Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 7, Issue 1 (June 2012), pp. 357 – 364

Applications and Applied Mathematics:
An International Journal
(AAM)

Solving Singular Boundary Value Problems Using Daftardar-Jafari Method

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Received: June 28, 2011; Accepted: May 08, 2012

Abstract

In this paper, we apply the suggested iterative method by Daftardar and Jafari hereafter called Daftardar-Jafari method for solving singular boundary value problems. In the implementation of this new method, one does not need the computation of the derivative of the so-called Adomian polynomials. The method is quite efficient and is practically well suited for use in these problems. Two illustrative examples has been presented.

Keywords: Singular boundary value problems; Daftardar-Jafari iterative method; non-linear problems.

MSC 2010: 34B16, 34K28

1. Introduction

An accurate and fast numerical solution of the two-point singular boundary value problems in ordinary differential equations is necessary in many important scientific and engineering applications, such as reactant concentration in a chemical reactor, boundary layer theory, control and optimization theory, and flow networks in biology, thermal distribution in human head, the study of astrophysics such as the theory of stellar interiors, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and the theory of thermionic currents. We consider the class of the singular two-point boundary value problems that arises in heat conduction through a solid with heat generation, the function $f(x, y) = f(y)$ represents the heat generation within the solid, with $\alpha = 0,1,2$

$$y''(x) + \frac{\alpha}{x} y'(x) + f(x, y) = 0 \tag{1}$$

where (1) represents the solid which could be a plate, a cylinder or a sphere with boundary conditions

$$y'(0) = 0 \,,\; A\, y(1) + B\, y'(1) = C \tag{2}$$

Equation (1) with α = 1, 2 and $f(x, y) = \upsilon e^{y}$, where υ is a physical parameter that arises in the theory of thermal explosions. The solution to these problems were obtained using numerical methods like the finite-difference method, cubic and quintic spline methods, finite-element methods, semi-analytical method based on Maple [Subramanian (2000), De Vidts (1992)] and the Adomian decomposition method Kumar (2010).

Recently Daftardar-Gejji and Jafari introduced an iterative method for solving functional equations [Daftardar-Gejji (2006)]. Koçak and Yildirim [Koçak (2011)] used this method for finding exact solutions of nonlinear time-fractional partial differential equations. They also have shown the method is convergent [Bhaleker (2011)]. This iterative method solves nonlinear equations without using Adomian polynomials and is considered an advantage over the Adomian decomposition method. In this work we have used their Iterative method for solving the singular non-linear problems.

2. Daftardar-Jafari Method

Consider the following general functional equation:

$$y = N(y) + f \,, \tag{3}$$

where N is a nonlinear operator from a Banach space $B \rightarrow B$ and f a known function. We are looking for a solution y of Equation (1) having the series form:

$$y = \sum_{i=0}^{\infty} y_i . \tag{4}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\} . \tag{5}$$

Substituting Equation (4) and (5) in Equation (3) leads to

$$\sum_{i=0}^{\infty} y_i = f + N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\} . \tag{6}$$

We define the recurrence relation:

$$\begin{cases} y_0 = f \\ y_1 = N(y_0) \\ y_{m+1} = N(y_0 + \cdots + y_m) - N(y_0 + \cdots y_{m-1}) \ , \ m = 1,2, \ldots \end{cases} \tag{7}$$

Then,

$$y_1 + \cdots + y_{m+1} = N(y_0 + \cdots + y_m) \ , \qquad m = 1,2, \ldots \tag{8}$$

and

$$y_i = f + \sum_{i=0}^{\infty} y_i \tag{9}$$

2.1. Convergence of Daftardar-Jafari Method

We present below the condition for convergence of the series $\sum y_i$. For details we refer the refer to [Bhalekar (2011)].

Theorem 1:

If N is $C^{(\infty)}$ in a neighborhood of y_0 and $\|N^n(y_0)\| \leq L$, for any n for some real $L > 0$ and $\| y_i \| \leq M < \frac{1}{e}$, $i = 1, 2, \ldots$, then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent and

$$\|G_n\| \leq LM^n e^{n-1}(e-1) \ , n = 1,2, \ldots \tag{10}$$

where

$$G_n = N\left(\sum_{i=0}^{n} y_i\right) - N\left(\sum_{i=0}^{n-1} y_i\right), \ n = 1,2, \ldots.$$

Theorem 2:

If N is $C^{(\infty)}$ and $\|N^n(y_0)\| \leq M \leq e^{-1} \ \forall n$, then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent.

3. Applications and Numerical Results

To give a clear overview of this method, we present the following illustrative examples.

Example 1.

