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Solving systems of nonlinear matrix equations involving Lipshitzian mappings

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Abstract

In this study, both theoretical results and numerical methods are derived for solving different classes of systems of nonlinear matrix equations involving Lipshitzian mappings.

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1 Introduction

Fixed point theory is a very attractive subject, which has recently drawn much attention from the communities of physics, engineering, mathematics, etc. The Banach contraction principle [1] is one of the most important theorems in fixed point theory. It has applications in many diverse areas.

Definition 1.1 *Let M be a nonempty set and f: $M \to M$ be a given mapping. We say that $x^* \in M$ is a fixed point of f if $fx^* = x^*$.*

Theorem 1.1 (Banach contraction principle [1]). *Let (M, d) be a complete metric space and f: $M \to M$ be a contractive mapping, i.e., there exists $\lambda \in [0, 1)$ such that for all $x, y \in M$,*

$$d(fx, fy) \leq \lambda\, d(x, y). \tag{1}$$

Then the mapping f has a unique fixed point $x^* \in M$. Moreover, for every $x_0 \in M$, the sequence (x_k) defined by: $x_{k+1} = fx_k$ for all $k = 0, 1, 2, \ldots$ converges to x^*, and the error estimate is given by:

$$d(x_k, x^*) \leq \frac{\lambda^k}{1-\lambda} d(x_0, x_1), \quad \textit{for all } k = 0, 1, 2, \ldots$$

Many generalizations of Banach contraction principle exists in the literature. For more details, we refer the reader to [2-4].

To apply the Banach fixed point theorem, the choice of the metric plays a crucial role. In this study, we use the Thompson metric introduced by Thompson [5] for the study of solutions to systems of nonlinear matrix equations involving contractive mappings.

We first review the Thompson metric on the open convex cone $P(n)$ $(n \geq 2)$, the set of all $n \times n$ Hermitian positive definite matrices. We endow $P(n)$ with the Thompson metric defined by:

$$d(A,B) = \max\left\{\log M(A/B), \log M(B/A)\right\},$$

where $M(A/B) = \inf\{\lambda > 0: A \leq \lambda B\} = \lambda^+(B^{-1/2}AB^{-1/2})$, the maximal eigenvalue of $B^{-1/2}AB^{-1/2}$. Here, $X \leq Y$ means that $Y - X$ is positive semidefinite and $X < Y$ means that $Y - X$ is positive definite. Thompson [5] (cf. [6,7]) has proved that $P(n)$ is a complete metric space with respect to the Thompson metric d and $d(A, B) = ||\log(A^{-1/2}BA^{-1/2})||$, where $||\cdot||$ stands for the spectral norm. The Thompson metric exists on any open normal convex cones of real Banach spaces [5,6]; in particular, the open convex cone of positive definite operators of a Hilbert space. It is invariant under the matrix inversion and congruence transformations, that is,

$$d(A,B) = d(A^{-1},B^{-1}) = d(MAM^*, MBM^*) \tag{2}$$

for any nonsingular matrix M. The other useful result is the nonpositive curvature property of the Thompson metric, that is,

$$d(X^r, Y^r) \leq r\, d(X,Y), \quad r \in [0,1]. \tag{3}$$

By the invariant properties of the metric, we then have

$$d(MX^rM^*, MY^rM^*) \leq |r| d(X,Y), \quad r \in [-1,1] \tag{4}$$

for any $X, Y \in P(n)$ and nonsingular matrix M.

Lemma 1.1 (see [8]). *For all $A, B, C, D \in P(n)$, we have*

$$d(A+B, C+D) \leq \max\{d(A,C), d(B,D)\}.$$

In particular,

$$d(A+B, A+C) \leq d(B,C).$$

2 Main result

In the last few years, there has been a constantly increasing interest in developing the theory and numerical approaches for HPD (Hermitian positive definite) solutions to different classes of nonlinear matrix equations (see [8-21]). In this study, we consider the following problem: Find $(X_1, X_2, ..., X_m) \in (P(n))^m$ solution to the following system of nonlinear matrix equations:

$$X_i^{r_i} = Q_i + \sum_{j=1}^{m}\left(A_j^* F_{ij}(X_j) A_j\right)^{\alpha_{ij}}, \quad i = 1, 2, \ldots, m, \tag{5}$$

where $r_i \geq 1$, $0 < |\alpha_{ij}| \leq 1$, $Q_i \geq 0$, A_i are nonsingular matrices, and $F_{ij}: P(n) \to P(n)$ are Lipshitzian mappings, that is,

$$\sup_{X,Y\in P(n), X\neq Y} \frac{d(F_{ij}(X), F_{ij}(Y))}{d(X,Y)} = k_{ij} < \infty. \tag{6}$$

If $m = 1$ and $\alpha_{11} = 1$, then (5) reduces to find $X \in P(n)$ solution to $X^r = Q + A^*F(X)A$. Such problem was studied by Liao et al. [15]. Now, we introduce the following definition.

