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Solving the main cosmological puzzles with a generalized time varying vacuum energy

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ABSTRACT

We study the dynamics of the FLRW flat cosmological models in which the vacuum energy density varies with time, $\Lambda(t)$. In particular, we investigate the dynamical properties of a generalized vacuum model, and we find that under certain circumstances the vacuum term in the radiation era varies as $\Lambda(z) \propto (1+z)^4$, while in the matter era we have $\Lambda(z) \propto (1+z)^3$ up to $z \simeq 3$ and $\Lambda(z) \simeq \Lambda$ for $z \leq 3$. The confirmation of such a behavior would be of paramount importance because it could provide a solution to the cosmic coincidence problem as well as to the fine-tuning problem, without changing the well known (from the concordance Λ-cosmology) Hubble expansion.

Key words. cosmology: theory – methods: analytical

1. Introduction

The analysis of the available high quality cosmological data (supernovae type Ia, CMB, galaxy clustering, etc.) have converged during the last decade towards a cosmic expansion history that involves a spatial flat geometry and a recent accelerating expansion of the universe (Spergel et al. 2007; Davis et al. 2007; Kowalski et al. 2008; Komatsu et al. 2009, and references therein). This expansion has been attributed to an energy component (dark energy) with negative pressure which dominates the universe at late times and causes the observed accelerating expansion. The simplest type of dark energy corresponds to the cosmological constant (see for review Peebles & Ratra 2003). The so called concordance Λ model accurately fits the current observational data and thus is an excellent candidate for the model which describes the observed universe.

However, the concordance model suffers from, among others (cf. Perivolaropoulos 2008), two fundamental problems: (a) *the fine-tuning problem* i.e., the fact that the observed value of the vacuum density ($\rho_\Lambda = \Lambda c^2/8\pi G$) is more than 120 orders of magnitude below that value found using quantum field theory (Weinberg 1989) and (b) *the coincidence problem* i.e., the matter energy density and the vacuum energy density are of the same order prior to the present epoch, despite the fact that the former is a function of time while the latter is not (Peebles & Ratra 2003). Attempts to solve the coincidence problem have been presented in the literature (see Egan & Lineweaver 2008, and references therein), in which an easy way to overcome the coincidence problem is to replace the constant vacuum energy with a dark energy that evolves with time. The simplest approach is to consider a tracker scalar field ϕ in which it rolls down the potential energy $V(\phi)$ and therefore could mimic the dark energy (see Ratra & Peebles 1988; Weinberg 1989; Turner & White 1997; Caldwell et al. 1998; Padmanabhan 2003). Nevertheless, the latter consideration does not really solve the problem because the initial value of the dark energy still needs to be fine-tuned (Padmanabhan 2003). Also, despite the fact that the current observations do not rule out the possibility of a dynamical dark energy (Tegmark et al. 2004), they strongly indicate that the dark energy equation of state parameter $w \equiv P_{\rm DE}/\rho_{\rm DE}$ is close to –1 (Spergel et al. 2007; Davis et al. 2007; Kowalski et al. 2008; Komatsu et al. 2009).

Alternatively, more than two decades ago, Ozer & Taha (1987) proposed a different pattern in which a time varying Λ parameter could be a possible candidate to solve the two fundamental cosmological puzzles (see also Bertolami 1986; Freese et al. 1987; Peebles & Ratra 1988; Carvalho et al. 1992; Overduin & Cooperstock 1998; Bertolami & Martins 2000; Opher & Pellison 2004; Bauer 2005; Barrow & Clifton 2006; Montenegro & Carneiro 2007, and references therein). In this cosmological paradigm, the dark energy equation of state parameter w is strictly equal to –1, but the vacuum energy density (or Λ) is not a constant but varies with time. Of course, the weak point in this theory is the unknown functional form of the $\Lambda(t)$ parameter. Also, in the $\Lambda(t)$ cosmological model there is a coupling between the time-dependent vacuum and matter (Wang & Meng 2005; Alcaniz & Lima 2005; Carneiro et al. 2008; Basilakos 2009; Basilakos et al. 2009). Indeed, using the combination of the conservation of the total energy with the variation of the vacuum energy, one can prove that the $\Lambda(t)$ model provides either a particle production process or that the mass of the dark matter particles increases (Basilakos 2009, and references therein). Despite the fact that most of the recent papers in dark energy studies are based on the assumption that the dark energy evolves independently of the dark matter, the unknown nature of both dark matter and dark energy implies that at the moment we cannot exclude the possibility of interactions in the dark sector (e.g., Zimdahl et al. 2001; Amendola et al. 2003; Cai & Wang 2005; Binder & Kremer 2006; Das et al. 2006; Olivares et al. 2008, and references therein).

In this work we attempt to generalize the main cosmological properties of the traditional Λ-cosmology by introducing a time varying vacuum energy, and specifically to investigate whether such models can yield a late accelerated phase of the cosmic expansion, without the need of the extreme fine-tuning required, in the classical Λ-model. The plan of the paper is as follows: The basic theoretical elements of the problem are presented in Sects. 2–4 by solving analytically (for a spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) geometry) the basic cosmological equations. In these sections we prove further that the concordance Λ-cosmology is a particular solution of the $\Lambda(t)$ models. In Sect. 5 we place constraints on the main parameters of our model by performing a likelihood analysis utilizing the recent Union08 SnIa data (Kowalski et al. 2008). Also, in Sect. 5 we compare the different time varying vacuum models with the traditional Λ cosmology. In this section we treat analytically, the basic cosmological puzzles (the fine-tuning and the cosmic coincidence problem) with the aid of the time varying $\Lambda(t)$ parameter. Finally, we draw our conclusions in Sect. 6.