Consider the non-linear BVP arising in theory of thermal explosions [Kumar (2010)]:

$$\begin{cases} y''(x) + \frac{y'(x)}{x} + \nu e^{y(x)} = 0\ , 0 \leq x \leq 1 \\ y'(0) = 1\ , y(1) = 0\ , \end{cases} \tag{11}$$

The exact solution is

$$y(x) = 2log\left[\frac{\left((8-2\nu) - \frac{[(8-2\nu)^2 - 4\nu^2]^{\frac{1}{2}}}{2}\right) + 1}{\left((8-2\nu) - \frac{[(8-2\nu)^2 - 4\nu^2]^{\frac{1}{2}}}{2}\right) x^2 + 1}\right].$$

For solving this example using DJM, we rewrite Equation (11) in following form

$$Ly + \nu e^{y(x)} = 0\ , \tag{12}$$

where $\mathrm{L} = x^{-1}\frac{\mathrm{d}}{\mathrm{dx}}\left(x^{1}\frac{\mathrm{d}}{\mathrm{dx}}\right)$. Assume $\nu = 1$, applying $L^{-1} = \int_1^x x^{-1} \int_0^x x(.)\, dx\, dx$ in (12) we get

$$y(x) = y(1) - \int_1^x {x_1}^{-1} \int_0^{x_1} t e^{y(t)}\, dt\, dx_1 \tag{13}$$

using the given iterative scheme (6), we have

$$N(y) = -\int_1^x {x_1}^{-1} \int_0^{x_1} t e^{y(t)}\, dt\, dx_1$$

$$y_0 = y(1) = 0$$

$$y_1 = N(y_0) = \frac{1}{4} - \frac{x^2}{4}$$

$$y_2 = N(y_0 + y_1) - N(y_0) = \frac{x^8}{24576} - \frac{5x^6}{4608} + \frac{41x^4}{2048} - \frac{109x^2}{1536} + \frac{3833}{73728}$$

$$\begin{aligned} y_3 &= N(y_0 + y_1 + y_2) - N(y_0 + y_1) \\ &= -\frac{x^{26}}{60204858690502656} + \frac{5x^{24}}{3206175906594816} \\ &- \frac{2707x^{22}}{32328940391497728} + \frac{204773x^{20}}{6011579824865280 0} \\ &- \frac{3133865x^{18}}{28855583159353344} + \frac{2018683x^{16}}{712483534798848} \\ &- \frac{1225596691x^{14}}{19637827427893248} + \frac{460596497x^{12}}{400771988324352} \\ &- \frac{5938558481x^{10}}{329853488332800} + \frac{637971498221x^8}{3206175906594816} \\ &- \frac{1224122727961x^6}{801543976648704} + \frac{1873838802013x^4}{267181325549568} \\ &- \frac{164291159413673x^2}{9618527719784448} + \frac{1649862916096709 5341817}{14456661590627605020672 00} \end{aligned}$$

and so on. In view of the above terms

$$y(x) = \frac{1}{4} - \frac{x^2}{4} + \frac{x^8}{24576} - \frac{5x^6}{4608} + \frac{41x^4}{2048} - \frac{109x^2}{1536} + \frac{3833}{73728} + \cdots$$

Table. 1. Exhibits the approximate solution obtained by using the Daftardar-Jafari method (DJM) and the Modified Adomian decomposition method (MADM). It is clear that the obtained results are in high agreement with the exact solutions. Higher accuracy can be obtained by using more terms.

Table 1. Results for Example 1

x	Exact Solution	DJM	MADM
0.1	0.313266	0.313521	0.314246
0.2	0.303015	0.303155	0.303986
0.3	0.286047	0.286015	0.287001
0.4	0.262531	0.262303	0.263462
0.5	0.232697	0.232286	0.2335998
0.6	0.196827	0.196283	0.197689
0.7	0.155248	0.154658	0.156056
0.8	0.108323	0.1078	0.109045
0.9	0.0564386	0.0561113	0.057015
1.0	0	5.08342(-17)	4.78(-7)

In Figure 1., approximate solution (Dashed line) (y(t) $\cong y_3(t)$) using the Daftardar-Jafari method and the exact solution have been plotted.

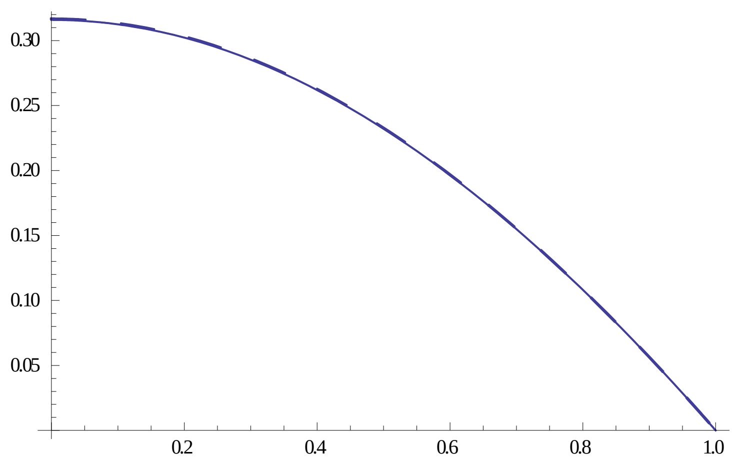

Figure1. Exact solution and solution by DJM for ν = 1

Example 2.