Definition 2.1 *We say that Problem (5) is Banach admissible if the following hypothesis is satisfied:*

$$\max_{1\le i\le m}\left\{\max_{1\le j\le m}\{|\alpha_{ij}|k_{ij}/r_i\}\right\} < 1.$$

Our main result is the following.

Theorem 2.1 *If Problem (5) is Banach admissible, then it has one and only one solution* $(X_1^*, X_2^*, \ldots, X_m^*) \in (P(n))^m$. *Moreover, for any* $(X_1(0), X_2(0), ..., X_m(0)) \in (P(n))^m$, *the sequences* $(X_i(k))_{k\ge 0}$, $1 \le i \le m$, *defined by:*

$$X_i(k+1) = \left(Q_i + \sum_{j=1}^{m}(A_j^* F_{ij}(X_j(k))A_j)^{\alpha_{ij}}\right)^{1/r_i}, \tag{7}$$

converge respectively to $X_1^*, X_2^*, \ldots, X_m^*$, *and the error estimation is*

$$\begin{aligned}&\max\{d(X_1(k), X_1^*), d(X_2(k), X_2^*), \ldots, d(X_m(k), X_m^*)\}\\ &\quad\le \frac{q_m^k}{1-q_m}\max\{d(X_1(1), X_1(0)), d(X_2(1), X_2(0)), \ldots, d(X_m(1), X_m(0))\},\end{aligned} \tag{8}$$

where

$$q_m = \max_{1\le i\le m}\left\{\max_{1\le j\le m}\{|\alpha_{ij}|k_{ij}/r_i\}\right\}.$$

Proof. Define the mapping G: $(P(n))^m \to (P(n))^m$ by:

$$G(X_1, X_2, \ldots, X_m) = (G_1(X_1, X_2, \ldots, X_m), G_2(X_1, X_2, \ldots, X_m), \ldots, G_m(X_1, X_2, \ldots, X_m)),$$

for all $X = (X_1, X_2, ..., X_m) \in (P(n))^m$, where

$$G_i(X) = \left(Q_i + \sum_{j=1}^{m}(A_j^* F_{ij}(X_j)A_j)^{\alpha_{ij}}\right)^{1/r_i},$$

for all $i = 1, 2, ..., m$. We endow $(P(n))^m$ with the metric d_m defined by:

$$d_m((X_1, X_2, \ldots, X_m), (Y_1, Y_2, \ldots, Y_m)) = \max\left\{d(X_1, Y_1), d(X_2, Y_2), \ldots, d(X_m, Y_m)\right\},$$

for all $X = (X_1, X_2, ..., X_m)$, $Y = (Y_1, Y_2, ..., Y_m) \in (P(n))^m$. Obviously, $((P(n))^m, d_m)$ is a complete metric space.

We claim that

$$d_m(G(X), G(Y)) \le q_m\, d_m(X, Y), \quad \text{for all } X, Y \in (P(n))^m. \tag{9}$$

For all X, $Y \in (P(n))^m$, We have

$$d_m(G(X), G(Y)) = \max_{1\le i\le m}\{d(G_i(X), G_i(Y))\}. \tag{10}$$

On the other hand, using the properties of the Thompson metric (see Section 1), for all $i = 1, 2, ..., m$, we have

