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Solving Partial Integro-Differential Equations Using Laplace Transform Method

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Abstract Partialintegro-differential equations (PIDE) occur naturally in various fields of science, engineering and social sciences. In this article, we propose a most general form of a linear PIDE with a convolution kernel. We convert the proposed PIDE to an ordinary differential equation (ODE) using a Laplace transform (LT). Solving this ODE and applying inverse LT an exact solution of the problem is obtained. It is observed that the LT is a simple and reliable technique for solving such equations. A variety of numerical examples are presented to show the performance and accuracy of the proposed method.

Keywords Partialintegro-differential equations, Laplace transform

1. Introduction

Real life phenomena are often modelled by ordinary/partial differential equations. Due to the local nature of ordinary differential operator(ODO), the models containing merely ODOs do not help in modelling memory and hereditary properties. One of the best remedies to overcome this drawback is the introduction of integral term in the model. The ordinary/partial differential equation along with the weighted integral of unknown function gives rise to an integro-differential equation (IDE) or a partial integro-differential equation (PIDE) respectively. Analysis of such equations can be found in[1-4].

Applications of PIDEs can be found in various fields. Dehghan and shakeri[5] have used variational iteration method (VIM) to solve PIDEs arising in heat conduction of materials with memory. Various numerical schemes are proposed by Dehghan[6] to solve PIDEs arising in viscoelasticity. Nonlinear PIDEs arising in nuclear reactor dynamics are solved by Pao[7] and Pachapatte[8]. PIDEs have been used in jump-diffusion models for pricing of derivatives in finance[9]. Abergel[10] used a nonlinear PIDE in financial modelling. Hepperger[11] proposed a PIDE in the model of electricity swaptions. A PIDE governing biofluid flow in fractured biomaterials is proposed by Zadeh in[12].

The most promising tool for solving linear equations is the Laplace transform (LT) method[13,14]. LT is used in[16] for calculations of water flow and heat transfer in fractured rocks. Alquran et al.[17] used LT to solve non -homogeneous partial differential equations. Merdan et al. [18] proposed a new method for nonlinear oscillatory systems using LT.

Stiff systems of ODEs are solved by Aminikhah[19] using a combined LT and HPM. Kexue and Jiger[20] have utilized LT to solve problems arising in fractional differential equations.

In this article we propose a most general form of a linear PIDE in two independent variables with a convolution kernel. In Section 2 we provide some preliminaries regarding LT. Section 3 is devoted to the proposed method and Section 4 provides an ample number of examples of various types.

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2. Preliminaries

2.1. Laplace Transform method:

Definition: TheLaplace transform of a function f(x), is defined by

$$\bar{f}(p) = \mathscr{L}\{ f(x)\} = \int_0^t e^{-px} f(x)dx\,;\; x \geq 0,$$

(whenever integral on RHS exists)
where, x ≥0, p is real and $\mathscr{L}$is the Laplace transform operator.

Convolution Theorem:

If $\bar{f}(p) = \mathcal{L}\{ f(t)\}$and $\bar{g}(p) = \mathcal{L}\{ g(t)\}$then

$$\mathscr{L}\{ f(t) * g(t)\} = \mathscr{L}\{ f(t)\}\mathscr{L}\{ g(t)\} = \bar{f}(p)\bar{g}(p),$$

where, $f(t) * g(t) = \int_0^t f(x-t)g(t)dt$.

3. Solving PIDEs using Laplace Transform Method

Consider PIDE,

$$\sum_{i=0}^{m} a_i \frac{\partial^i u}{\partial t^i} + \sum_{i=0}^{n} b_i \frac{\partial^i u}{\partial x^i} + cu + \sum_{i=0}^{r} d_i \int_0^t k_i(t -$$

$$s)\frac{\partial^i u(x,s)}{\partial x^i}ds + f(x,t) = 0, \qquad (*)$$

(with prescribed conditions)

where $f(x,t)$ and $k_i(t,s)$ are known functions. $a_i's$, b_i's, $d_i's$ and c are constants or the functions of x.