Consider the non-linear problem explosions [Kumar (2010)]:

$$\begin{cases} y''(x) + \frac{0.5}{x} y'(x) = e^{y(x)}\left(0.5 - e^{y(x)}\right), 0 \le x \le 1 \\ y(0) = Log[2] \ , y(1) = 0 \ , \end{cases}$$

The exact solution is

$$y(x) = Log[\tfrac{2}{1+x^2}] . \tag{15}$$

Operating $L = x^{-0.5} \frac{d}{dx}\left(x^{0.5} \frac{d}{dx}\right)$ and $L^{-1}(.) = \int_0^x x^{-0.5} \int_0^x x^{0.5} (.)\, dx\, dx$ then by using L^{-1} in (14) we get

$$y(x) = y(0) + \int_0^x {x_1}^{-0.5} \int_0^{x_1} t^{0.5} e^{y(t)}\left(0.5 - e^{y(t)}\right) dt\, dx_1 , \tag{16}$$

using the iterative scheme developed in section 2, we have

$$N(y) = \int_0^x {x_1}^{-0.5} \int_0^{x_1} t^{0.5} e^{y(t)}\left(0.5 - e^{y(t)}\right) dt\, dx_1$$

$$y_0 = y(0) = Log[2]$$

$$y_1 = N(y_0) = -0.987065x^2$$

$$y_2 = N(y_0 + y_1) - N(y_0) = -0.000135929x^{14} + 0.00191599x^{12}$$

$$-0.0136008x^{10} + 0.0617587x^8 - 0.197284x^6 + 0.474059x^4$$

.

In view of the above terms

$$y(x) = Log[2] - 0.987065x^2 - 0.000135929x^{14} + 0.00191599x^{12} + \ldots$$

Table 2. Exhibits the approximate solution obtained by using the Daftardar-Jafari method (DJM) and the Modified Adomian decomposition method (MADM). It is clear that the obtained results are in high agreement with the exact solutions. Higher accuracy can be obtained by using more terms.

x	Exact Solution	DJM	MADM
0.1	0.683197	0.683324	0.683172
0.2	0.653926	0.654399	0.653539
0.3	0.606969	0.607887	0.605083
0.4	0.544727	0.545943	0.53907
0.5	0.470004	0.470976	0.457062
0.6	0.385662	0.038531	0.360836
0.7	0.294371	0.290994	0.252475
0.8	0.198451	0.189757	0.13475
0.9	0.998203	0.0830512	0.0119924
1.0	0	-0.0278647	-0.108368

In Figure 2, Approximate solution (Dashed line) ($y(t) \cong y_4(t)$) of Equation (14) using the Daftardar-Jafari method and the exact solution have been plotted.

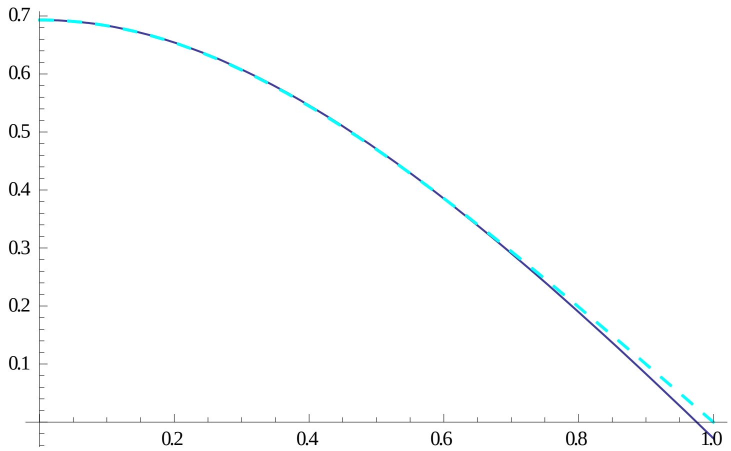

Figure 2. Exact solution and solution by DJM

4. Conclusion

The Daftardar-Jafari iterative method has been applied to give very reliable and accurate solutions to the singular problem. The method gives convergent approximations and handles non-linear problems. The fact that the DJM method solves nonlinear problems without using Adomians polynomials can be considered as an advantage of this method over the Adomian decomposition method.

Mathematica has been used for computations in this paper.

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