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Solving solitary waves with discontinuity by means of the homotopy analysis method

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Abstract

An analytic method, namely the homotopy analysis method (HAM), is applied to solve solitary waves governed by Camassa–Holm equation. Purely analytic solutions are given for soliton waves with and without continuity at crest. This provides with a new analytic approach to solve soliton waves with discontinuity.

1. Introduction

It is difficult to solve nonlinear problems, especially by analytic techniques. Currently, an analytic method for strongly nonlinear problems, namely the homotopy analysis method (HAM) [1,2], has been developed. Different from perturbation techniques [3], the homotopy analysis method does not depend upon any small or large parameters, and thus is valid for most of nonlinear problems in science and engineering. Besides, as pointed out by Liao [1], it logically contains other non-perturbation techniques such as Lyapunov's small parameter method [4], the δ-expansion method [5], and Adomian's decomposition method [6], and therefore is more general. The homotopy analysis method was successfully applied to solve many nonlinear problems such as the nonlinear waves [7], the magnetohydrodynamic flows of non-Newtonian fluids viscous flow past a porous plate [9], flows of an Oldroyd 6-constant fluid [10], soliton wave with one-loop [11], unsteady boundary-layer flows caused by an impulsively stretching plate [12], and so on.

All of the previous applications of the homotopy analysis method deal with solutions without discontinuity. However, many nonlinear problems have different types of discontinuity. In this paper, in order to verify the validity of the homotopy analysis method for nonlinear problems with discontinuation, we further apply it to solve shallow water solitary wave problems governed by Camassa–Holm equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \tag{1}$$

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where k is a constant related to the critical shallow water wave speed, c is the phase speed, u denotes the velocity, x and t denote the spatial and temporal variables, respectively. Camassa and Holm [13,14] pointed out that the solitary wave solution exists when $0 \leqslant k < 1/2$. Especially, when $k = 0$, the solitary wave can be simply expressed by

$$u(x,t) = c\exp(-|x-ct|), \tag{2}$$

whose first derivative is discontinue at the crest. But, when $k \neq 0$, the solitary wave is continuous at the crest. This is rather interesting.

In this article, we employ the homotopy analysis method to solve this solitary wave problem so as to provide a new analytic approach for nonlinear problems with discontinuity.

2. Mathematical formulation

Under the definition $\theta = x - ct$ and $u(\theta) = aw(\theta)$, Eq. (1) reads

$$2kw' - cw' + cw''' + 3aww' = 2aw'w'' + aww''', \tag{3}$$

where c is the wave speed, a is the wave amplitude, and the prime denotes the derivative with respect to θ, respectively. Writing $K = k/c$, we have

$$c(w''' - w' + 2Kw') + 3aww' - 2aw'w'' - aww''' = 0. \tag{4}$$

According to Camassa and Holm [13,14], the solitary waves exist for $0 \leqslant K < \frac{1}{2}$. Due to the symmetry of the solitons, we consider the wave profile only for $\theta \geqslant 0$.

Write

$$w(\theta) \sim B\exp(-\mu\theta), \text{ as } \theta \to +\infty, \tag{5}$$

where $\mu > 0$ and B are constants. Substituting it into Eq. (4) and balancing the main terms, we have

$$\mu^2 = 1 - 2K. \tag{6}$$

Writing $\eta = \mu\theta$, Eq. (4) becomes

$$c(1-2K)(w''' - w') + aw[3w' - (1-2K)w'''] - 2a(1-2K)w'w'' = 0, \tag{7}$$

where the prime denotes the derivative with respect to η. For simplicity, we choose $c = 1$.