2. The time dependent vacuum in the expanding universe

In the context of a spatially flat FLRW geometry the basic cosmological equations are:

$$\rho_{\rm tot} = \rho_{\rm f} + \rho_\Lambda = 3H^2 \quad (1)$$

and

$$\frac{{\rm d}(\rho_{\rm f} + \rho_\Lambda)}{{\rm d}t} + 3H(\rho_{\rm f} + P_{\rm f} + \rho_\Lambda + P_\Lambda) = 0, \quad (2)$$

where $\rho_{\rm f}$ is the density of the "cosmic" fluid:

$$\rho_{\rm f}(t) = \begin{cases} \rho_{\rm m}(t) & \text{matter era} \\ \rho_{\rm r}(t) & \text{radiation era} \end{cases} \quad (3)$$

and

$$P_{\rm f}(t) = \beta\rho_{\rm f} = \begin{cases} 0 & \text{matter era } \beta = 0 \\ \frac{\rho_r}{3} & \text{radiation era } \beta = 1/3 \end{cases} \quad (4)$$

is the corresponding pressure. Also ρ_Λ and P_Λ denote the density and the pressure of the vacuum component respectively. From a cosmological point of view, at an early enough epoch, the above generalized cosmic fluid behaves like radiation $P_{\rm f} = P_{\rm r} = \rho_r/3$ ($\beta = 1/3$), then behaves as matter $P_{\rm f} = P_{\rm m} = 0$ ($\beta = 0$) and as long as $P_\Lambda = -\rho_\Lambda$ it creates an accelerated phase of the cosmic expansion (see below). Notice that in order to simplify our formalism we use geometrical units ($8\pi G = c \equiv 1$) in which $\rho_\Lambda = \Lambda$. In the present work, we would like to investigate the potential of a time varying $\Lambda = \Lambda(t)$ parameter to account for the observed acceleration of the expansion of the universe. Within this framework it is interesting to mention that the equation of state takes the usual form of $P_\Lambda(t) = -\rho_\Lambda(t) = -\Lambda(t)$ (see Ozer & Taha 1987; Peebles & Ratra 1988). Also, introducing in the global dynamics the idea of the time-dependent vacuum, it is possible to explain the physical properties of the dark energy as well as the fine-tuning and the coincidence problem respectively (see Sects. 5.1 and 5.2). Using now Eq. (2) we have the following useful formula:

$$\dot{\rho}_{\rm f} + 3(\beta + 1)H\rho_{\rm f} = -\dot{\Lambda} \quad (5)$$

and considering Eq. (1) we find:

$$\dot{H} + \frac{3(\beta + 1)}{2}H^2 = \frac{\Lambda}{2} \quad (6)$$

where the over-dot denotes derivatives with respect to time. If the vacuum term is negligible, $\Lambda(t) \longrightarrow 0$, the solution of the above equation is reduced to $H(t) = 2(\beta + 1)^{-1}/3t$. Therefore, in the case of $\beta = 0$ (matter era) we get the Einstein-de Sitter model as we should, $H(t) = 2/3t$, while for $\beta = 1/3$ we trace the radiation phase of the universe i.e., $H(t) = 1/2t$. On the other hand, if we consider the case of $\Lambda(t) \neq 0$ it becomes evident (see Eq. (5)) that there is a coupling between the time-dependent vacuum and matter (or radiation) component.

Of course, in order to solve the above differential equation we need to define explicitly the functional form of the $\Lambda(t)$ component. Note that the traditional $\Lambda =$ const. cosmology can be described directly by the integration of the Eq. (6) (for more details see Sect. 3.1).

It is worth noting that the $\Lambda(t)$ scenario has the caveat of its unknown exact functional form, which however is also the case for the vast majority of the dark energy models. In the literature there have been different phenomenological parametrizations which treat the time-dependent $\Lambda(t)$ function. In particular, Freese et al. (1987) considered that $\Lambda(t) = 3c_1 H^2$, with the constant c_1 being the ratio of the vacuum to the sum of vacuum and matter density (see also Arcuri & Waga 1994). Chen & Wu (1990) proposed a different ansatz in which $\Lambda(t) \propto a^{-2}$.

Recently, many authors (see for example Ray et al. 2007; Sil & Som 2008, and references therein) have investigated the global dynamical properties of the universe considering that the vacuum energy density decreases linearly either with the energy density or with the square Hubble parameter. Attempts to provide a theoretical explanation for the $\Lambda(t)$ have also been presented in the literature (see Shapiro & Solá 2000; Babić et al. 2002; Grande et al. 2006; Solá 2008, and references therein). There it was found that a time dependent vacuum could arise from the renormalization group (RG) in quantum field theory. The corresponding solution for a running vacuum is found to be $\Lambda(t) = c_0 + c_1 H^2(t)$ (where c_0 and c_1 are constants; Grande et al. 2006) and it can mimic the quintessence or phantom behavior and a smooth transition between the two. Alternatively, Schutzahold (2002) used a different pattern in which the vacuum term is proportional to the Hubble parameter, $\Lambda(a) \propto H(a)$ (see also Carneiro et al. 2008), while Basilakos (2009) considered a power series form in H. Note that the linear pattern, $\Lambda(a) \propto H(a)$, has been motivated theoretically through a possible connection of cosmology with the QCD scale of strong interactions (Schutzhold 2002). In this context it has also been proposed that the vacuum energy density can be defined from a possible link of dark energy with QCD and the topological structure of the universe (Urban & Zhitnitsky 2009a–c).

In this paper we have phenomenologically identified a functional form of $\Lambda(a)$ for which we can solve the main differential equation (see Eq. (6)) analytically. This is:

$$\Lambda_{\gamma m}(t) = 3\gamma H^2(t) + 2mH(t) + 3n(\beta + 1 - \gamma){\rm e}^{2mt} \quad (7)$$

where the constants m and n are included for the consistency of units (see below). Although the above functional form was not motivated by some physical theory but rather phenomenologically by the fact that it provides analytical solutions to the Friedmann equation, its exact form can be physically justified a posteriori within the framework of the previously mentioned theoretical models (see Appendix A).

Using now Eq. (7), the generalized Friedmann's equation (see Eq. (6)) becomes

$$\dot{H} = -\frac{3(\beta + 1 - \gamma)}{2}H^2 + mH + \frac{3n(\beta + 1 - \gamma)}{2}{\rm e}^{2mt} \quad (8)$$

and indeed, it is routine to perform the integration of Eq. (8) to obtain (see Appendix B):

$$H(t) = \sqrt{n}e^{mt}\coth\left[\frac{3(\beta+1-\gamma)\sqrt{n}}{2}S(t)\right] \qquad (9)$$

where

$$S(t) = \begin{cases} (e^{mt}-1)/m & m \neq 0 \\ t & m = 0 \end{cases} \qquad (10)$$

while the range of values for which the above integration is valid is $n \in (0, +\infty)$ (for negative n values see the Appendix B). Using now the definition of the Hubble parameter $H \equiv \dot{a}/a$, the scale factor of the universe $a(t)$ evolves with time as

$$a(t) = a_1 \sinh^{\frac{2}{3(\beta+1-\gamma)}}\left[\frac{3(\beta+1-\gamma)\sqrt{n}}{2}S(t)\right]. \qquad (11)$$

The relevant units of $m \neq 0$ should correspond to time^{-1}, which implies that $m \propto H_0$. The parameter a_1 is the constant of integration given by

$$a_1 \equiv \left(\frac{\rho_{f0}}{\rho_{\Lambda 0}}\right)^{\frac{1}{3(\beta+1-\gamma)}} \qquad (12)$$

where ρ_{f0} and $\rho_{\Lambda 0}$ are the corresponding densities at the present time (for which $a(t_0) \equiv 1$).

