Solving QBF with Counterexample Guided Refinement

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- an extension of SAT with quantifiers
- PSPACE-complete
- formal verification
- planning

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$$\forall y_1 y_2 \exists x_1 x_2.\ (y_1 \leftrightarrow x_1) \wedge (y_2 \leftrightarrow x_2)$$

- we consider prenex form with maximal blocks of variables

$$QX_1\, \bar{Q}Y_1\, QX_2\, \bar{Q}Y_2 \ldots .\ \phi$$

$$\textit{where } Q \in \{\exists, \forall\}$$
$$\bar{\exists} = \forall, \bar{\forall} = \exists$$

A QBF as a Game

- it is useful to think about a QBF as a game between the universal and existential player
- universal player wins when the matrix becomes false
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$$\forall y_1 y_2 \exists x_1 x_2.\ (y_1 \leftrightarrow x_1) \wedge (y_2 \leftrightarrow x_2)$$

- $\exists$ always wins by playing $x_1 = y_1$, $x_2 = y_2$

Semantics with Winning Move

winning move, base case $QX.\phi$, for ϕ propositional

- for $Q = \exists$, an assignment that makes ϕ true (model of ϕ)
- for $Q = \forall$, an assignment that makes ϕ false

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countermove, for $QX.\ \Phi$, for Φ QBF

- an assignment μ is a countermove to the assignment τ if μ is a winning move for $\bar{Q}$ for $\Phi[\tau]$

Winning Move Semantics

QBF semantics

- $\exists X.\Phi$ is true if and only if there is a winning move for $\exists$
- $\forall X.\Phi$ is false if and only if there is a winning move for $\forall$

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Example

$$\forall y \exists x.\ x \wedge (y \vee \bar{x})$$

- $\{\bar{y}\}$ is a winning move for $\forall$, formula is false
- $\{y\}$ is not a winning move and $\{x\}$ is a countermove

Computing a Winning Move—Base Case

`Solve` ($\exists X.\ \phi$), where ϕ is a propositional
output : a winning move for $\exists$ if there is one; NULL otherwise
return `SAT`(ϕ)

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`Solve` ($\forall X.\ \phi$), where ϕ is a propositional
output : a winning move for $\forall$ if there is one; NULL otherwise
return `SAT`($\neg\phi$)

Naive Algorithm for a Winning Move

```
Function Solve (QX. Φ)
Λ ← {true, false}^X                      // consider all assignments
while true do
    if Λ = ∅ then
        return NULL                      // all assignments used up
    τ ← pick(Λ)                          // pick a candidate solution
    μ ← Solve(Φ[τ])                      // find a countermove
    if μ = NULL then
        return τ                         // winning move
    Λ ← Λ ∖ {τ}                          // remove bad candidate
end
```

Removing More Than One Candidate at a Time

Observation

- The naive algorithm does not avail of the countermove

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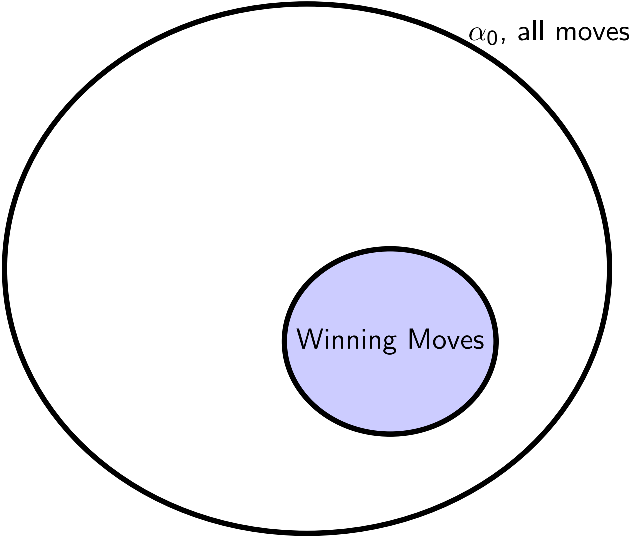
How?

