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Solving the one-loop soliton solution of the Vakhnenko equation by means of the Homotopy analysis method

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Abstract

A powerful, easy-to-use analytic technique for nonlinear problems, namely the Homotopy analysis method, is applied to solve the Vakhnenko equation, a nonlinear equation with loop soliton solutions governing the propagation of high-frequency waves in a relaxing medium. By means of the transformation of independent variables, an analysis one-loop soliton solution expressed by a series of exponential functions is obtained, which agrees well with the exact solution. This indicates the validity and great potential of the Homotopy analysis method in solving complicated solitary wave problems.

1. Introduction

Consider the propagation of high-frequency waves in a relaxing medium [1], governed by the so-called Vakhnenko equation [2–6]

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)u+u=0, \tag{1}$$

where u denotes the dimensionless pressure, x and t are spatial and temporal variables, respectively. The Vakhnenko equation has looplike soliton solution, and thus it is not easy to solve it.

Currently, an analytical method for strongly nonlinear problems, namely the Homotopy analysis method [7–13], has been developed and successfully applied to many kinds of nonlinear problems in science and engineering. In this paper, the Homotopy analysis method is applied to solve such a multiple-valued nonlinear problem with the one-loop solition solution. The soliton solution solved by the Homotopy analysis method is verified by the exact one given in [2,4]. This

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further demonstrates the validity and effectiveness of the Homotopy analysis method in solving complicated nonlinear solitary wave problems.

2. Transformation of the Vakhnenko equation for one-loop soliton solution

Following Vakhnenko et al. [4], we introduce new independent variables X and T, defined by

$$x = T + \int_{-\infty}^{X} U(\xi, T)\mathrm{d}\xi + x_0, \quad t = X, \tag{2}$$

where $u(x,t) = U(X,T)$, and x_0 is a constant. As pointed out by Vakhnenko and Parkes [14], the transformation (2) is similar to the transformation between Eulerian coordinates (x,t) and Lagrangian coordinates (T,X). Writing

$$W(X,T) = \int_{-\infty}^{X} U(\xi, T)\mathrm{d}\xi, \tag{3}$$

we have

$$W_X(X,T) = U(X,T). \tag{4}$$

From (2) it follows that

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial T} = \phi\frac{\partial}{\partial x}, \tag{5}$$

where

$$\phi(X,T) = 1 + W_T. \tag{6}$$

From (1) and (2), we obtain

$$U_{XT} + \phi U = 0. \tag{7}$$

Substituting (4) and (6) into (7) yields

$$W_{XXT} + W_X W_T + W_X = 0. \tag{8}$$

The solution of the above equation is looked for in the form

$$W(X,T) = W(\eta), \quad \eta = kX - \omega T, \tag{9}$$

where k and ω are constants. Substituting (9) into (8) yields

$$-k\omega W'''(\eta) - \omega[W'(\eta)]^2 + W'(\eta) = 0. \tag{10}$$

Write

$$W(\eta) \approx B\exp(\mu\eta), \quad \text{as } \eta \to -\infty, \tag{11}$$

where B is a constant. Substituting (11) into (10) and balancing the main term yields

$$\mu = \sqrt{\frac{1}{k\omega}}. \tag{12}$$

Under the transformation

$$\theta = \mu\eta = \sqrt{\frac{1}{k\omega}}\eta = \sqrt{\frac{k}{\omega}}X - \sqrt{\frac{\omega}{k}}T. \tag{13}$$

Eq. (10) becomes

$$W'''(\theta) + \sqrt{\frac{1}{v}}[W'(\theta)]^2 - W'(\theta) = 0, \tag{14}$$

where $v = k/\omega$. According to [4], v denotes the speed of the wave propagation.