In this context, the density of the cosmic fluid evolves with time (see Eq. (1)) as:

$$\rho_f(t) = 3H^2(t) - \Lambda_{\gamma m}(t) \qquad (13)$$

or

$$\rho_f(t) = 3(1-\gamma)H^2(t) - 2mH(t) - 3n(\beta+1-\gamma)e^{mt}. \qquad (14)$$

In the following sections, we investigate thoroughly whether such a generalized vacuum component in an expanding universe allows for a late accelerated phase of the universe, and under which circumstances such an approach provides a viable solution to the fine-tuning problem as well as to the cosmic coincidence problem.

3. The matter+vacuum scenario

In a matter+vacuum expanding universe ($\rho_f \equiv \rho_m$), we attempt to investigate the correspondence of the $\Lambda(t)$ pattern with the traditional Λ-cosmology in order to show the extent to which they compare. In particular, we will prove that the Hubble expansion, provided by the current time-dependent vacuum, is a generalization of the traditional Λ cosmology. Note that in the present formalism the matter era corresponds to $\beta = 0$.

3.1. The standard Λ-cosmology

Let us first investigate the solution for $(\gamma, m) = (0, 0)$. The vacuum term Eq. (7) of the problem becomes constant and is given by $\Lambda_{00}(a) = \Lambda = 3n$. In this framework, the Hubble function (see Eq. (9)) is

$$H_\Lambda(t) = \sqrt{\frac{\Lambda}{3}}\coth\left(\frac{3}{2}\sqrt{\frac{\Lambda}{3}}\,t\right). \qquad (15)$$

Now, using the well know parametrization

$$\Lambda = 3n = 3H_0^2\Omega_\Lambda \quad \Omega_\Lambda = 1-\Omega_m \qquad (16)$$

the scale factor of the universe is given by

$$a_\Lambda(t) = a_1 \sinh^{\frac{2}{3}}\left(\frac{3H_0\sqrt{\Omega_\Lambda}t}{2}\right) \qquad (17)$$

where (see Eq. (12))

$$a_1 = \left(\frac{\rho_{m0}}{\rho_{\Lambda 0}}\right)^{1/3} = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3}. \qquad (18)$$

The cosmic time is related with the scale factor as

$$t_\Lambda(a) = \frac{2}{3\sqrt{\Omega_\Lambda}H_0}\sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_m}}\,a^{3/2}\right). \qquad (19)$$

Combining the above equations we can define the Hubble expansion as a function of the scale factor:

$$H_\Lambda(a) = H_0\left[\Omega_\Lambda + \Omega_m a^{-3}\right]^{1/2}. \qquad (20)$$

In principle, H_0 and Ω_m are constrained by the recent WMAP data combined with the distance measurements from the type Ia supernovae (SNIa) and the Baryonic Acoustic Oscillations (BAOs) in the distribution of galaxies. Following the recent cosmological results provided by Komatsu et al. (2009), we fix the current cosmological parameters as $H_0 = 70.5$ km s^{-1} Mpc^{-1} and $\Omega_m = 1-\Omega_\Lambda = 0.27$. The current age of the universe ($a = 1$) is $t_{0\Lambda} \simeq 13.77$ Gyr, while the inflection point takes place at

$$t_{I\Lambda} = \frac{2}{3\sqrt{\Omega_\Lambda}H_0}\sinh^{-1}\left(\sqrt{\frac{1}{2}}\right), \quad a_{I\Lambda} = \left[\frac{\Omega_m}{2\Omega_\Lambda}\right]^{1/3}. \qquad (21)$$

Therefore, we estimate $t_{I\Lambda} \simeq 0.51t_{0\Lambda}$ and $a_{I\Lambda} \simeq 0.56$.

Finally, due to the fact that the traditional Λ cosmology is a particular solution of the current time varying vacuum models with (γ, m) strictly equal to $(0, 0)$, the constant value n is always defined by Eq. (16). That is why all relevant cosmological quantities are parametrized according to $n = \Omega_\Lambda H_0^2$ throughout the paper.

3.2. "The general" Λ (t) model

In this section, we examine a more general class of vacuum models with $(\gamma, m) \neq (0, 0)$ (hereafter $\Lambda_{\gamma m}$ model). The Hubble expansion and the corresponding evolution of the scale factor are (see Eqs. (9) and (11))

$$H(t) = \sqrt{\Omega_\Lambda}\,H_0\,e^{mt}\coth\left[\frac{3(1-\gamma)\sqrt{\Omega_\Lambda}H_0}{2m}(e^{mt}-1)\right] \qquad (22)$$

and

$$a(t) = a_1 \sinh^{\frac{2}{3(1-\gamma)}}\left[\frac{3(1-\gamma)\sqrt{\Omega_\Lambda}H_0}{2m}(e^{mt}-1)\right] \qquad (23)$$

or

$$t(a) = \frac{1}{m}\ln\left[1 + \frac{2m}{3(1-\gamma)\sqrt{\Omega_\Lambda}H_0}\sinh^{-1}\left(\frac{a}{a_1}\right)^{3(1-\gamma)/2}\right]. \qquad (24)$$

Obviously, if $(\gamma, m) \longrightarrow (0, 0)$ (or $e^{mt}-1 \approx mt$) then the $\Lambda_{\gamma m}$ model tends to the traditional Λ cosmology, which implies that the latter should be considered as a particular solution of the general $\Lambda_{\gamma m}$ model. Thus this limit together with Eq. (12) provides that

$$a_1 = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3(1-\gamma)}}. \qquad (25)$$

Taking the above expressions into account, the basic cosmological quantities as a function of the scale factor become

$$H(a) = H_0\,[1+g(a)]\left[\Omega_\Lambda + \Omega_m a^{-3(1-\gamma)}\right]^{1/2} \quad (26)$$

and

$$\Lambda_{\gamma m}(a) = 3\gamma H^2 + 2mH + 3H_0^2\Omega_\Lambda(1-\gamma)[1+g(a)]^2 \quad (27)$$

where

$$g(a) = \frac{2m}{3\sqrt{(1-\gamma)\Omega_\Lambda}H_0}\,\sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_m}}\,a^{3(1-\gamma)/2}\right). \quad (28)$$

If we take $(\gamma, m) = (0, m)$ with $m \neq 0$ (hereafter mild vacuum model or Λ_{0m}), the corresponding Hubble flow becomes:

$$H(a) = [1+g(a)]\,H_\Lambda(a) \quad (29)$$

which means that as long as the function $g(a)$ takes small values ($g(a) \ll 1$), the Λ_{0m} model has exactly the constant vacuum feature due to $H(a) \approx H_\Lambda(a)$. In this context, utilizing Eq. (27) we simply have

$$\Lambda_{0m}(a) = 2mH(a) + 3H_0^2\Omega_\Lambda[1+g(a)]^2. \quad (30)$$

Finally, the fact that the vacuum term has units of time^{-2} implies that the vacuum term is proportional to H_0^2 or the constant m has to satisfy the following scaling relation: $m \propto H_0$ (see also Sect. 2). Therefore, in the far future the condition $m \propto H_0 \neq 0$ represents a super-accelerated expansion of the universe because $a(t) \propto \exp(\frac{\sqrt{\Omega_\Lambda}H_0 e^{mt}}{m})$.