2.1. Solutions with continuity of derivative at crest

First of all, let us consider the solitary waves with continuous first derivative at crest, corresponding to $0 < K < 1/2$. Assume that the dimensionless wave elevation $w(\eta)$ arrives its maximum at the origin. Obviously, $w(\eta)$ and its derivatives tend to zero as $\eta \to +\infty$. Besides, due to the continuity, the first derivative of $w(\eta)$ at crest is zero. Thus, the boundary conditions of the solitary waves are

$$w(0) = 1, \quad w'(0) = 0, \quad w(+\infty) = 0. \tag{8}$$

According to the governing equation (7) and the boundary conditions (8), the solution can be expressed by a set of base functions

$$\{\exp(-n\eta) | n = 1, 2, 3, \ldots\}$$

in the form

$$w(\eta) = \sum_{n=1}^{+\infty} d_n \exp(-n\eta), \tag{9}$$

where d_n is a coefficient to be determined. This provides us with the so-called *rule of solution expression* (see [1,2]), i.e. the solution of Eqs. (7) and (8) must be expressed in the same form as (9) and other expressions such as $n^m\exp(-n\eta)$ must be avoided. Furthermore, according to the governing equation (7) and the rule of solution expression (9), we choose the linear operator

$$L[\Phi(n;q)] = \left(\frac{\partial^3}{\partial\eta^3} - \frac{\partial}{\partial\eta}\right)\Phi(\eta;q), \tag{10}$$

with the property

$$L[C_1\exp(-\eta)+C_2\exp(\eta)+C_3]=0, \tag{11}$$

where C_1, C_2, and C_3 are constants. From (7), we define a nonlinear operator

$$N[\Phi(n;q),A(q)]=c(1-2K)\left(\frac{\partial\Phi^3}{\partial^3\eta}-\frac{\partial\Phi}{\partial\eta}\right)+A(q)\Phi\left[3\frac{\partial\Phi}{\partial\eta}-(1-2K)\frac{\partial\Phi^3}{\partial^3\eta}\right]-2A(q)(1-2K)\frac{\partial\Phi}{\partial\eta}\frac{\partial^2\Phi}{\partial^2\eta}. \tag{12}$$

According to the boundary conditions (8) and the rule of solution expression (9), it is straightforward that the initial approximation should be in the form

$$w_0(\eta)=b_1\exp(-\eta)+b_2\exp(-2\eta).$$

Substituting it into (8), we have $b_1 = 2$ and $b_2 = -1$, which gives

$$w_0(\eta)=2\exp(-\eta)-\exp(-2\eta). \tag{13}$$

Using above definitions, we construct the *zeroth-order deformation equation*

$$(1-q)L[\Phi(\eta;q)-w_0(\eta)]=\hbar qN[\Phi(\eta;q),A(q)], \tag{14}$$

subject to the boundary conditions

$$\Phi(0,q)=1,\quad \Phi(+\infty,q)=0,\quad \Phi_n(0,q)=0, \tag{15}$$

where $\hbar$ is a nonzero auxiliary parameter, $q\in[0,1]$ is an embedding parameter, and the subscript denotes derivative with respect to η, respectively.

When $q = 0$, it is obvious that the zeroth-order deformation equation (14) has the solution

$$\Phi(x,0)=w_0(\eta). \tag{16}$$

When $q = 1$, Eq. (14) is the same as Eq. (7), provided

$$\Phi(x,1)=w(\eta),\quad A(1)=a. \tag{17}$$

Thus as q increases from 0 to 1, the solution $\Phi(\eta,q)$ varies from the initial solution $w_0(\eta)$ to $w_0(n)$, so does $A(q)$ from the initial guess a_0 to the wave amplitude a. Expand $\Phi(\eta,q)$ and $A(q)$ in Taylor series with respect to g, i.e.

$$\Phi(\eta;q)=w_0(\eta)+\sum_{m=1}^{+\infty}w_m(\eta)q^m, \tag{18}$$

$$A(q)=a_0+\sum_{m=1}^{+\infty}a_mq^m, \tag{19}$$

Where

$$w_m(\eta)=\frac{1}{m!}\left.\frac{\partial^m\Phi(\eta,q)}{\partial q^m}\right|_{q=0}, \tag{20}$$

$$a_m=\frac{1}{m!}\left.\frac{\partial^mA(q)}{\partial q^m}\right|_{q=0}. \tag{21}$$

