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## Solving System of Fractional Differential Equations by Fractional Complex Transform Method

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### ABSTRACT

In this study, we apply fractional complex transform to convert the fractional system of partial differential equations to the system of partial differential equations (PDEs).

**Key words:** Fractional complex transform, system of differential equation, Jumarie's derivative

### INTRODUCTION

To find the solutions of linear and nonlinear differential equations (Taiwo and Abubakar, 2011), many powerful methods have been used (Helal, 2005; Kazemnia *et al.*, 2008; Jin *et al.*, 2010; Mazidi *et al.*, 2011; Sharma and Methi, 2012).

In recent years (Zhang *et al.*, 2012), considerable interest in fractional differential equations (Avaji *et al.*, 2012) has been stimulated due to their numerous applications in the areas of physics and engineering.

Many important phenomena in electromagnetism, acoustics, viscoelasticity, electrochemistry (Abu-Haija *et al.*, 2007) and material science are well described by differential equations of fractional order (Wang, 2007; Momani and Odibat, 2007). To find the explicit solutions of linear and nonlinear fractional differential equations, many powerful methods have been used such as the variational iteration method (He *et al.*, 2010; Wu and Lee, 2010), homotopy perturbation method (Golbabai and Sayevand, 2010) and the Exp-function method (Zhang *et al.*, 2010). The fractional complex transform was first proposed by Li and He (2010). We extend the fractional complex transform method to solve the system of fractional partial differential equations.

### FRACTIONAL COMPLEX TRANSFORM

Jumarie's (Jumarie, 2007, 2010) derivative is a modified Riemann-Liouville derivative defined as:

$$D_z^\gamma f(z) = \begin{cases} \frac{1}{\Gamma(-\gamma)} \frac{d}{dz} \int_0^z (z-\tau)^{-\gamma-1} (f(\tau) - f(0)) d\tau, \gamma < 0, \\ \frac{1}{\Gamma(-\gamma)} \frac{d}{dz} \int_0^z (z-\tau)^{-\gamma} (f(\tau) - f(0)) d\tau, 0 < \gamma < 1, \\ (f^{(\gamma-n)}(z))^{(n)}, n \leq \gamma < n+1, n \geq 1, \end{cases} \quad (1)$$

where,  $f(z)$  is a real continuous (but not necessarily differentiable) function. The fundamental mathematical operations and results of Jumarie's derivative are given (Jumarie, 2007, 2010). Here, we review some of them:

$$D_z^\gamma c = 0, \quad \gamma > 0, \quad c = \text{constant}$$

$$D_z^\gamma (cf(z)) = cD_z^\gamma f(z), \quad \gamma > 0, \quad c = \text{constant}$$

$$D_z^\gamma z^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} z^{\beta-\gamma}, \quad \beta > \gamma > 0$$

$$D_z^\gamma (f(z)g(z)) = (D_z^\gamma f(z))g(z) + f(z)(D_z^\gamma g(z))$$

$$D_z^\gamma (f(z(t))) = f_z'(z) \bullet z^{(\gamma)}(t) = f_z^{(\gamma)}(z)(z_t)^\gamma$$

### EXAMPLES

The fractional complex transform (Li and He, 2010, 2011) and (He, 2011) can convert a fractional differential equation into its differential partner.

**Example 1:** Consider the linear system of PDEs:

$$D_t^\alpha u(x, t) + D_x^\beta v(x, t) = 0$$

$$D_t^\alpha v(x, t) + D_x^\beta u(x, t) = 0 \quad 0 < \alpha, \beta \leq 1 \tag{2}$$

with initial conditions:

$$u(x, 0) = \frac{e^{x^\beta}}{\Gamma(1+\beta)}$$

$$v(x, 0) = \frac{e^{-x^\beta}}{\Gamma(1+\beta)}$$

where,  $\alpha, \beta$  are parameters describing the order of the fractional Jumarie's derivative [8, 9]. By the fractional complex transform:

$$T = p \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad X = q \frac{x^\beta}{\Gamma(\beta+1)} \tag{3}$$

where,  $p$  and  $q$  are constants which are unknown to be further determined. Using Jumarie's chain rule, we have:

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial u}{\partial T} \frac{\partial^\alpha T}{\partial t^\alpha} = \rho \frac{\partial u}{\partial T}, & \frac{\partial^\alpha v}{\partial t^\alpha} &= \frac{\partial v}{\partial T} \frac{\partial^\alpha T}{\partial t^\alpha} = \rho \frac{\partial v}{\partial T} \\ \frac{\partial^\beta u}{\partial x^\beta} &= \frac{\partial u}{\partial X} \frac{\partial^\beta X}{\partial x^\beta} = \rho \frac{\partial u}{\partial X}, & \frac{\partial^\beta v}{\partial x^\beta} &= \frac{\partial v}{\partial X} \frac{\partial^\beta X}{\partial x^\beta} = \rho \frac{\partial v}{\partial X} \end{aligned} \tag{4}$$

By setting  $p = 1$  and  $q = 1$ , we have:

$$\begin{aligned} u_T + v_x &= 0, & v_T + u_x &= 0 \\ u(X, 0) &= e^X, & v(X, 0) &= e^{-X} \end{aligned} \tag{5}$$

The exact solutions are given by Ayaz (2004) as follows:

$$\begin{aligned} u(X, T) &= e^X \cosh T + e^{-X} \sinh T \\ v(X, T) &= e^{-X} \cosh T - e^X \sinh T \end{aligned} \tag{6}$$

Hence:

$$\begin{aligned} u(x, t) &= \frac{e^{x^\beta}}{\Gamma(1+\beta)} \cosh\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right) + \frac{e^{-x^\beta}}{\Gamma(1+\beta)} \sinh\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right), \\ v(x, t) &= \frac{e^{-x^\beta}}{\Gamma(1+\beta)} \cosh\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right) - \frac{e^{x^\beta}}{\Gamma(1+\beta)} \sinh\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right) \end{aligned} \tag{7}$$

**Example 2:** Consider the linear system of PDEs:

$$\begin{aligned} D_t^\alpha u(x, t) + D_x^\beta u(x, t) - 2v(x, t) &= 0 \\ D_t^\alpha v(x, t) + D_x^\beta v(x, t) - 2u(x, t) &= 0, \quad 0 < \alpha, \beta \leq 1 \end{aligned} \tag{8}$$

with initial conditions:

$$\begin{aligned} u(x, 0) &= \sin \frac{x^\beta}{\Gamma(1+\beta)} \\ v(x, 0) &= \cos \frac{x^\beta}{\Gamma(1+\beta)} \end{aligned}$$

By the fractional complex transform:

$$T = \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad X = \frac{x^\beta}{\Gamma(\beta+1)}$$

We have:

$$\begin{aligned} u_T + u_X - 2v &= 0, & v_T + v_X + 2u &= 0 \\ u(X, 0) &= \sin(X), & v(X, 0) &= \cos(X) \end{aligned} \tag{9}$$

The exact solutions are given by Ayaz (2004) as follows:

$$u(X, T) = \sin(X+T), \quad v(X, T) = \cos(X+T) \tag{10}$$

Hence:

$$\begin{aligned} u(x, t) &= \sin\left(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{t^\alpha}{\Gamma(1+\alpha)}\right) \\ v(x, t) &= \cos\left(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{t^\alpha}{\Gamma(1+\alpha)}\right) \end{aligned} \tag{11}$$

**Example 3:** Consider the linear system of PDEs:

$$\begin{aligned} D_t^\alpha u(x, t) + v D_x^\beta u(x, t) + u(x, t) &= 1 \\ D_t^\alpha v(x, t) + u D_x^\beta v(x, t) - v(x, t) &= -1, \quad 0 < \alpha, \beta \leq 1 \end{aligned} \tag{12}$$

with initial conditions:

$$\begin{aligned} u(x, 0) &= e^{\frac{x^\beta}{\Gamma(1+\beta)}} \\ v(x, 0) &= e^{-\frac{x^\beta}{\Gamma(1+\beta)}} \end{aligned}$$