- represent the set of considered candidates as the set of winning moves of a (simpler) QBF (abstraction)
- each time a countermove is found, strengthen the abstraction so that the same countermove cannot be used in the future (refinement)

CounterExample Guided Abstraction Refinement (CEGAR)



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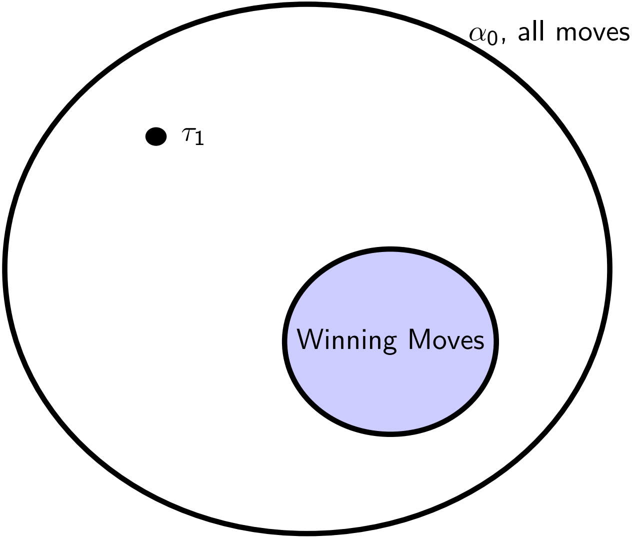

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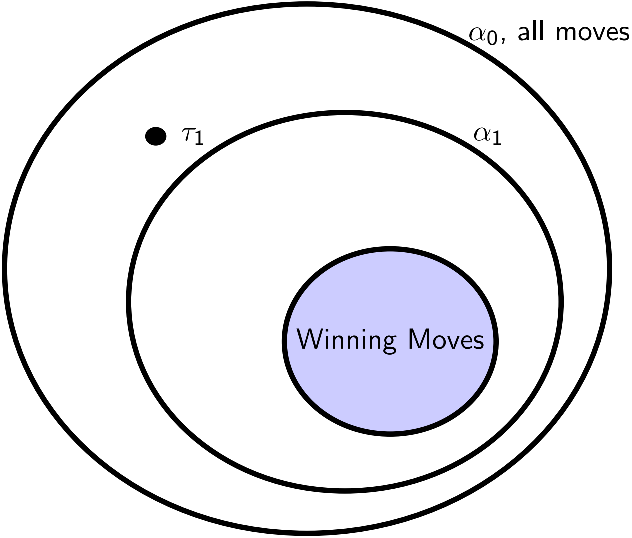

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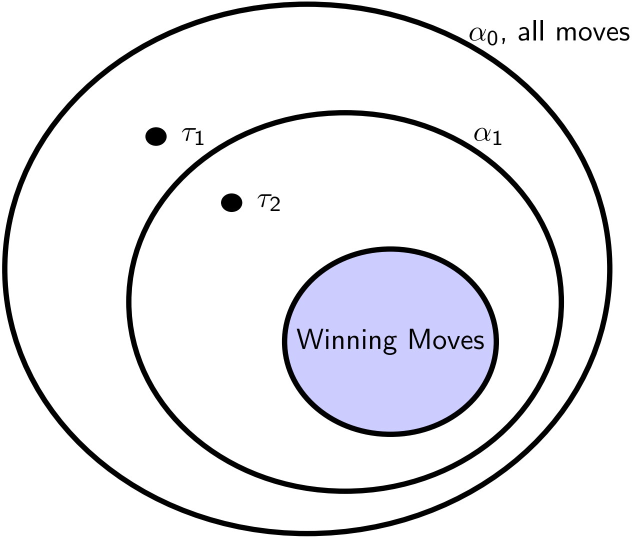

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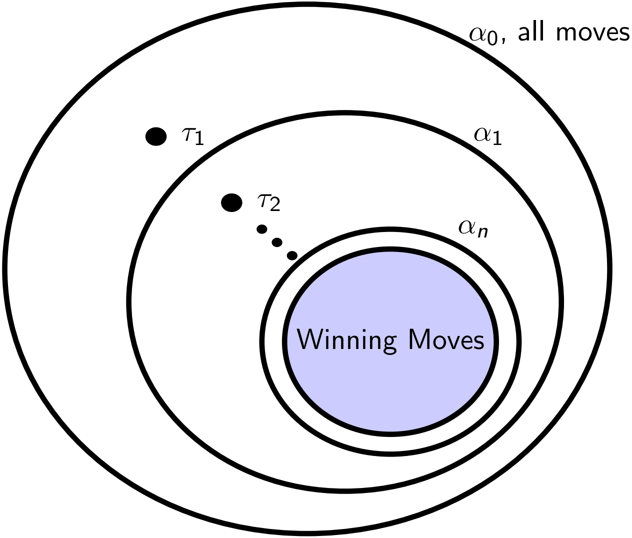