The boundary conditions of Eq. (14) are given below. Due to the definition (3), we have

$$W(-\infty) = 0. \tag{15}$$

Considering the symmetry of $U(X,T)$ in $X-T$ space and the continuation of its 1st-order derivative, and taking account of (4), we have

$$W'(\theta) = W'(-\theta), \tag{16}$$

$$W''(0) = 0. \tag{17}$$

Integrating (16) gives

$$W(\theta) + W(-\theta) = A, \tag{18}$$

where A is a constant. From (18) and (15) we obtain

$$W(0) = \frac{A}{2}, \; W(+\infty) = A. \tag{19}$$

Eqs. (15), (17) and (19) are the boundary conditions of Eq. (14).

3. Homotopy analysis method

Then, we apply the Homotopy analysis method to obtain $W(\theta)$ on $\theta > 0$, because $W(\theta)$ on $\theta < 0$ can be obtained from (18) by the symmetry.

Under the transformation

$$W(\theta) = A + \frac{A}{2}g(\theta), \tag{20}$$

Eq. (14) becomes

$$g'''(\theta) + \gamma[g'(\theta)]^2 - g'(\theta) = 0, \tag{21}$$

subject to the boundary conditions

$$g(0) = -1, \quad g''(0) = 0, \quad g(+\infty) = 0, \tag{22}$$

where

$$\gamma = \frac{A}{2}\sqrt{\frac{1}{v}} \tag{23}$$

is a constant to be determined.

According to the governing Eq. (21) and the boundary conditions (22), the solution $g(\theta)$ can be expressed by

$$g(\theta) = \sum_{m=1}^{+\infty} \alpha_m \exp(-m\theta), \tag{24}$$

where $\alpha_m (m = 1, 2, \ldots)$ is a coefficient. This provides us with the so-called Rule of Solution Expression, as mentioned by Liao [7]. Thereafter, it is straightforward to choose

$$g_0(\theta) = -\frac{4}{3}\exp(-\theta) + \frac{1}{3}\exp(-2\theta) \tag{25}$$

as the initial guess of $g(\theta)$ and

$$\mathscr{L}[\Phi(\theta,q)] = \left(\frac{\partial^3}{\partial\theta^3} - \frac{\partial}{\partial\theta}\right)\Phi(\theta,q) \tag{26}$$

as the auxiliary linear operator, respectively. Then, we construct the so-called zero-order deformation equation

$$(1-q)\mathscr{L}[G(\theta,q) - g_0(\theta)] = \hbar q \mathscr{N}[G(\theta,q), \Gamma(q)], \tag{27}$$

subject to the boundary conditions

$$G(0,q) = -1, \quad G''(0,q) = 0, \quad G(+\infty,q) = 0, \tag{28}$$

where q is an embedding parameter, $\hbar$ is a non-zero parameter, and

$$\mathscr{N}[G(\theta,q), \Gamma(q)] = G'''(\theta,q) + \Gamma(q)[G'(\theta,q)]^2 - G'(\theta,q), \tag{29}$$

with primes denoting derivatives with respect to θ.

When $q = 0$, the solution of Eqs. (27) and (28) is

$$G(\theta, 0) = g_0(\theta). \tag{30}$$

When $q = 1$, Eqs. (27) and (28) are equivalent to Eq. (21), provided

$$G(\theta, 1) = g(\theta), \quad \Gamma(1) = \gamma. \tag{31}$$

Thus, as q increases from 0 to 1, $G(\theta, q)$ varies from the initial approximation $g_0(\theta)$ to the exact solution $g(\theta)$ of Eqs. (21) and (22), so does $\Gamma(q)$ from its initial approximation γ_0 to the exact value γ. Note that we have great freedom to choose the auxiliary parameter $\hbar$. Assume that $\hbar$ is properly chosen so that the zero-order deformation Eqs. (27) and (28) have solutions for all $q \in [0, 1]$ and that the terms

$$g_m(\theta) = \frac{1}{m!}\frac{\partial^m G(\theta, q)}{\partial q^m}\bigg|_{q=0}, \quad \gamma_m = \frac{1}{m!}\frac{\partial^m \Gamma(q)}{\partial q^m}\bigg|_{q=0} \tag{32}$$