3.3. "The modified" Λ model

Now we consider $(\gamma, m) = (\gamma, 0)$ with $\gamma \neq 0$ (hereafter $\Lambda_{\gamma 0}$ model). From Eq. (9) we can easily write the corresponding Hubble flow as a function of time

$$H(t) = \sqrt{\Omega_\Lambda}\,H_0 \coth\left[\frac{3(1-\gamma)\sqrt{\Omega_\Lambda}H_0}{2}\,t\right]. \quad (31)$$

Using now Eqs. ((10), (11)), the scale factor of the universe $a(t)$ evolves with time as

$$a(t) = a_1 \sinh^{\frac{2}{3(1-\gamma)}}\left[\frac{3(1-\gamma)\sqrt{\Omega_\Lambda}H_0}{2}\,t\right] \quad (32)$$

where

$$a_1 = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3(1-\gamma)}. \quad (33)$$

Inverting Eq. (32) we estimate the cosmic time:

$$t(a) = \frac{2}{3(1-\gamma)\sqrt{\Omega_\Lambda}H_0}\sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_m}}\,a^{3(1-\gamma)/2}\right). \quad (34)$$

The corresponding inflection point ($\ddot{a}(t_I) = 0$) is found to be

$$t_I = \frac{2}{3(1-\gamma)\sqrt{\Omega_\Lambda}H_0}\sinh^{-1}\left(\sqrt{\frac{1-3\gamma}{2}}\right) \quad (35)$$

or

$$a_I = \left[\frac{(1-3\gamma)\Omega_m}{2\Omega_\Lambda}\right]^{1/3(1-\gamma)} \quad (36)$$

which implies that the condition for which an inflection point is present in the evolution of the scale factor is $\gamma < 1/3$.

As expected, for $\gamma \ll 1$ the above solution tends to the concordance model, $a_{\gamma 0}(t) \longrightarrow a_\Lambda(t)$. Now from Eqs. ((31), (32)), using the well known hyperbolic formula $\coth^2 x - 1 = 1/\sinh^2 x$, we arrive after some algebra:

$$H(a) = H_0\left[\Omega_\Lambda + \Omega_m a^{-3(1-\gamma)}\right]^{1/2}. \quad (37)$$

From this analysis it becomes clear that the Hubble expansion predicted by the $\Lambda_{\gamma 0}$ model extents well beyond that of the usual Λ cosmology. To this end, utilizing Eq. (27) we can obtain the vacuum energy density

$$\Lambda_{\gamma 0}(a) = 3\gamma H^2(a) + 3\Omega_\Lambda H_0^2(1-\gamma). \quad (38)$$

As we have previously mentioned in Sect. 2, the above phenomenological functional form (see Eq. (38)) is motivated theoretically by the renormalization group (RG) in the quantum field theory (Shapiro & Solá 2000; Babić et al. 2002; Solá 2008). Moreover, recent studies (see Grande et al. 2006; and Grande et al. 2009) find that this solution alleviates the cosmic coincidence problem (see Sect. 5.1). Obviously, at late enough times ($a \gg 1$) the above solution asymptotically reaches the de Sitter regime $\Lambda \sim H^2$.

4. The radiation+vacuum scenario

In this section, we consider a universe that is spatially flat but contains both radiation and a time vacuum term. This crucial period in the cosmic history corresponds to $\beta = 1/3$. For clarity reasons we re-formulate our approach by using $\rho_f \equiv \rho_r$ and $P_f \equiv \rho_r/3$ in the following sections. These restrictions imply that

$$\frac{\rho_{f0}}{\rho_{\Lambda 0}} \equiv \frac{\rho_{r0}}{\rho_{\Lambda 0}} = \frac{\Omega_r}{\Omega_\Lambda}$$

where, $\Omega_r \simeq 10^{-4}$ is the radiation density parameter at the present epoch derived by the CMB data (see Komatsu et al. 2009). Within this context, based on Eqs. (7), (11), and (12) we present briefly the following cosmological situations:

- **radiation+constant vacuum**: $(\gamma, m) = (0, 0)$: The scale factor is

$$a(t) = \left(\frac{\Omega_r}{\Omega_\Lambda}\right)^{\frac{1}{4}}\sinh^{\frac{1}{2}}\left(\sqrt{\Omega_\Lambda}H_0 t\right). \quad (39)$$

  Owing to the fact that in this period $t \ll 1$, the above solution reduces to the following simple analytic approximation:

$$a(t) \approx (2\sqrt{\Omega_r}H_0 t)^{1/2} \quad \text{with} \quad H(t) \equiv \frac{\dot{a}}{a} \approx \frac{1}{2t}. \quad (40)$$

- **radiation+general vacuum**: $(\gamma, m) \neq (0, 0)$: this general scenario provides

$$a(t) = \left(\frac{\Omega_r}{\Omega_\Lambda}\right)^{\frac{1}{4\gamma_1}}\sinh^{\frac{1}{2\gamma_1}}\left[\frac{2\gamma_1\sqrt{\Omega_\Lambda}H_0}{m}(e^{mt}-1)\right] \quad (41)$$

  where $\gamma_1 = 1 - 3\gamma/4$. The vacuum component as a function of time (see Eq. (7)) is

$$\Lambda_{\gamma m}(t) \approx \frac{4(1-\gamma_1)}{4\gamma_1^2 t^2} + \frac{m}{\gamma_1 t} \quad (42)$$

or

$$\Lambda_{\gamma m}(a) \approx \frac{4(1-\gamma_1)\Omega_r H_0^2}{a^{4\gamma_1}} + \frac{2m\sqrt{\Omega_r}H_0^2}{a^{2\gamma_1}} \cdot \quad (43)$$

It is very interesting that during the radiation epoch $\Lambda_{\gamma m}(a) \propto a^{-4\gamma_1}$. For small values of γ or $\gamma_1 \simeq O(1)$, the latter relation implies that as long as the scale factor tends to zero the vacuum term moves rapidly to infinity (see Sect. 6). In the case of $(\gamma, m) = (0, m)$ (or $\gamma_1 = 1$), the vacuum term (see Eqs. (42) and (43)) varies with time as

$$\Lambda_{0m}(t) \approx \frac{m}{t} \approx \frac{2m\sqrt{\Omega_r}H_0^2}{a^2} \cdot \quad (44)$$

Now the vacuum component evolves as $\Lambda_{\gamma 0}(a) \propto a^{-2}$, in agreement with the Chen & Wu (1990) model.