$$
\begin{aligned}
d(G_i(X), G_i(Y)) &= d\left(\left(Q_i + \sum_{j=1}^{m}(A_j^*F_{ij}(X_j)A_j)^{\alpha_{ij}}\right)^{1/r_i}, \left(Q_i + \sum_{j=1}^{m}(A_j^*F_{ij}(Y_j)A_j)^{\alpha_{ij}}\right)^{1/r_i}\right)\\
&\le \frac{1}{r_i} d\left(Q_i + \sum_{j=1}^{m}(A_j^*F_{ij}(X_j)A_j)^{\alpha_{ij}}, Q_i + \sum_{j=1}^{m}(A_j^*F_{ij}(Y_j)A_j)^{\alpha_{ij}}\right)\\
&\le \frac{1}{r_i} d\left(\sum_{j=1}^{m}(A_j^*F_{ij}(X_j)A_j)^{\alpha_{ij}}, \sum_{j=1}^{m}(A_j^*F_{ij}(Y_j)A_j)^{\alpha_{ij}}\right)\\
&\le \frac{1}{r_i} d\left((A_1^*F_{i1}(X_1)A_1)^{\alpha_{i1}} + \sum_{j=2}^{m}(A_j^*F_{ij}(X_j)A_j)^{\alpha_{ij}}, (A_1^*F_{i1}(Y_1)A_1)^{\alpha_{i1}} + \sum_{j=2}^{m}(A_j^*F_{ij}(Y_j)A_j)^{\alpha_{ij}}\right)\\
&\le \frac{1}{r_i} \max\left\{d((A_1^*F_{i1}(X_1)A_1)^{\alpha_{i1}}, (A_1^*F_{i1}(Y_1)A_1)^{\alpha_{i1}}), d\left(\sum_{j=2}^{m}(A_j^*F_{ij}(X_j)A_j)^{\alpha_{ij}}, \sum_{j=2}^{m}(A_j^*F_{ij}(Y_j)A_j)^{\alpha_{ij}}\right)\right\}\\
&\le \cdots\\
&\le \frac{1}{r_i} \max\left\{d((A_1^*F_{i1}(X_1)A_1)^{\alpha_{i1}}, (A_1^*F_{i1}(Y_1)A_1)^{\alpha_{i1}}), \ldots, d((A_m^*F_{im}(X_m)A_m)^{\alpha_{im}}, (A_m^*F_{im}(Y_m)A_m)^{\alpha_{im}})\right\}\\
&\le \frac{1}{r_i} \max\left\{|\alpha_{i1}| d(A_1^*F_{i1}(X_1)A_1, A_1^*F_{i1}(Y_1)A_1), \ldots, |\alpha_{im}| d(A_m^*F_{im}(X_m)A_m, A_m^*F_{im}(Y_m)A_m)\right\}\\
&\le \frac{1}{r_i} \max\left\{|\alpha_{i1}| d(F_{i1}(X_1), F_{i1}(Y_1)), \ldots, |\alpha_{im}| d(F_{im}(X_m), F_{im}(Y_m))\right\}\\
&\le \frac{1}{r_i} \max\left\{|\alpha_{i1}| k_{i1} d(X_1, Y_1), \ldots, |\alpha_{im}| k_{im} d(X_m, Y_m)\right\}\\
&\le \frac{\max_{1\le j\le m}\{|\alpha_{ij}| k_{ij}\}}{r_i} \max\left\{d(X_1, Y_1), \ldots, d(X_m, Y_m)\right\}\\
&\le \max_{1\le j\le m}\{|\alpha_{ij}| k_{ij}/r_i\}\ d_m(X, Y).
\end{aligned}
$$

Thus, we proved that for all $i = 1, 2, \ldots, m$, we have

$$d(G_i(X), G_i(Y)) \le \max_{1\le j\le m}\{|\alpha_{ij}| k_{ij}/r_i\}\ d_m(X, Y). \tag{11}$$

Now, (9) holds immediately from (10) and (11). Applying the Banach contraction principle (see Theorem 1.1) to the mapping G, we get the desired result. □

3 Examples and numerical results

3.1 The matrix equation: $X = \left(((X^{1/2} + B_1)^{-1/2} + B_2)^{1/3} + B_3\right)^{1/2}$

We consider the problem: Find $X \in P(n)$ solution to

$$X = \left(\left((X^{1/2} + B_1)^{-1/2} + B_2\right)^{1/3} + B_3\right)^{1/2}, \tag{12}$$

where $B_i \ge 0$ for all $i = 1, 2, 3$.

