Solving the Sextic by Iteration: A Study in Complex Geometry and Dynamics

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CONTENTS

We use the Valentiner action of $\mathcal{A}_6$ on $\mathbb{CP}^2$ to develop an iterative algorithm for the solution of the general sextic equation over $\mathbb{C}$, analogous to Doyle and McMullen's algorithm for the quintic.

1. INTRODUCTION

1A. Overview

Doyle and McMullen [1989] devised an iterative solution to the fifth degree polynomial. At the core of the method is a rational mapping f of $\mathbb{CP}^1$ with the icosahedral symmetry of a general quintic. Algebraically, this means that f commutes with a group of Möbius transformations that is isomorphic to the alternating group $\mathcal{A}_5$. Moreover, this $\mathcal{A}_5$-*equivariant* posseses *reliable* dynamics: for almost any initial point $a \in \mathbb{CP}^1$, the sequence of iterates $f^k(a)$ converges to one of the periodic cycles that comprise an icosahedral orbit; see [Doyle and McMullen 1989, p. 163] for a geometric description. This breaking of $\mathcal{A}_5$-symmetry provides for a *generally convergent* quintic-solving algorithm: with almost any fifth-degree equation, associate a rational mapping that has reliable dynamics and whose attractor consists of a single orbit from which one computes a root.

An algorithm that solves the sixth-degree equation calls for a dynamical system with $\mathcal{S}_6$ or $\mathcal{A}_6$ symmetry. Since neither $\mathcal{S}_6$ nor $\mathcal{A}_6$ acts on $\mathbb{CP}^1$, attention turns to higher dimensions. Acting on $\mathbb{CP}^2$ is an $\mathcal{A}_6$-isomorphic group of projective transformations found by Valentiner in the late nineteenth century. The present work exploits this two-dimensional $\mathcal{A}_6$ soccer ball in order to discover a Valentiner-symmetric rational mapping of $\mathbb{CP}^2$ whose dynamics *experimentally appear* to be nice in the above sense—

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transferred to the $\mathbb{CP}^2$ setting. This map provides the central feature of a conjecturally generally convergent sextic-solving algorithm analogous to that employed in the quintic case.

1B. Solving Equations by Iteration

For $n \leq 4$, the symmetric groups $\mathcal{S}_n$ act faithfully on $\mathbb{CP}^1$. Corresponding to each action is a map whose nice dynamics provides for algorithmic convergence to roots of a given nth-degree equation. For instance, Newton's method provides a direct iterative solution to quadratic polynomials, but, due to a lack of symmetry, not to higher degree equations. My interests here are the geometric and dynamical properties of complex projective mappings rather than numerical estimates.

The search for elegant complex geometry and dynamics continues into degree five where $\mathcal{A}_5$ is the appropriate group, since $\mathcal{S}_5$ fails to act on the sphere. This reduction in the galois group requires the extraction of the square root of a polynomial's discriminant. Such root-taking is itself the result of a reliable iteration, namely, Newton's method. In practical terms, the Doyle-McMullen algorithm solves a family of fifth-degree *resolvents* the members of which possess $\mathcal{A}_5$ symmetry. A map with icosahedral symmetry and nice dynamics plays the leading role.

Pressing on to the sixth-degree leads to the two-dimensional $\mathcal{A}_6$ action of the Valentiner group $\mathcal{V}$. Here the problem shifts to one of finding a nice $\mathcal{V}$-symmetric mapping of $\mathbb{CP}^2$ from whose attractor one calculates a given sextic's root. (The solution procedure follows that of the quintic algorithm: see Section 4.) Providing the overall framework is the two-dimensional $\mathcal{A}_6$ analogue of the icosahedron.

1C. Proofs and Computations

At the moment, many of this work's results have only computational support; accordingly, I call them "facts". Furthermore, its conjectural nature calls for a deeper understanding of Valentiner geometry and dynamics. As the theory of complex dynamics in several dimensions develops more sophisticated weaponry, the barricades to understanding might become assailable. For now, I hope that these discoveries provide a stimulus to such development.

2. VALENTINER'S GROUP: THE $\mathcal{A}_6$ ACTION ON $\mathbb{CP}^2$

In 1884, Klein wrote

> If ... any equation $f(x) = 0$ is given, we will investigate what is the smallest number of variables with which we can construct a group of linear substitutions which is isomorphic with the Galois group of $f(x) = 0$. [Klein 1913, p. 138]

In the wake of the mid-nineteenth century abstractionist turn in mathematics the theory of group representations began to emerge. Part of the concrete yield from work on symmetric groups was Valentiner's discovery [1889] of a complex projective group that is isomorphic to $\mathcal{A}_6$ — the alternating group of six things. Then Wiman [1895] explored some of the geometric and invariant structure determined by this action on the complex projective plane. A more thorough exposition appeared in [Fricke 1926].

Here I take a new approach to the generation of the Valentiner group and then explore some of its rich combinatorial geometry. The core of this work involves the development of a combinatorially sensitive description of the basic geometric structures. In so doing, I reproduce some of the Wiman and Fricke results. The study culminates in an account of the system of Valentiner-invariant polynomials and, thereby, lays the algebraic foundation for constructing Valentiner-symmetric mappings of $\mathbb{CP}^2$.