– **radiation+modified vacuum**: $(\gamma, m) = (\gamma, 0)$, $\gamma \neq 0$: in this cosmological model we have

$$a(t) = \left(\frac{\Omega_r}{\Omega_\Lambda}\right)^{\frac{1}{4\gamma_1}} \sinh^{\frac{1}{2\gamma_1}}\left[2\gamma_1\sqrt{\Omega_\Lambda}H_0\, t\right] \quad (45)$$

where $\gamma_1 = 1-3\gamma/4$. The approximate solution now becomes

$$a(t) \approx (2\gamma_1\sqrt{\Omega_r}H_0 t)^{1/2\gamma_1} \quad \text{with} \quad H(t) \approx \frac{1}{2\gamma_1 t} \cdot \quad (46)$$

The vacuum component (see Eq. (7)) evolves with time as

$$\Lambda_{\gamma 0}(t) \approx \frac{4(1-\gamma_1)}{4\gamma_1^2 t^2} \quad (47)$$

or

$$\Lambda_{\gamma 0}(a) \approx \frac{4(1-\gamma_1)\Omega_r H_0^2}{a^{4\gamma_1}} \simeq \Lambda_{\gamma m}(a). \quad (48)$$

Obviously, for $a \longrightarrow 0$ ($\gamma_1 \simeq O(1)$) the vacuum energy density goes rapidly to infinity.

5. Tackling the cosmological puzzles

As we have stated already in the introduction, there is a possibility for the vacuum energy to be a function of time rather than having a constant value. Therefore, in this section we compare the cosmic phases of the $\Lambda(t)$ scenarios (described in the previous sections) and the concordance Λ-cosmology. The aim here is to investigate the consequences of such a comparison on the basic cosmological puzzles, namely the cosmic coincidence problem and fine-tuning problem.

5.1. The coincidence problem

In order to investigate the coincidence problem we define the time-dependent proximity parameter of $\rho_m(a)$ (see Eq. (14)) and $\rho_\Lambda(a)$ (see Egan & Lineweaver 2008, and references therein):

$$r(a) \equiv \min\left[\frac{\rho_\Lambda(a)}{\rho_m(a)}, \frac{\rho_m(a)}{\rho_\Lambda(a)}\right] \quad (49)$$

where in this work we use $\rho_\Lambda(a) \equiv \Lambda(a)$ (see Eq. (7)). If the two densities differ by many orders of magnitude then $r \simeq 0$. If on the other hand the two densities are equal the proximity parameter is $r = 1$. The current observational data shows that the proximity parameter at the present time ($a = 1$) is $r_0 = \frac{\rho_m(1)}{\rho_\Lambda(1)} = \frac{\Omega_m}{\Omega_\Lambda} \simeq 0.37$. A cosmological model may therefore suffer from the so called coincidence problem if its proximity parameter is close to zero before the inflection point, $r(a < a_I) \sim 0$. As an example, for the traditional Λ-cosmology we have $r(a < 0.56) \sim 0$. On the other hand, if for a particular model we find that $r(a < a_I) = O(1)$ then this model possibly does not suffer from the cosmic coincidence problem.

In particular, suppose that we have a cosmological model which accommodates a late time accelerated expansion and contains n-free parameters, described by the vector $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, ..., \epsilon_n)$. The main question that we should address here is the following: "what is the range of input $(\epsilon_1, \epsilon_2, ..., \epsilon_n)$ parameters for which the coincidence problem can be avoided?" Below we implement the following tests.

(i) We find the range of the free parameters of the considered cosmological model that implies $r \simeq r_0$ for at least two different epochs, one of which is precisely the present epoch.

(ii) We know that for epochs between the inflection point and the present time $a_I \leq a \leq 1$, the proximity parameter is $r(a) \geq r_0$. As an example, for the traditional Λ-cosmology we have $r(a) \geq 0.37$. Thus, the goal here is to define the range of the free parameters in which at least a second region with $r(a < a_I) \geq r_0$ occurs before the inflection point ($a < a_I$).

(ii) Once steps (i) and (ii) are accomplished, we finally check whether the remaining parameters fit the recent SnIa data by performing a standard χ^2 minimization. In this work, we use the so called Union08 sample of 307 supernovae of Kowalski et al. (2008). In particular, the χ^2 function can be written as:

$$\chi^2(\boldsymbol{\epsilon}) = \sum_{j=1}^{307}\left[\frac{\mu^{\rm th}(a_j, \boldsymbol{\epsilon}) - \mu^{\rm obs}(a_j)}{\sigma_j}\right]^2 \cdot \quad (50)$$

where $a_j = (1+z_j)^{-1}$ is the observed scale factor of the universe, z_j is the observed redshift, μ is the distance modulus $\mu = m-M = 5\log d_L + 25$ and $d_L(a, \boldsymbol{\epsilon})$ is the luminosity distance, given by

$$d_L(a, \boldsymbol{\epsilon}) = \frac{c}{H_0 a}\int_a^1 \frac{{\rm d}x}{x^2 E(x)}, \quad (51)$$

where $\boldsymbol{\epsilon}$ is the vector containing the unknown free parameters and c is the speed of light (≡1 here).

A cosmological model for which the present tests are successfully passed should not suffer from the coincidence problem. Below we apply our tests to the current $\Lambda(t)$ cosmological models (see also Table 1).

– The modified vacuum model with $\boldsymbol{\epsilon} = (\gamma, 0, ...0)$: We sample the unknown γ parameter as follows: $\gamma \in (-1, 1/3)$ in steps of 10^{-4}. We confirm that in the range of $\gamma \in [0.004, 0.03]$ the $\Lambda_{\gamma 0}$ model[1] satisfies both the criteria (i); and (ii) respectively. Also, we verify that this range of values fits the SnIa data, $\chi^2_{\rm min}$/d.o.f. $\simeq$ 1.01 very well. Notice that for $\gamma > 0.03$ the criterion (i) is not satisfied. As an example, in the upper panel of Fig. 1 we present the evolution of the proximity parameter for $\gamma = 0.004$ (solid line) and 0.03 (dashed line). It becomes

[1] Note that from a theoretical viewpoint the predicted value of the γ parameter is $|\gamma| = \frac{1}{12\pi}\frac{M^2}{M_P^2}$, where M_P is the Planck mass and M is an effective mass parameter representing the average mass of the heavy particles of the Grand Unified Theory (GUT) near the Planck scale, after taking into account their multiplicities. In the case of $M \sim M_P$ we can derive an upper limit of $|\gamma| \leq 1/12\pi$ (for more details see Basilakos et al. 2009).