By the fractional complex transform:

$$T = \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad X = \frac{x^\beta}{\Gamma(\beta+1)}$$

We have:

$$\begin{aligned} u_T + v u_X + u &= 1, & v_T + u v_X - v &= -1 \\ u(X, 0) &= e^X, & v(X, 0) &= e^{-X} \end{aligned} \tag{13}$$

The exact solutions are given by Ayaz (2004) as follows:

$$u(X, T) = e^{X+T}, \quad v(X, T) = e^{-X+T} \tag{14}$$

Hence:

$$u(x, t) = e^{\left(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{t^\alpha}{\Gamma(1+\alpha)}\right)}, \quad v(x, t) = e^{\left(-\frac{x^\beta}{\Gamma(1+\beta)} + \frac{t^\alpha}{\Gamma(1+\alpha)}\right)} \tag{15}$$

**Example 4:** Consider the system of PDEs:

$$\begin{aligned} D_t^\alpha u + D_x^\beta v D_y^\gamma w - D_y^\gamma v D_x^\beta w &= -u \\ D_t^\alpha v + D_x^\beta w D_y^\gamma u + D_y^\gamma w D_x^\beta u &= v \\ D_t^\alpha w + D_x^\beta u D_y^\gamma v + D_y^\gamma u D_x^\beta v &= w, \quad 0 < \alpha, \beta, \gamma \leq 1 \end{aligned} \tag{16}$$

with initial conditions:

$$\begin{aligned} u(x, y, 0) &= e^{\frac{x^\beta / \Gamma(1+\beta) + x^\gamma}{\Gamma(1+\gamma)}} \\ v(x, y, 0) &= e^{\frac{x^\beta / \Gamma(1+\beta) - x^\gamma}{\Gamma(1+\gamma)}} \\ w(x, y, 0) &= e^{-\frac{x^\beta / \Gamma(1+\beta) + x^\gamma}{\Gamma(1+\gamma)}} \end{aligned} \tag{17}$$

By the fractional complex transform:

$$T = \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad X = \frac{x^\beta}{\Gamma(\beta+1)}, \quad Y = \frac{x^\gamma}{\Gamma(\gamma+1)}$$

We have:

$$\begin{aligned} u_T + v_X w_Y - v_Y w_X &= -u \\ v_T + w_X u_Y + w_Y u_X &= v \\ w_T + u_X v_Y + u_Y v_X &= w \end{aligned} \tag{18}$$

with initial conditions:

$$u(X, Y, T) = e^{X+Y-T}, \quad v(X, Y, T) = e^{X-Y+T}, \quad w(X, Y, T) = e^{-X+Y+T} \tag{19}$$

The exact solutions are given by Ayaz (2004) as follows:

$$u(X, Y, T) = e^{X+Y-T}, \quad v(X, Y, T) = e^{X-Y+T}, \quad w(X, Y, T) = e^{-X+Y+T} \tag{20}$$

Hence:

$$\begin{aligned} u(x, y, t) &= e^{\left(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{x^\gamma}{\Gamma(1+\gamma)} - \frac{t^\alpha}{\Gamma(1+\alpha)}\right)} \\ v(x, y, t) &= e^{\left(\frac{x^\beta}{\Gamma(1+\beta)} - \frac{x^\gamma}{\Gamma(1+\gamma)} + \frac{t^\alpha}{\Gamma(1+\alpha)}\right)} \\ w(x, y, t) &= e^{\left(-\frac{x^\beta}{\Gamma(1+\beta)} + \frac{x^\gamma}{\Gamma(1+\gamma)} + \frac{t^\alpha}{\Gamma(1+\alpha)}\right)} \end{aligned} \tag{21}$$

## CONCLUSIONS

The fractional complex transform is very simple and use of this method does not need the knowledge of fractional calculus.

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