If the auxiliary parameter $\hbar$ is so properly chosen that these two series are convergent at $q = 1$, we have from (18) and (19) that

$$w(\eta)=w_0(\eta)+\sum_{m=1}^{+\infty}w_m(\eta), \tag{22}$$

$$a=a_0+\sum_{m=1}^{+\infty}a_m. \tag{23}$$

Fore the sake of simplicity, define

$$\vec{w}_k = \{w_0, w_1, w_2, \ldots, w_k\}, \tag{24}$$

$$\vec{a}_k = \{a_0, a_1, a_2, \ldots, a_k\}. \tag{25}$$

Differentiating Eqs. (14) and (15) m times with respect to q, then setting $q = 0$, and finally dividing them by $m!$, we gain the *mth-order deformation equation*

$$L[w_m(\eta) - \chi_m w_{m-1}(\eta)] = \hbar R_m(\vec{w}_{m-1}, \vec{a}_{m-1}), \tag{26}$$

subject to the boundary conditions

$$w_m(0) = w_m(+\infty) = w_m'(0) = 0, \tag{27}$$

where

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1 \end{cases} \tag{28}$$

and

$$\begin{aligned} R_m(\vec{w}_{m-1}, \vec{a}_{m-1}) = {} & c(1-2K)(w_{m-1}''' - w_{m-1}') + \sum_{i=0}^{m-1} \sum_{j=0}^{i} a_j w_{i-j} [3w_{m-i-1}' - (1-2K) w_{m-i-1}'''] \\ & - 2(1-2K) \sum_{i=0}^{m-1} \sum_{j=0}^{i} a_j w_{i-j}' w_{m-i-1}''. \end{aligned} \tag{29}$$

It is easy to solve above linear differential equation. Due to (11), the solution of $w_m(\eta)$ can be expressed in the form

$$w_m(\eta) = C_1 \exp(-\eta) + C_2 \exp(\eta) + C_3 + \hat{w}_m(\eta), \tag{30}$$

where $\hat{w}_m(\eta)$ is a special solution of Eq. (26) which contains the unknown term a_{m-1}. According to the boundary conditions (27) and the rule of solution expression (9), we have

$$C_2 = C_3 = 0,$$

and besides the unknown $a_{m-1}(\eta)$ and C_1 are governed by

$$\hat{w}_m(0) + C_1 = 0, \quad \hat{w}_m'(0) - C_1 = 0.$$

Thus, the unknown a_{m-1} is obtained by solving the linear algebraic equation

$$\hat{w}_m(0) + \hat{w}_m'(0)$$

and thereafter C_1 is given by

$$C_1 = -\hat{w}_m(0).$$

2.2. Solutions with discontinuity of derivative at crest

Then, let us consider the case that the first derivative at crest of the solitary waves has not continuity, corresponding to $K = 0$. In this special case, Eq. (7) reads

$$c(w''' - w') + aw(3w' - w''') - 2aw'w'' = 0. \tag{31}$$

The corresponding boundary conditions are

$$w(0) = 1, \quad w(+\infty) = 0. \tag{32}$$

It should be emphasized that the boundary condition $w'(0) = 0$ is invalid now. It is clear that we can choose the same solution expression as (9). However, because we have now only one boundary condition at crest, we should choose a new linear operator

$$\widetilde{L}[\Phi(\eta; q)] = \left(\frac{\partial^3}{\partial \eta^3} - 3\frac{\partial^2}{\partial \eta^2} + 2\frac{\partial}{\partial \eta} \right) \Phi(\eta; q), \tag{33}$$

with the property

$$\widetilde{L}[C_1 \exp(\eta) + C_2 \exp(2\eta) + C_3] = 0, \tag{34}$$

where C_1, C_2 and C_3 are constants. Similarly, from (31), the nonlinear operator is now defined by

$$\widetilde{N}[\Phi(\eta;q),A(q)] = c\left(\frac{\partial\Phi^3}{\partial^3\eta} - \frac{\partial\Phi}{\partial\eta}\right) + A(q)\Phi\left[3\frac{\partial\Phi}{\partial\eta} - \frac{\partial\Phi^3}{\partial^3\eta}\right] - 2A(q)\frac{\partial\Phi}{\partial\eta}\frac{\partial^2\Phi}{\partial^2\eta}. \tag{35}$$

And the initial approximation is chosen as

$$w_0(\eta) = \exp(-\eta) - \varepsilon[\exp(-2\eta) - \exp(-3\eta)], \tag{36}$$

where ε is a parameter to be determined later. Then, we can construct the *zeroth-order deformation equation*

$$(1-q)\tilde{L}[\Phi(n;q) - w_0(\eta)] = \hbar q\tilde{N}[\Phi(\eta;q),A(q)], \tag{37}$$

subject to the boundary conditions

$$\Phi(0,q) = 1, \quad \Phi(+\infty,q) = 0, \tag{38}$$

where q and $\hbar$ are defined as before.