Table 1. Numerical results.

Model	γ	m/H_0	t_0	$\frac{\Lambda(t_{\rm inf})}{\Lambda(t_0)}$	$\frac{\Lambda(t_{\rm pl})}{\Lambda(t_0)}$
Λ	0	0	13.77	1	1
$\Lambda_{\gamma 0}$	0.004	0	13.82	10^{102}	10^{124}
Λ_{0m}	0	2.4×10^{-3}	13.75	10^{51}	10^{63}
$\Lambda_{\gamma m}$	0.004	2.8×10^{-3}	13.80	10^{102}	10^{124}

The 1st column indicates the vacuum model used; the last two rows correspond to the fine-tuning problem.

clear is that for $0.1 \le a \le 0.34$ (or $2 \le z \le 10$) the vacuum density is low enough ($r \sim 0$) to allow galaxies and galaxy clusters to form (Garriga et al. 1999; Basilakos et al. 2009). From now on, we will utilize $\gamma \simeq 0.004$ that corresponds to the best fit parameter. Thus it becomes clear that the $\Lambda_{\gamma 0}$ model passes the above criteria and does not suffer from the cosmic coincidence problem.

– The mild vacuum model with $\epsilon = (0, m, ...0)$: In this cosmological model we find that for $m \ge 0.17H_0$, the corresponding age of the universe is $t_0 \le 12.7$ Gyr. The latter appears to be ruled out by the ages of the oldest known globular clusters (Krauss 2003; Hansen et al. 2004). Using this constraint the unknown m parameter has an upper limit of $0.17H_0$, and we perform the following sampling: $m \in [5\times10^{-4}H_0, 0.17H_0)$ in steps of $5\times10^{-4}H_0$. Within this range, we find that the required (i) and (ii) criteria are not satisfied. Thus, the Λ_{0m} cosmological model suffers from the coincidence problem. The resulting minimization provides: $m = 2.4^{+6}_{-1}\times10^{-3}H_0$ with $\chi^2_{\rm min}/{\rm d.o.f.} \simeq 1.01$. Note that the errors of the fitted parameters represent 1σ uncertainties.
– The general vacuum model with $\epsilon = (\gamma, m, ...0)$: This vacuum cosmological model contains 2 free parameters. Using the sampling mentioned previously, we obtain that our main criteria for the $\Lambda_{\gamma m}$ scenario are fullfilled for $\gamma \in [0.004, 0.02]$, $m \in [1.4\times10^{-3}H_0, 9\times10^{-3}H_0]$ with $\chi^2_{\rm min}/{\rm d.o.f.} \in [1.01, 1.02]$. Throughout the rest of the paper we will use the best fit parameters. These are: $m \simeq 2.8\times10^{-3}H_0$ and $\gamma \simeq 0.004$

In addition to the SnIa data, we further check our statistical results using the dimensionless distance to the surface of the last scattering $R = 1.71 \pm 0.019$ (Komatsu et al. 2009), and the baryon acoustic oscillation (BAO) distance at $z = 0.35$, $A = 0.469 \pm 0.017$ (Eisenstein et al. 2005; Padmanabhan et al. 2007). We find that the above results remain unaltered.

5.2. The cosmic evolution – fine-tuning problem

Using now our best fit parameters for the different kind of vacuums, we present in Fig. 1 the corresponding normalized energy densities, vacuum $\Lambda(a)/H_0^2$, matter $\rho_m(a)/H_0^2$ and radiation $\rho_{\rm r}(a)/H_0^2$ as a function of the scale factor. We verify that both the $\Lambda_{\gamma 0}$ (solid line) and $\Lambda_{\gamma m}$ (open stars) solutions are models that provide large values for the vacuum energy density at early epochs, in contrast with the usual Λ cosmology (open circles) in which the vacuum energy density remains constant everywhere. Also, within a Hubble time ($0 < a \le 1$) and for each (γ, m) pair we find the well known cosmic behavior for the matter density $\rho_m(a) \propto a^{-3}$ and the radiation density $\rho_{\rm r}(a) \propto a^{-4}$ respectively. As an example, in Fig. 1 we present the density evolution of the cosmic fluid for the $\Lambda_{\gamma 0}$ cosmological model: matter (long dashed line) and radiation (dashed line). For a comparison we also plot the predictions of the traditional Λ cosmology: matter

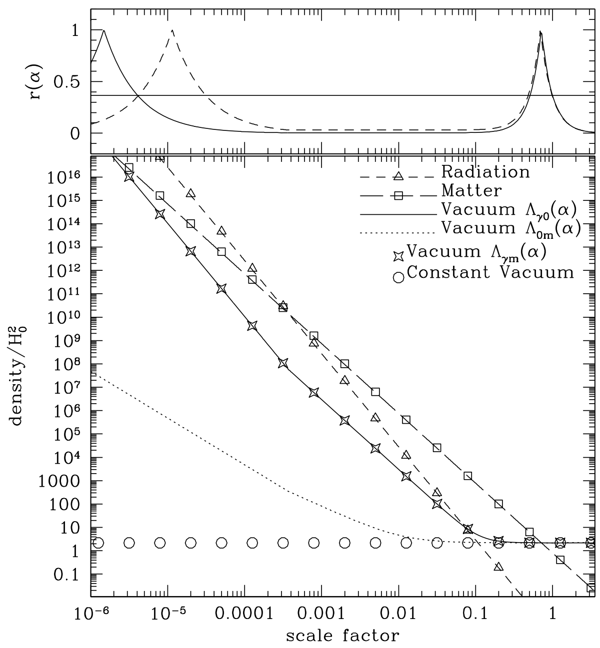

Fig. 1. *Upper panel*: the evolution of the proximity parameter for the $\Lambda_{\gamma 0}$ cosmological model. Note that the scale factor is normalized to unity at the present time. The lines correspond to $\gamma = 0.004$ (solid) and $\gamma = 0.03$ (dashed). *Bottom panel*: the evolution of the radiation, matter and vacuum density considering different kind of vacuums (after fitting the constants using the Union08 SnIa data and $\Omega_m = 0.27$, $H_0 = 70.5$ km s^{-1} Mpc^{-1}). I) Traditional Λ-cosmology: radiation density (open triangles), matter density (open squares) and constant vacuum density (open circles). II) Modified Λ-cosmology, $\gamma \ne 0$, $\Lambda_{\gamma 0}$: radiation density (dashed line), matter density (long-dashed line) and vacuum density (solid line). III) The evolution of the mild vacuum, $m \ne 0$, Λ_{0m} and IV) the evolution of the general vacuum, $\Lambda_{\gamma m}$ (open stars).

