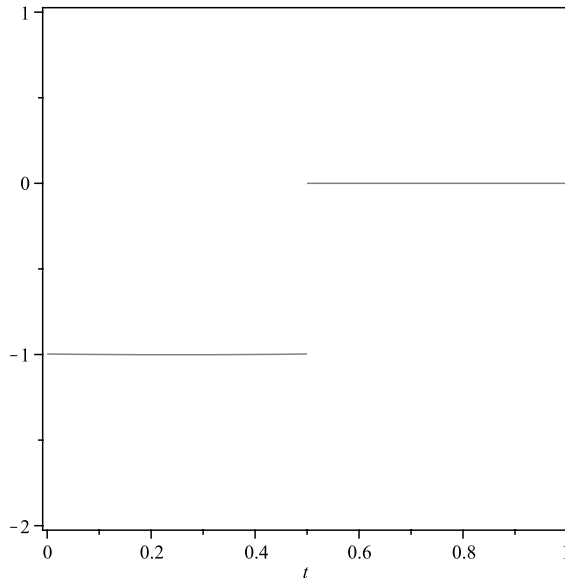
$$\text{Min } I = 0.2500003248.$$

The exact trajectories and control functions are obtained by Kurina, [8] as follows:

$$x_1^*(t) = \begin{cases} -2t + 1, & 0 \le t \le \frac{1}{2}, \\ 0, & \frac{1}{2} \le t \le 1, \end{cases} \quad , \quad x_2^*(t) = u^*(t) = \begin{cases} -1, & 0 \le t \le \frac{1}{2}, \\ 0, & \frac{1}{2} \le t \le 1, \end{cases}$$

$$\text{Min } I = I^* = \frac{1}{4}.$$

The graphs of trajectories and control functions which we obtain, are shown in Figures 1-3.

Figure 1 – Graph of trajectory $x_1(t)$.

Figure 2 – Graph of trajectory $x_2(t)$.

Figure 3 – Graph of control $u(t)$.

Example 2. In second example, we consider a variational problem. We want to find two functions $x_1(t)$ and $x_2(t)$ which minimize the following objective function, [5]-page 223,

$$I = \int_0^{\frac{\pi}{2}} (\dot{x}_1^2 + \dot{x}_2^2 + 2x_1x_2)dt,$$

subject to:

$$x_1(0) = 0, \quad x_1\left(\tfrac{\pi}{2}\right) = 1,$$
$$x_2(0) = 0, \quad x_2\left(\tfrac{\pi}{2}\right) = 1.$$

This problem is solved by $k = 1$ and we have obtained $x_1(t) = x_2(t)$ and Min $I = 2.180665422$ for objective function. The approximate solutions of the problem are obtained as:

$$x_1(t) = x_2(t) = \begin{cases} 0.70632t \left(\tfrac{\pi}{4} - t\right)^2 + 1.40387t^2 \left(\tfrac{\pi}{4} - t\right) \\ \qquad\qquad + 0.77943t^3, & 0 \le t \le \tfrac{\pi}{4}, \\ 0.77943 \left(\tfrac{\pi}{2} - t\right)^3 + 3.27272 \left(\tfrac{\pi}{2} - t\right)^2 \left(t - \tfrac{\pi}{4}\right) \\ + 4.43014 \left(\tfrac{\pi}{2} - t\right) \left(t - \tfrac{\pi}{4}\right)^2 + 2.06410 \left(t - \tfrac{\pi}{4}\right)^3, & \tfrac{\pi}{4} \le t \le \tfrac{\pi}{2}. \end{cases}$$

The exact solution of problem and it's value of objective function are:

$$x_1^*(t) = x_2^*(t) = \frac{e^{-\frac{\pi}{2}}(e^t - e^{-t})}{(1 - e^{-\pi})}, \quad 0 \le t \le \frac{\pi}{2},$$

$$\text{Min } I = I^* = \frac{2(e^\pi + 1)}{e^\pi - 1} = 2.180662822.$$

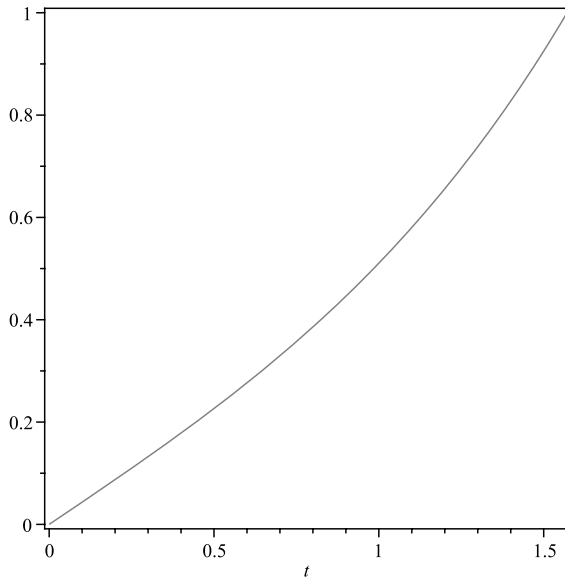The graphs of x_1 and error function are shown in Figures 4-5.

Figure 4 – Graph of trajectory $x_1(t) = x_2(t)$.