Taking Laplace transform on both sides of PIDE (*) with respect to t we get,

$$\sum_{i=0}^{m} a_i \mathscr{L}\{\frac{\partial^i u}{\partial t^i}\} + \sum_{i=0}^{n} b_i \mathscr{L}\left\{\frac{\partial^i u}{\partial x^i}\right\} + c\mathscr{L}\{u\} + \sum_{i=0}^{r} d_i \mathscr{L}\left\{k_i(t) * \frac{\partial^i u(x,t)}{\partial x^i}\right\} + \mathscr{L}\{\mathrm{f(x,t)}\} = 0,$$

Using convolution theorem for Laplace transform we get,

$$\sum_{i=0}^{m} a_i \left(p^i \bar{u}(x,p) - \sum_{j=1}^{i}(p^{j-1}u^{(i-j)}(x,0)) \right) + \sum_{i=0}^{n} b_i \frac{d^i \bar{u}(x,p)}{dx^i} + c\,\bar{u}(x,p) + \sum_{i=0}^{r} d_i \,\widetilde{k_i}(p) \frac{d^i \bar{u}(x,p)}{dx^i} + f(x,p)=0, \qquad (**)$$

where, $\bar{u}(x,p) = \mathscr{L}\{u(x,t)\}$,

$$\tilde{f}(x,p) = \mathscr{L}\{\mathrm{f(x,t)}\}$$

And $\widetilde{k_i}(p) = \mathscr{L}\{k_i(t)\}$.

Equation (**) is an ordinary differential equation in $\bar{u}(x,p)$. Solving this ordinary differential equation and taking inverse Laplace transform of $\bar{u}(x,p)$, we get a solution $u(x,t)$ of (*).

4. Illustrative examples

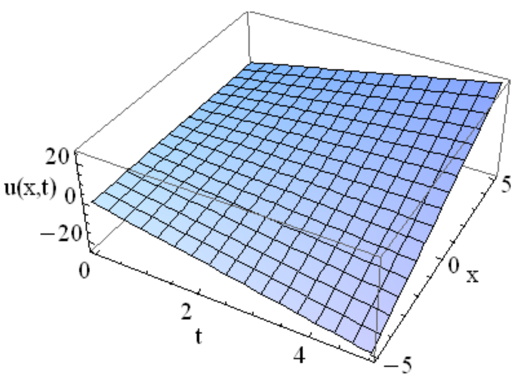

Fig. 1.Solution of (1.1)$u(x,t)$=xt.

Example 1. Consider the PIDE

$$xu_x = u_{tt} + xsint + \int_0^t sin(t-s)u(x,s)ds, \qquad (1)$$

with initial conditions

$$u(x,0) = 0, \; u_t(x,0) = x, \qquad (2)$$

and boundary condition

$$u(1,t) = \mathrm{t}. \qquad (3)$$

Taking Laplace transform with respect to t on both sides of (1),

$$x\frac{d\bar{u}}{dx} = p^2\bar{u}(x,p) - pu(x,0) - u_t(x,0) + \frac{x}{p^2+1} + \frac{1}{p^2+1}\bar{u}.$$

$$\Rightarrow \frac{d\bar{u}}{dx} + \frac{-1}{x}(p^2 + \frac{1}{p^2+1})\bar{u} = \frac{-p^2}{p^2+1}. \qquad (4)$$

Solution of (1) is

$$\bar{u}(x,p) = \frac{1}{p^2}x + \mathrm{C}\, x^{(p^2+\frac{1}{p^2+1})}. \qquad (5)$$

Where, C is a constant to be determined.

From (3),

$$\bar{u}(1,p) = \frac{1}{p^2}.$$

From (4) C = 0.

$$\therefore \bar{u}(x,p) = \frac{1}{p^2}x. \qquad (6)$$

Taking inverse Laplace transform on both the sides of (6), we get

$$u(x,t) = xt.$$

The solution of (1) is plotted in the Fig.1.