(open squares) and radiation (open triangles). From Fig. 1 it becomes clear that the radiation-matter equality takes place close to $a_{\rm rm} \simeq 3.7\times10^{-4} \simeq \Omega_{\rm r}/\Omega_m$. For those vacuum models where $m \ne 0$ (Λ_{0m} and $\Lambda_{\gamma m}$), we verify that the behavior of their cosmic fluid (matter+radiation) deviates from the Λ solution in the far future ($t \gg t_0$), since the exponential term ${\rm e}^{mt}$ in Eq. (14) plays an important role in the global dynamics (see Sect. 3.4 and below).

In particular, for the $\Lambda_{\gamma 0}$ vacuum scenario (the same behavior holds for $\Lambda_{\gamma m}$) we have revealed the following phases: (a) at early enough times ($\alpha < a_{\rm rm}$) the scale factor of the universe tends to its minimum value, $a \longrightarrow 0$, which means that the vacuum energy density initially moves quickly to infinity. So, as long as the scale factor increases the vacuum energy rolls down rapidly as $\Lambda_{\gamma 0}(a) \propto a^{-4\gamma_1}$ (where $\gamma_1 \sim O(1)$). This evolution may solve the fine-tuning problem. Indeed, for $\gamma \in (0, 1/3)$, we find that prior to the inflation point ($t_{\rm inf} \sim 10^{-32}$ s), the vacuum energy density divided by its present value is $\Lambda(t_{\rm inf})/\Lambda(t_0) \sim 10^{102}$ Finally, if we consider that the functional form of $\Lambda(a) \propto a^{-4\gamma_1}$ is still valid during the Planck time ($t_{\rm pl} \sim 10^{-43}$ s), then $\Lambda(t_{\rm pl})/\Lambda(t_0) \sim 10^{124}$ (see the last rows in Table 1); and (b) in the matter era the vacuum density continues to roll down but with a different power law $\Lambda_{\gamma 0}(a) \propto a^{-3(1-\gamma)}$ and it tends to a constant value close to $a \sim 0.25$ ($z \sim 3$). Finally, for $a \ge 0.25$ the vacuum energy density is effectively frozen to the nominal value, $\Lambda_{\gamma 0}(a) \simeq \Lambda = 3\Omega_\Lambda H_0^2$, which implies that the considered time varying vacuum model explains why the matter energy density and the dark energy density are

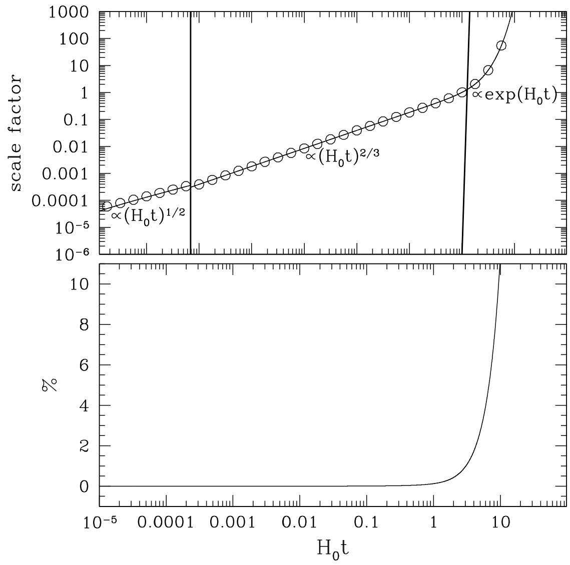

Fig. 2. *Upper panel*: comparison of the scale factor provided by our $\Lambda_{\gamma 0}$ model with the traditional Λ cosmology (open points). Note that we use $\Omega_m = 0.27$ and $H_0 = 70.5$ km s^{-1} Mpc^{-1} model. In the bottom panel we present the deviation of the scale factors between the $\Lambda_{\gamma 0}$ and $\Lambda_{\gamma m}$ model respectively. Note that the scale factor is normalized to unity at the present time.

of the same order prior to the present epoch. The moment of radiation-vacuum equality occurs at $a_{rv} \simeq 0.1 \simeq (\Omega_r/\Omega_\Lambda)^{1/4}$. Similarly, the moment of matter-vacuum equality takes place at $a_{mv} \simeq 0.72 \simeq (\Omega_m/\Omega_\Lambda)^{1/3}$. From the observational viewpoint, in order to investigate whether the vacuum energy density follows the above evolution, we need a robust cosmological probe at redshifts $z \geq 3$. In a recent paper (Basilakos et al. 2009) we have investigated how realistic it would be to detect differences among the vacuum models. In particular, we have found that the Sunayev-Zeldovich cluster number-counts (as expected from the survey of the South Pole Telescope, Staniszewski et al. 2009, and the Atacama Cosmology Telescope, Hincks et al. 2009) indicate that we may be able to detect significant differences among the vacuum models in the redshift range $2.5 \leq z \leq 3$ at a level of ∼6−12%, which translates in number count differences over the whole sky of ∼100 clusters (see Fig. 6 in Basilakos et al. 2009).

Finally, in Fig. 1 we also show the evolution of the mild vacuum model $\Lambda_{0m}(a)$ (dot line), in which $\gamma = 0$. Briefly, we get the following dependence: (a) $\Lambda_{0m} \propto a^{-2\gamma_1}$ for $a < a_{\rm rm}$, while we estimate that $\Lambda_{0m}(t_{\rm inf})/\Lambda_{0m}(t_0) \sim 10^{51}$ and $\Lambda_{0m}(t_{pl})/\Lambda_{0m}(t_0) \sim 10^{63}$; (b) between $a_{\rm rm} \leq a \leq 0.08$ we have $\Lambda_{0m} \propto a^{-3/2}$; and (c) for $a \geq 0.08$ the Λ_{0m} becomes constant.