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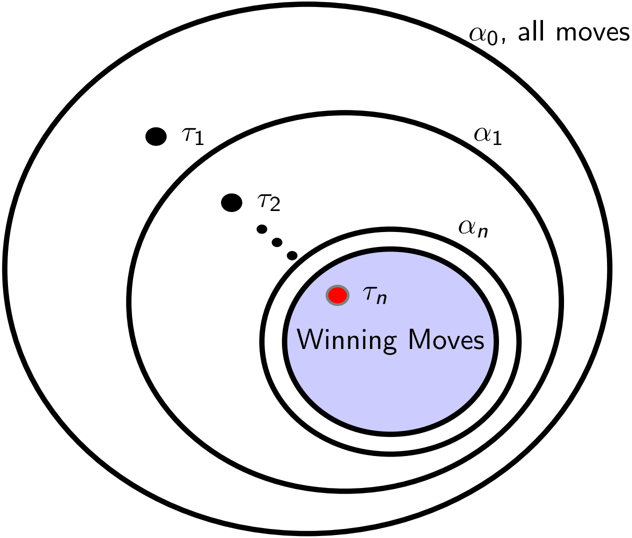

Refinement

for a bad candidate τ

- $\bar{Q}$ wins by μ, i.e. $QX\bar{Q}Y.\ \Phi[\tau][\mu]$ is losing for Q

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for a set of countermoves $\omega = \{\mu_1, \ldots, \mu_n\}$

- $\bigwedge_{\mu\in\omega} \Phi[\mu]$, $Q = \exists$
- $\bigvee_{\mu\in\omega} \Phi[\mu]$, $Q = \forall$

Refinement—Example

- $\forall y \exists x.\ x \wedge (y \vee \bar{x})$
- candidate: $\{y\}$, countermove: $\{x\}$
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Abstraction-Based Algorithm for a Winning Move

Function Solve $(QX.\, \Phi)$
begin
  if Φ *has no quant* **then**
    return $(Q = \exists)$? SAT(ϕ) : SAT$(\neg\phi)$
  $\omega \leftarrow \emptyset$
  while true **do**
    $\alpha \leftarrow (Q = \exists)\ ?\ \bigwedge_{\mu \in \omega} \Phi[\mu]\ :\ \bigvee_{\mu \in \omega} \Phi[\mu]$ // abstraction
    $\tau' \leftarrow$ Solve(Prenex$(QX.\, \alpha)$) // find a candidate
    if $\tau' =$ NULL **then return** NULL // no winning move
    $\tau \leftarrow \{l \mid l \in \tau' \wedge \mathrm{var}(l) \in X\}$ // filter a move for X
    $\mu \leftarrow$ Solve$(\Phi[\tau])$ // find a countermove
    if $\mu =$ NULL **then return** τ // winning move
    $\omega \leftarrow \omega \cup \{\mu\}$ // refine
  end
end

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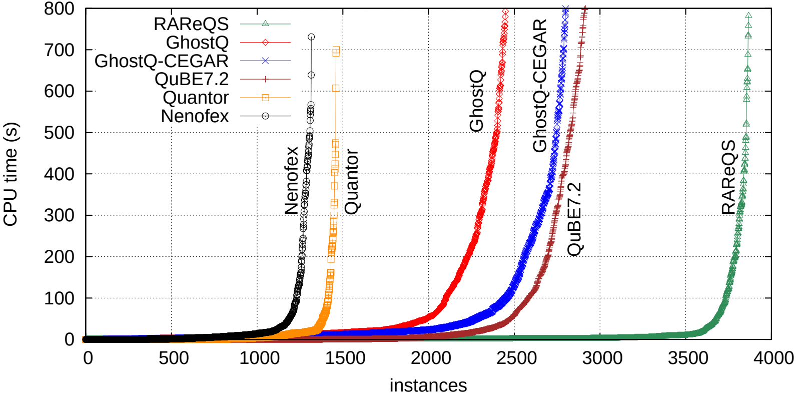
Results for planning and Formal Verification families



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- in the future we plan to further develop the integration between DPLL and CEGAR
- in RAReQS we plan to investigate how to integrate techniques used in other solvers (e.g. dependency detection)

Questions?

Results

Total instances solved (out of 4669):

RAReQS	GhostQ	GhostQ-C	Qube	Quantor	Nenofex
3868	2449	2801	2916	1462	1317