Example 2. Consider the PIDE

$$u_{tt} = u_x + 2\int_0^t (t-s)u(x,s)ds - 2e^x, \qquad (7)$$

with initial condition

$$u(x,0) = e^x, \; u_t(x,0) = 0, \qquad (8)$$

and boundary condition

$$u(0,t) = cost. \qquad (9)$$

Taking Laplace transform w.r.t. t on both sides of (7),

$$p^2\bar{u}(x,p) - pu(x,0) - u_t(x,0) = \frac{d\bar{u}}{dx} + 2(\frac{1}{p^2})\bar{u} - 2e^x(\frac{1}{p}).$$

$$\therefore \frac{d\bar{u}}{dx} + (\frac{2}{p^2} - p^2)\bar{u} = \frac{2}{p} - p. \qquad (10)$$

$$\bar{u}(x,p) = \frac{1}{e^{\int(\frac{2}{p^2}-p^2)dx}}\left[\int e^{\int(\frac{2}{p^2}-p^2)dx}(\frac{2}{p} - p)dx + \mathrm{C}\right].$$

Therefore the solution of (10) is

$$\bar{u}(x,p) = (\frac{p}{p^2+1})\,e^x + \mathrm{C}e^{(p^2-\frac{2}{p^2})x}. \qquad (11)$$

From the boundary condition (9)

$$\bar{u}(0,p) = \frac{p}{p^2+1}. \qquad (12)$$

Using (11) and (12), we get C = 0.

∴ Equation (11) becomes,

$$\bar{u}(x,p) = (\frac{p}{p^2+1})\,e^x. \qquad (13)$$

Taking inverse Laplace transform of (13)

$$u(x,t) = e^x \cos t. \qquad (14)$$

The solution (14) is plotted in Figure 2.

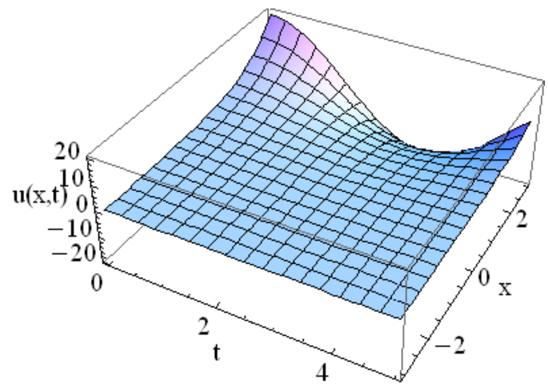

Figure 2.Solution of$u(x,t)$=$e^x \cos t$.

Example 3. Consider

$$u_t - u_{xx} + u + \int_0^t e^{t-s}u(x,s)ds = (x^2+1)e^t - 2 \qquad (15)$$

$$u(x,0) = x^2, \; u_t(x,0) = 1, \qquad (16)$$

$$u(0,t) = t, \; u_x(0,t) = 0. \qquad (17)$$

Taking Laplace transform of (15) w.r.t. t we get

$$p\bar{u}\ (x,p) - u(x,0) - \frac{d^2\bar{u}}{dx^2} + \bar{u} + \frac{1}{(p-1)}\bar{u} = \frac{1}{(p-1)}(x^2+1) - \frac{2}{p}. \quad (18)$$

$$\frac{d^2\bar{u}}{dx^2} - \left(\frac{p^2}{p-1}\right)\bar{u} = -x^2\left(\frac{p}{p-1}\right) - \frac{1}{p-1} + \frac{2}{p}. \quad (19)$$

Solving (19) we get,

$$\bar{u}\ (x,p) = C_1 e^{\sqrt{\frac{p^2}{p-1}}x} + C_2 e^{-\sqrt{\frac{p^2}{p-1}}x} + \frac{x^2}{p} + \frac{1}{p^2}. \quad (20)$$

Now,

$$u(0,t) = t \ \therefore \bar{u}(0,p)\frac{1}{p^2}, \quad (21)$$

$$\text{And } u_x(0,t) = 0 \ \therefore \frac{d\bar{u}(0,p)}{dx} = 0 \quad (22)$$

Using (21) and (22) in (20) we get,

$$C_1 + C_2 = 0. \quad (23)$$

$$\text{And } C_1 - C_2 = 0. \quad (24)$$

Solving (23) and (24) we get,

$$C_1 = 0, C_2 = 0.$$

∴ Equation (20) becomes,

$$\bar{u}\ (x,p) = \frac{x^2}{p} + \frac{1}{p^2}.$$

Taking inverse Laplace transform we get,

$$u(x,t) = x^2 + t.$$

This is an exact solution of (15).