Similarly, as q increases from 0 to 1, the solution $\Phi(\eta,q)$ varies from the initial solution $w_0(\eta)$ to $w(\eta)$, so does $A(q)$ from the initial guess a_0 to a. We can also expand $\Phi(\eta,q)$ and $A(q)$ in Taylor series with respect to q as (18) and (19). If $\hbar$ and ε are so properly chosen that these two series are convergent at $q = 1$, we also have (22) and (23). And in the similar way, we have the *mth-order deformation equation*

$$\tilde{L}[w_m(\eta) - \chi_m w_{m-1}(\eta)] = \hbar\tilde{R}_m(\vec{w}_{m-1},\vec{a}_{m-1}), \tag{39}$$

subject to the boundary conditions

$$w_m(0) = w_m(+\infty) = 0, \tag{40}$$

where

$$\tilde{R}_m(\tilde{w}_{m-1},\vec{a}_{m-1}) = c(w'''_{m-1} - w'_{m-1}) + \sum_{i=0}^{m-1}\sum_{j=0}^{i} a_j w_{i-j}[3w'_{m-i-1} - w'''_{m-i-1}] - 2\sum_{i=0}^{m-1}\sum_{j=0}^{i} a_j w'_{i-j} w''_{m-i-1} \tag{41}$$

and $\vec{a}_k$, $\vec{w}_k$ and χ_k are defined by (24), (25) and (28), respectively.

It is easy to get the solution of the above linear differential equation. Due to (34), its solution $w_m(\eta)$ can be expressed in the form

$$w_m(\eta) = C_1\exp(\eta) + C_2\exp(2\eta) + C_3 + \tilde{w}_m(\eta), \tag{42}$$

where $\tilde{w}_m(\eta)$ is a special solution of Eq. (39), which contains the unknown term a_{m-1}. According to the boundary condition (34) at infinity and the rule of solution expression (9), the constants C_1, C_2 and C_3 must be zero. And due to the boundary condition (34) at $\eta = 0$, the unknown term a_{m-1} is determined by the linear algebraic equation

$$\tilde{w}_m(0) = 0.$$

3. Result analysis

3.1. The solution with continuity of derivative at crest

Note that the two series (22) and (23) contain the auxiliary parameter $\hbar$, which influences the convergent rate and region of the two series. To ensure that these two series converge, we first focus on how to choose a proper value of $\hbar$.

Since a represents the wave amplitude, we can first investigate the influence of $\hbar$ on the series of a by means of the so-called $\hbar$-curve, i.e. a curve of a versus $\hbar$. As pointed by Liao [1], the valid region of $\hbar$ is a horizontal line segment. Thus, the valid region of $\hbar$ in this case is $-22 < \hbar < -7$, as shown in Fig. 1 for $K = 9/20$. As proved by Liao [1], the solution series (23) must be exact solution, as long as it is convergent. It is found that, in general, as long as the series (23) for the amplitude a is convergent, the corresponding series (22) is also convergent. For example, when $K = 9/20$ and $\hbar = -10$, our analytic solution converges, as shown in Fig. 2. Due to Liao's proof (see [1]), the convergent series is one solution of the considered problem. We also check the residual error of the corresponding governing equation, and find that this is indeed true. So, the solitary waves with continuous first derivative at crest can be successfully solved by the homotopy analysis method.