Problem (12) is equivalent to: Find $X_1 \in P(n)$ solution to

$$X_1^{r_1} = Q_1 + (A_1^*F_{11}(X_1)A_1)^{\alpha_{11}}, \tag{13}$$

where $r_1 = 2$, $Q_1 = B_3$, $A_1 = I_n$ (the identity matrix), $\alpha_{11} = 1/3$ and $F_{11} : P(n) \to P(n)$ is given by:

$$F_{11}(X) = (X^{1/2} + B_1)^{-1/2} + B_2.$$

Proposition 3.1 *F_{11} is a Lipshitzian mapping with $k_{11} \le 1/4$.*

Proof. Using the properties of the Thompson metric, for all $X, Y \in P(n)$, we have

$$\begin{aligned} d(F_{11}(X), F_{11}(Y)) &= d((X^{1/2} + B_1)^{-1/2} + B_2, (Y^{1/2} + B_1)^{-1/2} + B_2) \\ &\leq d((X^{1/2} + B_1)^{-1/2}, (Y^{1/2} + B_1)^{-1/2}) \\ &\leq \frac{1}{2} d(X^{1/2} + B_1, Y^{1/2} + B_1) \\ &\leq \frac{1}{2} d(X^{1/2}, Y^{1/2}) \leq \frac{1}{4} d(X, Y). \end{aligned}$$

Thus, we have $k_{11} \leq 1/4$. □

Proposition 3.2 *Problem (13) is Banach admissible.*

Proof. We have

$$\frac{|\alpha_{11}|k_{11}}{r_1} \leq \frac{\frac{1}{3}\frac{1}{4}}{2} = \frac{1}{24} < 1.$$

This implies that Problem (13) is Banach admissible. □

Theorem 3.1 *Problem (13) has one and only one solution $X_1^* \in P(n)$. Moreover, for any $X_1(0) \in P(n)$, the sequence $(X_1(k))_{k \geq 0}$ defined by:*

$$X_1(k+1) = \left(\left((X_1(k)^{1/2} + B_1)^{-1/2} + B_2 \right)^{1/3} + B_3 \right)^{1/2}, \tag{14}$$

converges to X_1^*, and the error estimation is

$$d(X_1(k), X_1^*) \leq \frac{q_1^k}{1 - q_1} d(X_1(1), X_1(0)), \tag{15}$$

where $q_1 = 1/4$.

Proof. Follows from Propositions 3.1, 3.2 and Theorem 2.1. □

Now, we give a numerical example to illustrate our result given by Theorem 3.1.

We consider the 5 × 5 positive matrices B_1, B_2, and B_3 given by:

$$B_1 = \begin{pmatrix} 1.0000 & 0.5000 & 0.3333 & 0.2500 & 0 \\ 0.5000 & 1.0000 & 0.6667 & 0.5000 & 0 \\ 0.3333 & 0.6667 & 1.0000 & 0.7500 & 0 \\ 0.2500 & 0.5000 & 0.7500 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1.4236 & 1.3472 & 1.1875 & 1.0000 & 0 \\ 1.3472 & 1.9444 & 1.8750 & 1.6250 & 0 \\ 1.1875 & 1.8750 & 2.1181 & 1.9167 & 0 \\ 1.0000 & 1.6250 & 1.9167 & 1.8750 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$B_3 = \begin{pmatrix} 2.7431 & 3.3507 & 3.3102 & 2.9201 & 0 \\ 3.3507 & 4.6806 & 4.8391 & 4.3403 & 0 \\ 3.3102 & 4.8391 & 5.2014 & 4.7396 & 0 \\ 2.9201 & 4.3403 & 4.7396 & 4.3750 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We use the iterative algorithm (14) to solve (12) for different values of $X_1(0)$:

$$X_1(0) = M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}, \quad X_1(0) = M_2 = \begin{pmatrix} 0.02 & 0.01 & 0 & 0 & 0 \\ 0.01 & 0.02 & 0.01 & 0 & 0 \\ 0 & 0.01 & 0.02 & 0.01 & 0 \\ 0 & 0 & 0.01 & 0.02 & 0.01 \\ 0 & 0 & 0 & 0.01 & 0.02 \end{pmatrix}$$

and

$$X_1(0) = M_3 = \begin{pmatrix} 30 & 15 & 10 & 7.5 & 6 \\ 15 & 30 & 20 & 15 & 12 \\ 10 & 20 & 30 & 22.5 & 18 \\ 7.5 & 15 & 22.5 & 30 & 24 \\ 6 & 12 & 18 & 24 & 30 \end{pmatrix}.$$