exist for $m \geqslant 1$. Then, by Taylor's theorem and using (30), we can expand $G(\theta, q)$ and $\Gamma(q)$ in power series of q as follows

$$g(\theta) = g_0(\theta) + \sum_{m=1}^{+\infty} g_m(\theta) q^m, \tag{33}$$

$$\gamma = \gamma_0 + \sum_{m=1}^{+\infty} \gamma_m q^m. \tag{34}$$

Furthermore, assuming that $\hbar$ are so properly chosen that the power series (33) and (34) are convergent at $q = 1$, we have from (31) the solution series

$$g(\theta) = g_0(\theta) + \sum_{m=1}^{+\infty} g_m(\theta), \tag{35}$$

$$\gamma = \gamma_0 + \sum_{m=1}^{+\infty} \gamma_m. \tag{36}$$

For brevity, define the vectors

$$\vec{g}_k = \{g_0(\theta), g_1(\theta), g_2(\theta), \ldots, g_k(\theta)\}, \quad \vec{\gamma}_k = \{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_k\}. \tag{37}$$

Differentiating the zero-order deformation equations (27) and (28) m times with respect to q and then dividing them by $m!$ and finally setting $q = 0$, we have the high-order deformation equation

$$\mathscr{L}[g_m(\theta) - \chi_m g_{m-1}(\eta)] = \hbar \mathscr{R}_m(\vec{g}_{m-1}, \vec{\gamma}_{m-1}), \quad m \geqslant 1, \tag{38}$$

subject to boundary conditions

$$g_m(0) = g''_m(0) = g_m(+\infty) = 0, \tag{39}$$

where

$$\mathscr{R}_m(\vec{g}_{m-1}, \vec{\gamma}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathscr{N}[G(\theta, q), \Gamma(q)]}{\partial q^{m-1}}\bigg|_{q=0} = \sum_{n=0}^{m-1}\gamma_{m-1-n}\left[\sum_{k=0}^{n} g'_k(\theta) g'_{n-k}(\theta)\right] + g'''_{m-1}(\theta) - g'_{m-1} \tag{40}$$

and

$$\chi_m = \begin{cases} 1, & m > 1, \\ 0, & m = 1. \end{cases} \tag{41}$$

It should be emphasized that Eq. (38) is linear, and $\mathscr{R}_m(\vec{g}_{m-1}, \vec{\gamma}_{m-1})$ is dependent upon $\vec{g}_{m-1}$ and $\vec{\gamma}_{m-1}$ that contain the unknown γ_{m-1}. The solution of Eq. (38) can be expressed by

$$g_m(\theta) = g^*(\theta) + C_1 \exp(-\theta) + C_2 \exp(\theta) + C_3, \tag{42}$$

where C_1, C_2, and C_3 are the integral constants, $g^*(\eta)$ is a special solution of Eq. (38) and contains the unknown γ_{m-1}. Due to the boundary condition (39) at infinity, C_2 and C_3 must be zero. The unknown γ_{m-1} and the constant C_1 are determined by the two boundary conditions (39) at $\theta = 0$.

We apply the symbol computation software MATHEMATICA and solve the linear Eq. (38) one after the other in order $m = 1,2,3,\ldots$ At the Mth-order of approximation, we have the analytic solutions

$$g(\theta) \approx \sum_{m=0}^{M} g_m(\theta) = \sum_{k=1}^{2M+2} \beta_k \exp(-k\theta), \tag{43}$$

$$\gamma \approx \sum_{m=0}^{M} \gamma_m, \tag{44}$$

where β_k is coefficient.