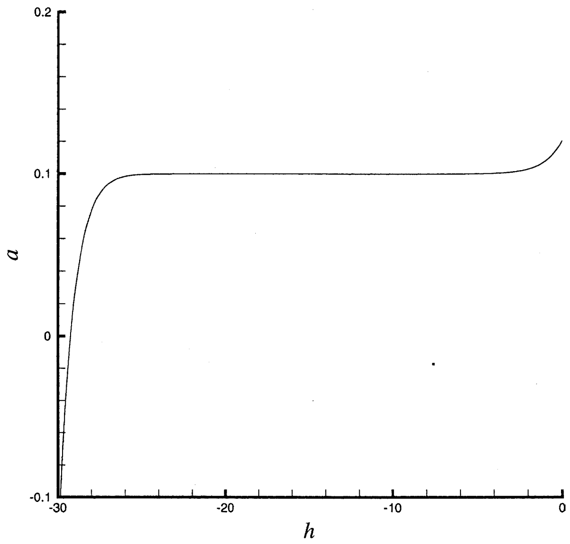

Fig. 1. The $\hbar$-curve of the wave amplitude a at the 20th-order approximation when $K = 9/20$.

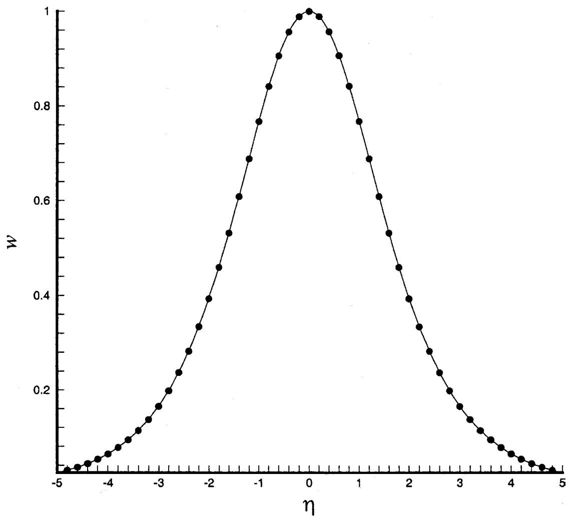

Fig. 2. The analytic approximation of $w(\eta)$ when $K = 9/20$ by means of $\hbar = -10$. Symbols: the 10th-order of approximation; solid line: the 20th-order of approximation.

3.2. The solution with discontinuity of derivative at crest

Camassa and Holm pointed out [13] that the solitary waves with discontinuity at crest exist in the case of $K = 0$, and the corresponding exact solution is

$$w(\eta) = \exp(|-t|). \tag{43}$$

Our solution series contain the auxiliary parameter $\hbar$ and the parameter ε. Similarly, we could choose proper values of $\hbar$ and ε to ensure that the two solution series converge. For a given $\hbar$, we can investigate the influence of ε on the convergence of a by plotting the correspondent curve of a versus ε, as shown in Fig. 3. We find that the valid region of ε is $-1/2 < \varepsilon < 1/2$. In the same way, we can plot the $\hbar$-curve for any given ε, as shown in Fig. 4. Obviously, we can choose $\hbar = -2$ and $\varepsilon = -1/4$. The corresponding 30th-order approximation of $w(\eta)$ agrees well with the exact

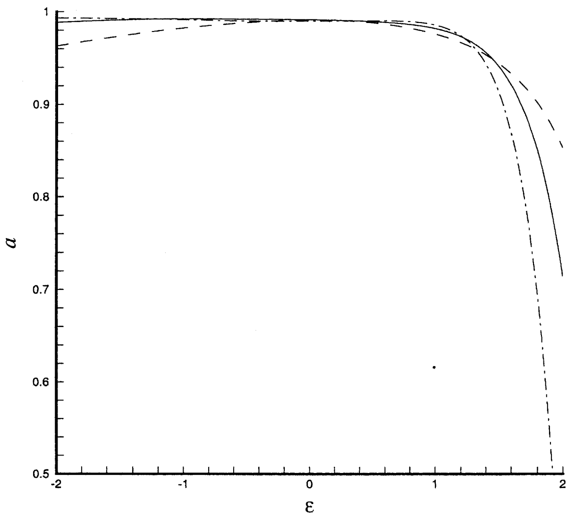

Fig. 3. The curves of the wave amplitude a versus ε when $K = 0$. Solid line: 10th-order of approximation when $\hbar = -2$; dashed line: 10th-order of approximation when $\hbar = -3$; dash-dotted line: 10th-order of approximation when $\hbar = -1$.