We would like to end this section with a discussion of the evolution of the scale factor. In particular, our approach provides an evolution of the scale factor in the $\Lambda_{\gamma 0}$ model seen in the upper panel of Fig. 2 as the solid line, which mimics the corresponding scale factor of the Λ cosmological model (open points), despite the fact that they describe the vacuum term differently. On the other hand, in the bottom panel of Fig. 2 we present the corresponding deviation $[(a_{\gamma m} - a_{\gamma 0})/a_{\gamma 0}]$%, of the growth factors. It becomes evident that within the range $0 < H_0 t < 5$ the evolution of the scale factor provided by the $\Lambda_{\gamma m}$ model closely resembles, the corresponding scale factor of the $\Lambda_{\gamma 0}$ model (the same result holds also for the Λ cosmology). However, for models where $m \neq 0$ the situation is somewhat different in the far future. Indeed, for $H_0 t \geq 5$ the $\Lambda_{\gamma m}$ (or Λ_{0m}) cosmological scenario deviates from the $\Lambda_{\gamma 0}$ (or Λ) model by ∼5−10%. Thus, we conclude that the models with $m \neq 0$ give a super-accelerated expansion of the universe in the far future with respect to those vacuum models where $m = 0$.

6. Conclusions

The reason why a cosmological constant leads to a late cosmic acceleration is because it introduces in Friedmann's equation a component which has an equation of state with negative pressure, $P_\Lambda = -\rho_\Lambda$. In the last decade the so called concordance Λ-cosmology is considered to be the model which describes the cosmological properties of the observed universe because it fits the current observational data accurately. However, the traditional Λ cosmology suffers from two fundamental puzzles. These are the fine-tuning and the cosmic coincidence problems. An avenue through which the above cosmological problems could be solved is via the time varying vacuum energy which has the same equation of state as the traditional Λ-cosmology.

Below we wish to present the basic assumptions and conclusions of our analysis.

- We are assuming a time varying vacuum pattern in which the specific functional form is: $\Lambda(t) = 3\gamma H^2(t) + 2mH(t) + 3n(\beta + 1 - \gamma)e^{2mt}$, where $\beta = 0$ (matter era) or $\beta = 1/3$ (radiation era), $n = 3\Omega_\Lambda H_0^2$, while the pair (γ, m) characterizes the different types of vacuum. Note that the above functional form includes the effect of the quantum field theory (for $m = 0$) (Shapiro & Solá 2000; Babić et al. 2002; Grande et al. 2006; Solá 2008) and it also extents recent studies (see for example Ray et al. 2007; Carneiro et al. 2008; Sil & Som 2008; Basilakos 2009). In this context we can easily prove that the cosmological constant is a particular solution of the general vacuum, that $(\gamma, m) = (0, 0)$. We have also investigated the following models: (a) modified vacuum in which $(\gamma, m) = (\gamma, 0)$, mild vacuum with $(\gamma, m) = (0, m)$ and general vacuum in which $(\gamma, m) \neq (0, 0)$. In this framework we find that the time evolution of the basic cosmological functions (scale factor and Hubble flow) is described in terms of hyperbolic functions which can accommodate a late time accelerated expansion equivalent to the standard Λ model.
- We find that within the framework of either the modified or general vacuum models the corresponding vacuum term in the radiation era varies as $\Lambda(a) \propto a^{-4}$ while in the matter-dominated era we have $\Lambda(a) \sim a^{-3}$ up to $z = a^{-1} - 1 \simeq 3$ while $\Lambda(a) \simeq \Lambda = 3\Omega_\Lambda H_0^2$ for $z \leq 3$. This vacuum mechanism simultaneously sets (a) the value of Λ at the present time to its observed value; and (b) at the Planck time to a value which is 10^{124} at its present value $(\Lambda(t_{\rm pl})/\Lambda(t_0) \sim 10^{124})$. Additionally, we verify that our models appear to overcome the cosmic coincidence problem. Finally, in order to confirm the above results, we need to define a robust cosmological probe at high redshifts ($z \geq 3$). In Basilakos et al. (2009) we propose that the future cluster surveys based on the Sunayev-Zeldovich detection method will possibly distinguish the closely resembling vacuum models at high redshifts.

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Appendix A

In this appendix we provide a physical justification of the functional form of $\Lambda(a)$ used in our paper. As we have already mentioned in section two the vacuum energy density can take several forms, depending on the theoretical approach. Briefly, the renormalization group from the quantum field theory introduces only even powers of H out of which the H^2 is the leading term (Grande et al. 2006; Solá 2008, and references therein). In another vein, the aforementioned possibility that the vacuum energy could be evolving linearly with H has been motivated theoretically through a possible connection of cosmology with the QCD scale of strong interactions (see Schutzhold 2002; Carneiro et al. 2008). In this framework it has also been proposed a possible link of dark energy with QCD and the topological structure of the universe (Urban & Zhitnitsky 2009). The simplest approach therefore to introduce the effects of the DE is to consider a potential $V(\phi) \simeq V_0 + m^2\phi^2/2$, where the homogeneous scalar field ϕ obeys the Klein-Gordon equation. It is well known that for $H \simeq$ const. the corresponding ϕ evolves with time as $\phi(t) \simeq \phi_0 \mathrm{e}^{mt}$ (where in general m is a complex number). In this context, one would expect that the functional form of the $\Lambda(t)$ should contain also an additional term of $\phi^2(t) \propto \mathrm{e}^{2mt}$ in order to take into account the possible link between dark energy and QCD.

All the above options have merits and demerits. In the current paper the functional form of $\Lambda(t)$ is motivated by a combination of the above possibilities, namely $H^2(t)$ [RG], $H(t)$ [QCD] and e^{2mt} (dark energy). In particular, the linear combination reads as follows:

$$\Lambda(t) = n_1 H^2(t) + n_2 H(t) + n_3 \mathrm{e}^{2mt}$$

which obviously is very similar to the original (phenomenologically selected) form of $\Lambda(t)$ (Eq. (7)). Finally, from a mathematical point of view we can select the constants n_1, n_2 and n_3 to match those presented in the original Eq. (7).

Appendix B

With the aid of the differential equation theory we present solutions that are relevant to our Eq. (8). If we have a Riccati differential equation which is given by the following special form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)y^2(x) + my(x) - n\mathrm{e}^{2mx} f(x) \tag{52}$$

then the general solution of Eq. (52) for $n > 0$ is

$$y(x) = \sqrt{n}\mathrm{e}^{mx}\coth\left[-\sqrt{n}\int_{x_0}^{x} \mathrm{e}^{mu} f(u)\mathrm{d}u\right]. \tag{53}$$

On the other hand, if $n < 0$ then the solution of Eq. (52) is

$$y(x) = \sqrt{|n|}\mathrm{e}^{mx}\cot\left[-\sqrt{|n|}\int_{x_0}^{x} \mathrm{e}^{mu} f(u)\mathrm{d}u\right]. \tag{54}$$

Note that in our formulation the function $f(x)$ is a constant: $f(x) = -3(\beta + 1 - \gamma)/2$. Also, $n < 0$ implies that $\Omega_m > 1$ (or $\Lambda < 0$).

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