By means of transformation related to the original independent u and t, the solution of the Vakhnenko equation (1) is then given in parametric form by

$$u(x,t) = W_X(\theta) = \sqrt{v}W_\theta(\theta) = \sqrt{v}\frac{A}{2}g'(\theta), \tag{45}$$

$$x - vt = T + W + x_0 - vX = A + \frac{A}{2}g(\theta) + x_0 - \sqrt{v}\theta, \tag{46}$$

where $A = 2\gamma\sqrt{v}$ and x_0 is a constant. For the symmetry in $x - t$ space, we have

$$x_0 = -A - \frac{A}{2}g(0) = -\frac{A}{2}. \tag{47}$$

Therefore, the Mth-order approximation solution of the Vakhnenko equation (1) is given by a series of exponential functions as

$$u \approx -v\gamma \sum_{k=1}^{2M+2} k\beta_k \exp(-k\theta), \tag{48}$$

$$x - vt \approx \sqrt{v}\left[\gamma + \gamma \sum_{k=1}^{2M+2} \beta_k \exp(-k\theta) - \theta\right]. \tag{49}$$

4. Result analysis

In [4], the exact one-loop soliton solution is given by

$$u = \frac{3v}{2}\operatorname{sech}^2\left(\frac{\sqrt{v}\zeta}{2}\right), \quad x - vt = 3\sqrt{v}\tanh\left(\frac{\sqrt{v}\zeta}{2}\right) - v\zeta, \tag{50}$$

where ζ plays the role of the parameter in these dependences. From [4] it is easy to derive that the exact value of γ equals to 3.

Note that the series (35) and (36) contain the auxiliary parameter $\hbar$, which provides us with a simply way to adjust and control the convergence region and rate of the solution series. As pointed out by Liao [7], in general, by means of the so-called $\hbar$-curve (the $\gamma \sim \hbar$ curve for the considered problem), it is straightforward to choose a proper value of $\hbar$ which ensures that the solution series is convergent. In this way, it is found that our series converge when $\hbar = -1$.

To verify the correctness of our solution given by the Homotopy analysis method, we substitute our analysis approximate solution $g(\theta)$ expressed by (44) and (43) into Eq. (21) to evaluate the corresponding residual error. The residual error of our 40th-order approximation solution under $\hbar = -1$ is as shown in Fig. 1. Note that the maximum magnitude of the residual error of the 40th-order approximation when $\hbar = -1$ is less than 5×10^{-6}, as shown in Fig. 1. Our approximate results of γ are as listed in Table 1. Obviously, our solution series (36) converges to the exact value $\gamma = 3$. Besides, using the so-called Homotopy–Padé technique (see page 38 and Section 3.5.2 in [7]), we can greatly accelerate the convergence of the series (36), as shown in Table 2.

Our approximate one-loop soliton solution of the Vakhnenko equation (1) is expressed by a series of exponential functions (48) and (49). For instance, at 10th-order of approximation (when $\hbar = -1$), the coefficient β_k are as follows:

$\beta_1 = -1.9713308559345335$	$\beta_2 = 1.8741124232730177$
$\beta_3 = -1.688995582562311$	$\beta_4 = 1.4156399534626933$
$\beta_5 = -1.0857623593637347$	$\beta_6 = 0.7525346878125345$
$\beta_7 = -0.46677089965620133$	$\beta_8 = 0.25705676045984427$
$\beta_9 = -0.1248151101756921$	$\beta_{10} = 0.05307270253073546$
$\beta_{11} = -0.019621378720951158$	$\beta_{12} = 0.006256735459929639$
$\beta_{13} = -0.0017046514230563638$	$\beta_{14} = 0.00039234284621390814$
$\beta_{15} = -0.000075225805394 6054$	$\beta_{16} = 0.000011805598044 97769$
$\beta_{17} = -1.4822626210812044\times10^{-6}$	$\beta_{18} = 1.444037499504032\times10^{-7}$
$\beta_{19} = -1.0451783764835563\times10^{-8}$	$\beta_{20} = 5.255674886127573\times10^{-10}$
$\beta_{21} = -1.6284562979132532\times10^{-11}$	$\beta_{22} = 2.3280111040532965\times10^{-13}$