For $X_1(0) = M_1$, after 9 iterations, we get the unique positive definite solution

$$X_1(9) = \begin{pmatrix} 1.6819 & 0.69442 & 0.61478 & 0.51591 & 0 \\ 0.69442 & 1.9552 & 0.96059 & 0.84385 & 0 \\ 0.61478 & 0.96059 & 2.0567 & 0.9785 & 0 \\ 0.51591 & 0.84385 & 0.9785 & 1.9227 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and its residual error

$$R(X_1(9)) = \left\| X_1(9) - \left(\left(\left(X_1(9)^{1/2} + B_1 \right)^{-1/2} + B_2 \right)^{1/3} + B_3 \right)^{1/2} \right\| = 6.346 \times 10^{-13}.$$

For $X_1(0) = M_2$, after 9 iterations, the residual error

$$R(X_1(9)) = 1.5884 \times 10^{-12}.$$

For $X_1(0) = M_3$, after 9 iterations, the residual error

$$R(X_1(9)) = 1.1123 \times 10^{-12}.$$

The convergence history of the algorithm for different values of $X_1(0)$ is given by Figure 1, where c_1 corresponds to $X_1(0) = M_1$, c_2 corresponds to $X_1(0) = M_2$, and c_3 corresponds to $X_1(0) = M_3$.

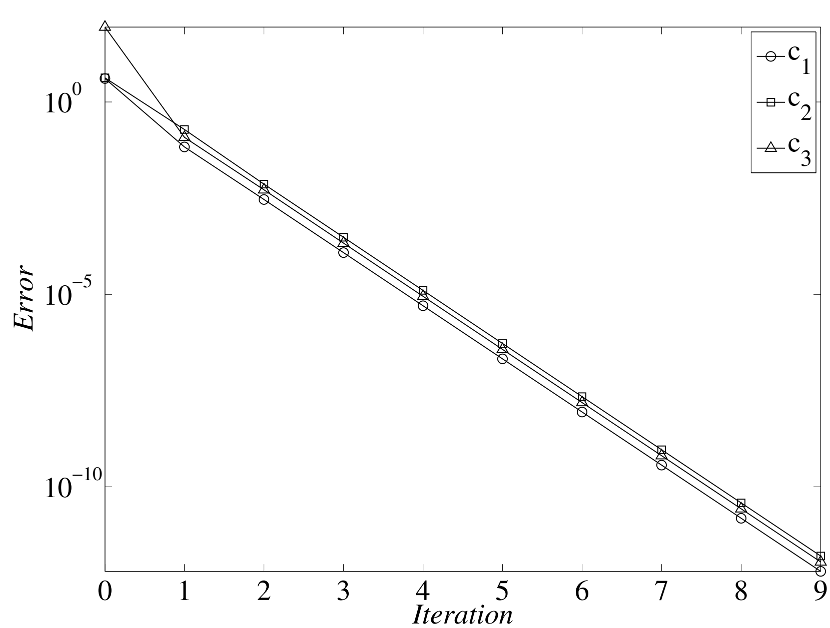

Figure 1 Convergence history for Eq. (12).

3.2 System of three nonlinear matrix equations

We consider the problem: Find $(X_1, X_2, X_3) \in (P(n))^3$ solution to

$$\begin{cases} X_1 = I_n + A_1^*(X_1^{1/3} + B_1)^{1/2}A_1 + A_2^*(X_2^{1/4} + B_2)^{1/3}A_2 + A_3^*(X_3^{1/5} + B_3)^{1/4}A_3, \\ X_2 = I_n + A_1^*(X_1^{1/5} + B_1)^{1/4}A_1 + A_2^*(X_2^{1/3} + B_2)^{1/2}A_2 + A_3^*(X_3^{1/4} + B_3)^{1/3}A_3, \\ X_3 = I_n + A_1^*(X_1^{1/4} + B_1)^{1/3}A_1 + A_2^*(X_2^{1/5} + B_2)^{1/4}A_2 + A_3^*(X_3^{1/3} + B_3)^{1/2}A_3, \end{cases} \tag{16}$$

where A_i are $n \times n$ singular matrices.