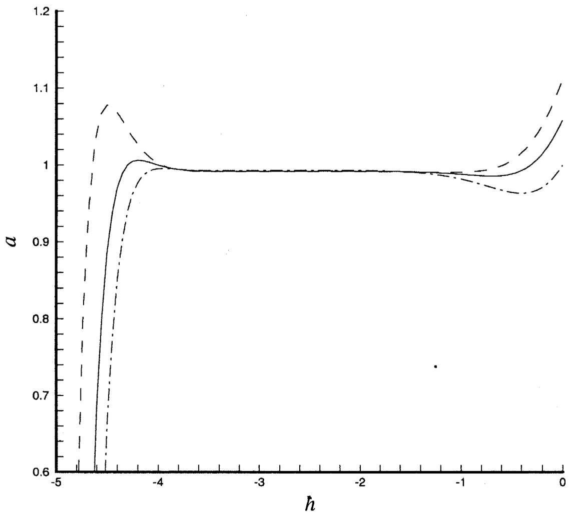

Fig. 4. The curves of the wave amplitude a versus ħ when $K = 0$. Solid line: 20th-order of approximation when $\varepsilon = -1/2$; dashed line: 20th-order of approximation when $\varepsilon = -1/10$; dash-dotted line: 20th-order of approximation when $\varepsilon = -1$.

solution, as shown in Fig. 5. The value of the first derivative at crest is as shown in Table 1. The so-called homotopy-Padé technique (see [1,7]) is employed, which greatly accelerates the convergence. Clearly, the first derivative converges to the exact value -1.

All of these verify that the homotopy analysis method is valid for the solitary wave problems with discontinuity.

4. Conclusion

In this paper, the homotopy analysis method [1] is applied to obtain the analytic solution of the solitary waves governed by Camassa–Holm equation. Analytic solutions are obtained for solitary waves with and without continuity at crest. This provides us with a new analytic way to solve solitary wave problems with discontinuity.

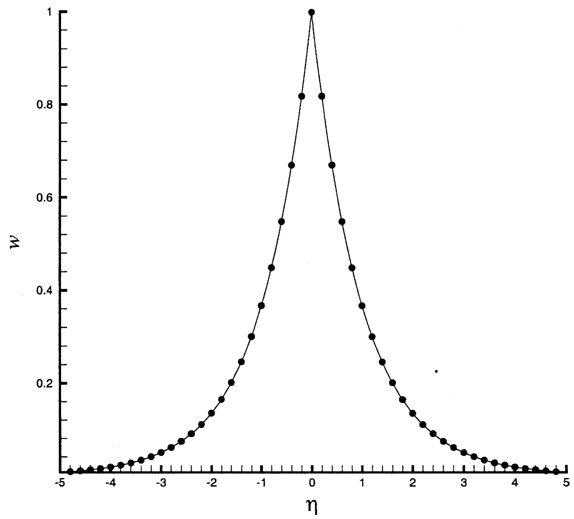

Fig. 5. Comparison of the exact solution with the analytic approximation of $w(\eta)$ when $K = 0$ by means of $\hbar = -2$ and $\varepsilon = -1/4$. Solid line: 30th-order of approximation; symbols: exact solution.

Table 1
The analytic approximation and the $[m, m]$ homotopy-Padé approximation of the first derivative of $w(\eta)$ at crest when $K = 0$ by means of $\hbar = -2$ and $\varepsilon = -1/4$

Order of approximation	$w'(0)$	$[m, m]$	$w'(0)$
2	−0.6517	[1, 1]	−0.3494
4	−0.5883	[2, 2]	−0.5852
6	−0.5602	[3, 3]	−0.5845
8	−0.5577	[4, 4]	−0.6565
10	−0.5712	[5, 5]	−1.0470
20	−0.7020	[10, 10]	−0.9828
30	−0.8044	[15, 15]	−1.0037
40	−0.8699	[20, 20]	−1.0008
50	−0.91213	[25, 25]	−1.0000

The auxiliary linear operator (33) for solitary waves with discontinuous derivative at crest is different from the auxiliary linear operator (11) for those with continuous derivative at crest. This is the only difference for the two cases. It should be emphasized that the homotopy analysis method provides us with such kind of freedom to choose different auxiliary linear operators according to the different properties of solutions of considered nonlinear problems. So, this example also shows the flexibility and potential of the homotopy analysis method for complicated nonlinear problems.

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