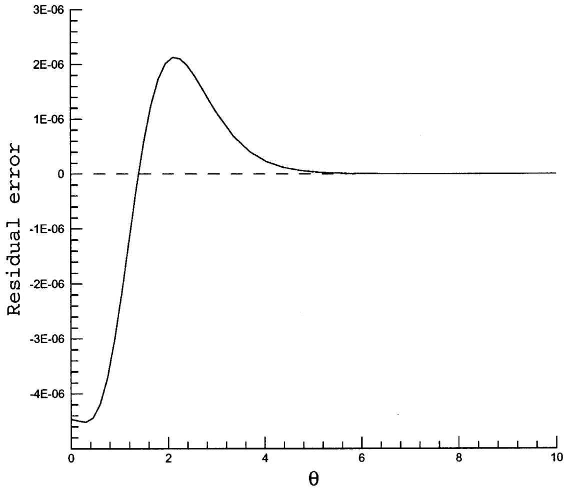

Fig. 1. Residual error of the 40th-order approximation of Eq. (21) given by means of $\hbar = -1$.

Table 1
The analytic approximations of γ by means of $\hbar = -1$

Order of approximation	γ
5	2.9069507
10	2.9795901
15	2.9942622
20	2.9981896
25	2.9993887
30	2.9997841
35	2.9999212
40	2.9999706
45	2.9999888
50	2.9999957
55	2.9999983
60	2.9999993

Table 2
The $[m,m]$ Homotopy–Padé approximation of γ

m	γ
2	2.9959286
4	2.9993598
6	2.9999717
8	2.9999997
10	2.9999997
12	2.9999999
14	3.0000000
16	3.0000000
18	3.0000000
20	3.0000000

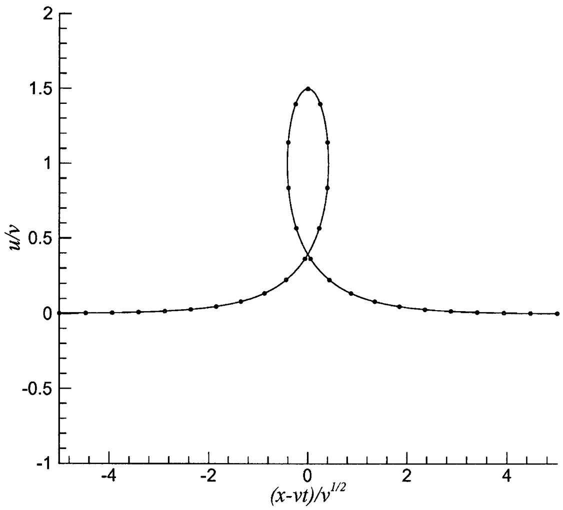

Fig. 2. Comparison of the exact one-loop solitor solution of Vakhnenko equation [4] with the 10th-order homotopy analysis approximation when $\hbar = -1$. Solid line 10th-order appproximation; Circle: exact solution given by Vakhnenko et al. [4].

The comparison of our 10th-order approximation when $\hbar = -1$ with the exact solution (50) is as shown in Fig. 2. Obviously, our analytic approximation agrees well with the exact one. This verifies the validity and effectiveness of the Homotopy analysis method to complicated nonlinear solitary wave problems.

5. Conclusion

In this paper, a powerful, easy-to-use analytic technique for nonlinear problems in general, namely the Homotopy analysis method [7], is applied to solve to a nonlinear problem with one-loop soliton solution, i.e. the propagation of high-frequency waves in a relaxing medium governed by the Vakhnenko equation (1). We use the independent variables transformation mentioned in [4] to obtain a nonlinear equation and then solve it by means of the Homotopy analysis method. By means of the transformation back to the original independent variables, an analysis one-loop soliton solution expressed by a series of exponential functions to the Vakhnenko equation is gained. Our analytic solution agrees well with the exact solution given by Vakhnenko [2,4]. This verifies the validity and great potential of the Homotopy analysis method in solving complicated solitary wave problems in science and engineering.

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