Problem (16) is equivalent to: Find $(X_1, X_2, X_3) \in (P(n))^3$ solution to

$$X_i^{r_i} = Q_i + \sum_{j=1}^{3} (A_j^* F_{ij}(X_j)A_j)^{\alpha_{ij}}, \quad i = 1, 2, 3, \tag{17}$$

where $r_1 = r_2 = r_3 = 1$, $Q_1 = Q_2 = Q_3 = I_n$ and for all $i, j \in \{1, 2, 3\}$, $\alpha_{ij} = 1$,

$$F_{ij}(X_j) = (X_j^{\theta_{ij}} + B_j)^{\gamma_{ij}}, \quad \theta = (\theta_{ij}) = \begin{pmatrix} 1/3 & 1/4 & 1/5 \\ 1/5 & 1/3 & 1/4 \\ 1/4 & 1/5 & 1/3 \end{pmatrix}, \quad \gamma = (\gamma_{ij}) = \begin{pmatrix} 1/2 & 1/3 & 1/4 \\ 1/4 & 1/2 & 1/3 \\ 1/3 & 1/4 & 1/2 \end{pmatrix}.$$

Proposition 3.3 *For all $i, j \in \{1, 2, 3\}$, F_{ij}: $P(n) \to P(n)$ is a Lipshitzian mapping with $k_{ij} \le \gamma_{ij}\theta_{ij}$.*

Proof. For all $X, Y \in P(n)$, since $\theta_{ij}, \gamma_{ij} \in (0, 1)$, we have

$$\begin{aligned} d(F_{ij}(X), F_{ij}(Y)) &= d((X^{\theta_{ij}} + B_j)^{\gamma_{ij}}, (Y^{\theta_{ij}} + B_j)^{\gamma_{ij}}) \\ &\le \gamma_{ij} d(X^{\theta_{ij}} + B_j, Y^{\theta_{ij}} + B_j) \\ &\le \gamma_{ij} d(X^{\theta_{ij}}, Y^{\theta_{ij}}) \\ &\le \gamma_{ij}\theta_{ij} d(X, Y). \end{aligned}$$

Then, F_{ij} is a Lipshitzian mapping with $k_{ij} \le \gamma_{ij}\theta_{ij}$. □

Proposition 3.4 *Problem (17) is Banach admissible.*

Proof. We have

$$\begin{aligned} \max_{1\le i\le 3}\left\{\max_{1\le j\le 3}\{|\alpha_{ij}|k_{ij}/r_i\}\right\} &= \max_{1\le i,j\le 3} k_{ij} \\ &\le \max_{1\le i,j\le 3} \gamma_{ij}\theta_{ij} \\ &= 1/6 < 1. \end{aligned}$$

This implies that Problem (17) is Banach admissible. □

Theorem 3.2 *Problem (16) has one and only one solution $(X_1^*, X_2^*, X_3^*) \in (P(n))^3$. Moreover, for any $(X_1(0), X_2(0), X_3(0)) \in (P(n))^3$, the sequences $(X_i(k))_{k\ge 0}$, $1 \le i \le 3$, defined by:*

$$X_i(k + 1) = I_n + \sum_{j=1}^{3} A_j^* F_{ij}(X_j(k))A_j, \tag{18}$$

converge respectively to X_1^*, X_2^*, X_3^*, and the error estimation is

$$\begin{aligned} &\max\{d(X_1(k), X_1^*), d(X_2(k), X_2^*), d(X_3(k), X_3^*)\} \\ &\quad \le \frac{q_3^k}{1 - q_3}\max\{d(X_1(1), X_1(0)), d(X_2(1), X_2(0)), d(X_3(1), X_3(0))\}, \end{aligned} \tag{19}$$

where $q_3 = 1/6$.

Proof. Follows from Propositions 3.3, 3.4 and Theorem 2.1. □

Now, we give a numerical example to illustrate our obtained result given by Theorem 3.2.

We consider the 3 × 3 positive matrices B_1, B_2 and B_3 given by:

$$B_1 = \begin{pmatrix} 1. & 0.5 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1.25 & 1 & 0 \\ 1 & 1.25 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B_3 = \begin{pmatrix} 1.75 & 1.625 & 0 \\ 1.625 & 1.75 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We consider the 3 × 3 nonsingular matrices A_1, A_2 and A_3 given by:

$$A_1 = \begin{pmatrix} 0.3107 & -0.5972 & 0.7395 \\ 0.9505 & 0.1952 & -0.2417 \\ 0 & -0.7780 & -0.6282 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1.5 & -2 & 0.5 \\ 0.5 & 0 & -0.5 \\ -0.5 & 2 & -1.5 \end{pmatrix}$$

and

$$A_3 = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix}.$$

We use the iterative algorithm (18) to solve Problem (16) for different values of $(X_1(0), X_2(0), X_3(0))$:

$$X_1(0) = X_2(0) = X_3(0) = M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$X_1(0) = X_2(0) = X_3(0) = M_2 = \begin{pmatrix} 0.02 & 0.01 & 0 \\ 0.01 & 0.02 & 0.01 \\ 0 & 0.01 & 0.02 \end{pmatrix}$$

and

$$X_1(0) = X_2(0) = X_3(0) = M_3 = \begin{pmatrix} 30 & 15 & 10 \\ 15 & 30 & 20 \\ 10 & 20 & 30 \end{pmatrix}.$$