Figure 3 represents the graph of $u(x,t) = x^2 + t$.

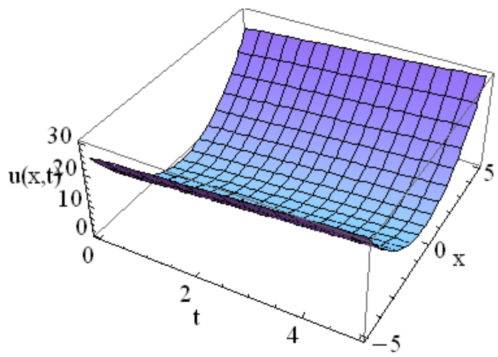

Figure 3. Solution of (15) $u(x,t) = x^2 + t$.

Example 4. Consider

$$u_t + u_{ttt} + u_t - u + xt - \int_0^t \sinh(t-s)u_{xxx}(x,s)ds = 0 \quad (25)$$

$$u(x,0) = 0, u_t(x,0) = x, u_{tt}(x,0) = 0, \quad (26)$$

$$u(0,t) = 0, u_x(0,t) = \sin t, u_{xx}(0,t) = 0. \quad (27)$$

Taking Laplace transform of (25) with respect to t and using equation (26) we get,

$$\frac{d^3\bar{u}}{dx^3} + \frac{1}{p^2}(p^3 + p^4 - p^7 - 1)\bar{u} = \frac{1}{p^2}(p^2-1)(1-p^3)x \quad (28)$$

Solving (28) we get,

$$\bar{u}(x,p) = e^{\frac{(-1)^{\frac{2}{3}}(1-p^3-p^4+p^7)^{\frac{1}{3}}x}{p^{\frac{2}{3}}}}C_1 + e^{-\frac{(-1)^{\frac{1}{3}}(1-p^3-p^4+p^7)^{\frac{1}{3}}x}{p^{\frac{2}{3}}}}C_2 + e^{\frac{(1-p^3-p^4+p^7)^{\frac{1}{3}}x}{p^{\frac{2}{3}}}}C_3 + \frac{1}{p^2+1}x. \quad (29)$$

Using (27) we get,

$C_1 = 0, C_2 = 0$ and $C_3 = 0$.

Therefore equation (29) becomes,

$$\bar{u}(x,p) = \frac{1}{p^2+1}x.$$

Taking inverse Laplace transform we get,

$$u(x,t) = x\sin t. \quad (30)$$

Figure 4 shows the graph of (30).

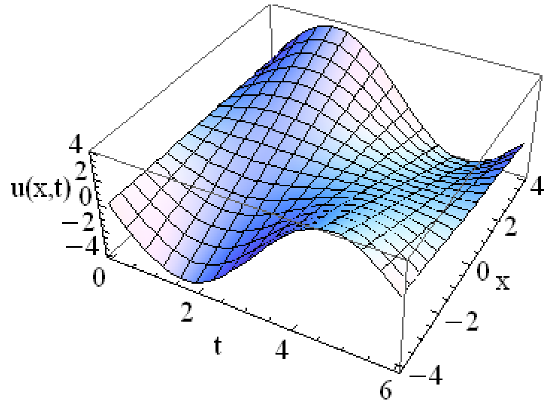

Figure 4. Solution of (25) $u(x,t) = x\sin t$.

5. Conclusions

PIDEs are used in modelling various phenomena in science, engineering and social sciences. The LT technique is successfully used to solve a general linear PIDE involving a convolution kernel. We get exact solutions of such PIDEs after a few steps of calculations. We hope some other types of PIDEs and these equations can be used in modelling real life phenomena.

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