Example 3. As a last example, consider a time varying quadratic optimal control problem which is a model of aerospace trajectory control, [20]. The system data is given as follows:

$$A(t) = \begin{bmatrix} 2t & 1 \\ 0 & t+1 \end{bmatrix}, B(t) = \begin{bmatrix} 1 \\ \frac{2t+2}{2t+3} \end{bmatrix},$$

$$P(t) = I, \quad Q(t) = 1, x_0 = (1, 1)^T,$$

$$H(t) = K(t) = R(t) = F(t) = 0, \quad t \in [0, 4].$$

We solve this problem by proposed method and obtain the approximate solutions of the problem. For $k = 4$, the minimum of the objective function is
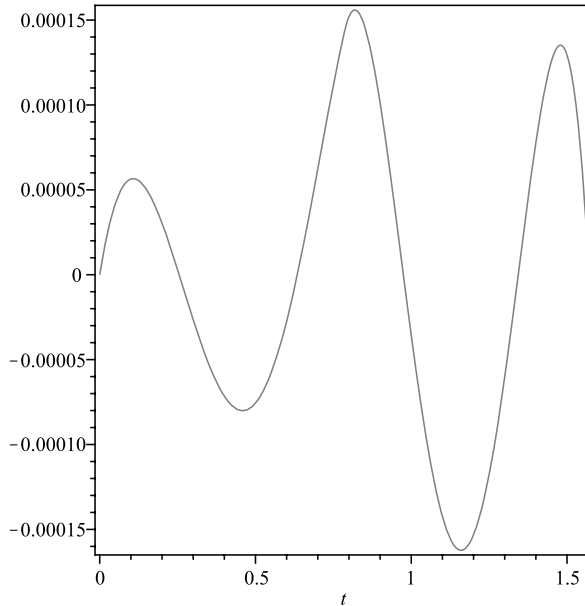
Figure 5 – Graph of error.

6.573994915. Note that, the five terms of Taylor series of b_{21} at $t = 2$ is used instead it. The graphs of x_1, x_2 and u are shown in figures 6-8. The asymptotic behaviors of trajectories of the system are discussed in [17] and we see this behaviors in figures.

4   Conclusions

We have used least squares methods for numerical solutions of linear quadratic optimal control problems and problems of calculus of variation in quadratic form using Bézier control points instead of computing integrals or performing discretization. The computation is simple and intuitive. The control point structure provides a bound on the residual function. Numerical examples show that the approximate functions are satisfactory for large h. Clearly, if we could solve systems of nonlinear equations, this technique will be quite general and can be easily extended to solve other problems, for example, nonlinear quadratic optimal control problems, general problems of calculus of variation.
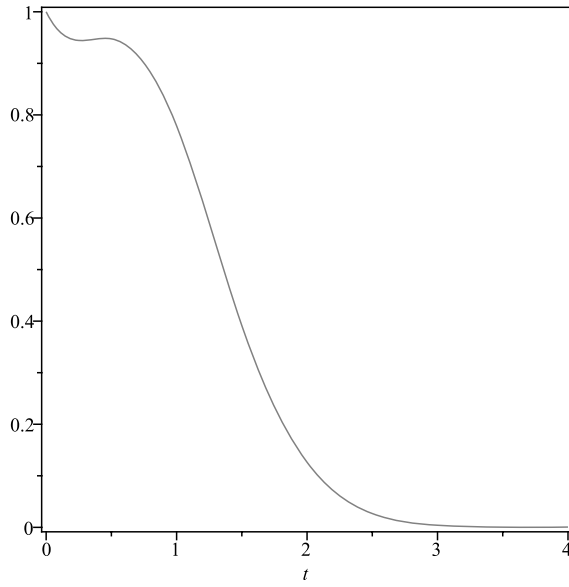
Figure 6 – Graph of trajectory $x_1(t)$.
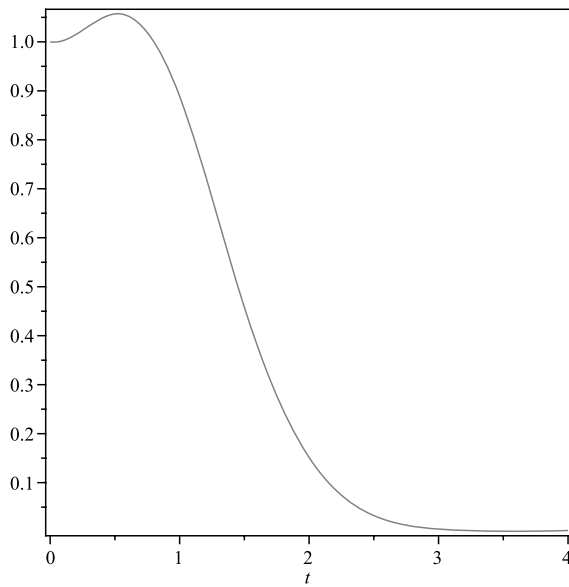
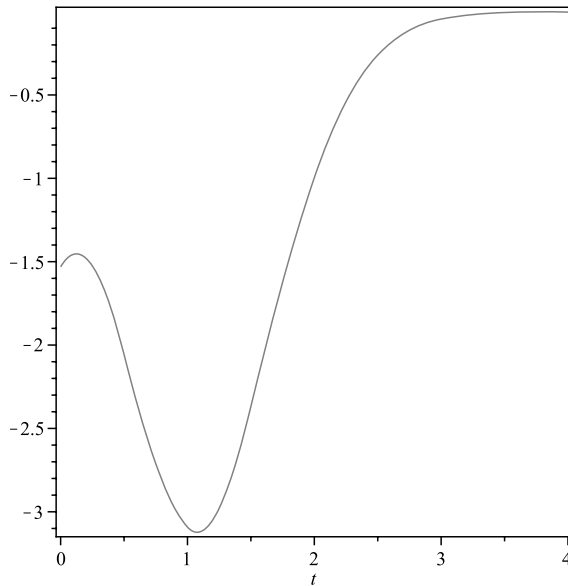Figure 7 – Graph of trajectory $x_2(t)$.

Figure 8 – Graph of control $u(t)$.

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