The error at the iteration k is given by:

$$R(X_1(k), X_2(k), X_3(k)) = \max_{1 \le i \le 3} \left\| X_i(k) - I_3 - \sum_{j=1}^{3} A_j^* F_{ij}(X_j(k)) A_j \right\|.$$

For $X_1(0) = X_2(0) = X_3(0) = M_1$, after 15 iterations, we obtain

$$X_1(15) = \begin{pmatrix} 10.565 & -4.4081 & 2.7937 \\ -4.4081 & 16.883 & -6.6118 \\ 2.7937 & -6.6118 & 9.7152 \end{pmatrix}, \quad X_2(15) = \begin{pmatrix} 11.512 & -5.8429 & 3.1922 \\ -5.8429 & 19.485 & -7.9308 \\ 3.1922 & -7.9308 & 10.68 \end{pmatrix}$$

and

$$X_3(15) = \begin{pmatrix} 11.235 & -3.5241 & 3.2712 \\ -3.5241 & 17.839 & -7.8035 \\ 3.2712 & -7.8035 & 11.618 \end{pmatrix}.$$

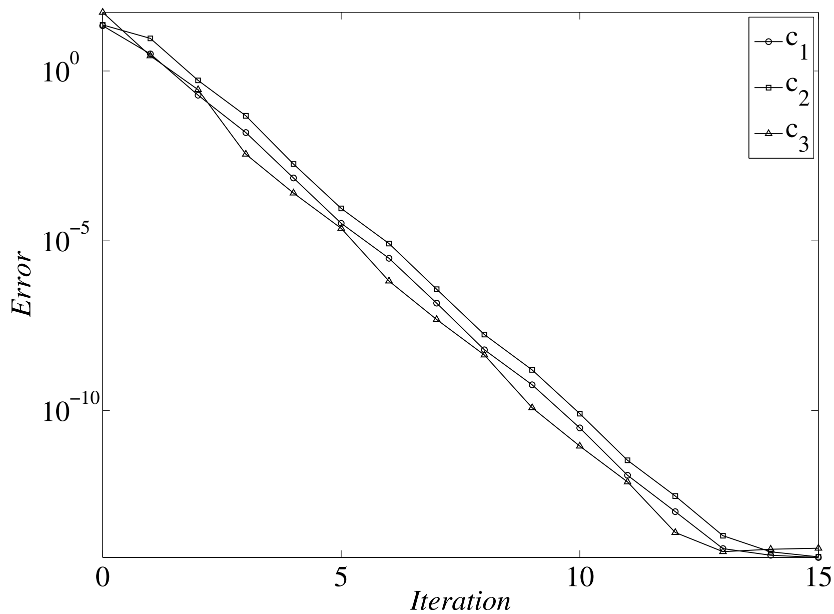

Figure 2 Convergence history for Sys. (16).

The residual error is given by:

$$R(X_1(15), X_2(15), X_3(15)) = 4.722 \times 10^{-15}.$$

For $X_1(0) = X_2(0) = X_3(0) = M_2$, after 15 iterations, the residual error is given by:

$$R(X_1(15), X_2(15), X_3(15)) = 4.911 \times 10^{-15}.$$

For $X_1(0) = X_2(0) = X_3(0) = M_3$, after 15 iterations, the residual error is given by:

$$R(X_1(15), X_2(15), X_3(15)) = 8.869 \times 10^{-15}.$$

The convergence history of the algorithm for different values of $X_1(0)$, $X_2(0)$, and $X_3(0)$ is given by Figure 2, where c_1 corresponds to $X_1(0) = X_2(0) = X_3(0) = M_1$, c_2 corresponds to $X_1(0) = X_2(0) = X_3(0) = M_2$ and c_3 corresponds to $X_1(0) = X_2(0) = X_3(0) = M_3$.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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