2A. Basics of $\mathcal{A}_6$

These are the nonidentity elements of $\mathcal{A}_6$ classified by order and cycle-structure:

order	structure	# of elements
2	$(ab)(cd)$	$45 = \frac{1}{2}\binom{6}{4} \cdot 3!$
3	(abc)	$40 = \binom{6}{3} \cdot 2$
3	$(abc)(def)$	$40 = \binom{6}{3} \cdot 2$
4	$(abcd)(ef)$	$90 = \binom{6}{4} \cdot 3!$
5	$(abcde)$	$144 = 6 \cdot 4!$

Sitting inside $\mathcal{A}_6$ are twelve versions of $\mathcal{A}_5$ that decompose into two conjugate systems of six:

(1) the stabilizers $\mathrm{Stab}\{k\}$ of one thing, and
(2) the permutations of the six pairs of antipodal icosahedral vertices under rotation.

Acting by conjugation, $\mathcal{A}_6$ permutes each of the two systems individually. A given $\mathcal{A}_5$ subgroup fixes itself set-wise and permutes its five conjugates according to the rotational icosahedral group's action on the five cubes found in the icosahedron. Meanwhile, the other system of six subgroups undergo the permutations of the six pairs of antipodal vertices. Consequently, the intersection of two $\mathcal{A}_5$ subgroups in the *same* system is isomorphic to $\mathcal{A}_4$ — the tetrahedral rotations — while two in *different* systems give a dihedral group $\mathcal{D}_5$.

2B. Generating the Valentiner Group

An $\mathcal{A}_5$ subgroup of $\mathcal{A}_6$, say $\mathrm{Stab}\{1\}$, extends to $\mathcal{A}_6$ by addition of the generator $(12)(3456)$. Furthermore, this order-four element generates an $\mathcal{S}_4$ over the $\mathcal{A}_4$ subgroup

$$\langle (35)(46),\, (456)\rangle \subset \mathrm{Stab}\{1\}.$$

This structure suggests a method for producing an $\mathcal{A}_6$-isomorphic group $\mathcal{V}$ in $\mathrm{PGL}_3(\mathbb{C})$:

(1) take a tetrahedral subgroup $\mathcal{T}$ of an icosahedral group $\mathcal{I}$;
(2) by addition of an order-4 transformation Q, extend $\mathcal{T}$ to an octahedral group $\mathcal{O} = \langle \mathcal{T}, Q\rangle$;
(3) generate $\mathcal{V} = \langle \mathcal{I}, Q\rangle \simeq \mathcal{A}_6$.

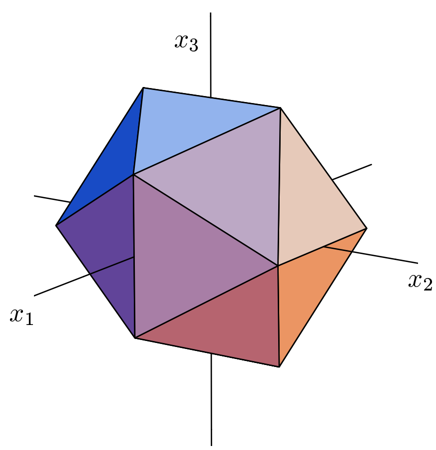

FIGURE 1. The icosahedron in octahedral coordinates.

The 15 pairs of antipodal edges of the standard icosahedron decompose into five triples such that three lines joining antipodal edge-midpoints are mutually perpendicular. Stabilizing each such triple is one of the five tetrahedral subgroups of the icosahedral group. Alternatively, the lines in such a triple correspond to the two-fold rotational axes of a tetrahedron whose four vertices are face-centers of the icosahedron, as in Figure 1. With such a triple of lines as coordinate axes, in octahedral coordinates $\{x_1, x_2, x_3\}$, the points

$$A = \big\{[1,1,1],\ [-1,-1,1],\ [1,-1,-1],\ [-1,1,-1]\big\}$$

constitute a set of tetrahedral vertices, where the brackets indicate homogeneous coordinates for projective space, as usual. The corresponding tetrahedral group $\mathcal{T} = \mathrm{Stab}(A)$ consists of the identity and the 11 *orthogonal* transformations:

$$Z_1 = \begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix},\quad Z_2 = \begin{pmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},\quad Z_3 = \begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$T_1 = \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\quad T_1^2 = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix},$$

$$T_2 = \begin{pmatrix} 0 & 0 & -1\\ 1 & 0 & 0\\ 0 & -1 & 0\end{pmatrix},\quad T_2^2 = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & -1\\ -1 & 0 & 0\end{pmatrix},$$

$$T_3 = \begin{pmatrix} 0 & 0 & -1\\ -1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\quad T_3^2 = \begin{pmatrix} 0 & -1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{pmatrix},$$

$$T_4 = \begin{pmatrix} 0 & 0 & 1\\ -1 & 0 & 0\\ 0 & -1 & 0\end{pmatrix},\quad T_4^2 = \begin{pmatrix} 0 & -1 & 0\\ 0 & 0 & -1\\ 1 & 0 & 0\end{pmatrix}.$$

Being orthogonal, $\mathcal{T}$ preserves the quadratic form

$$C(x) = x_1^2 + x_2^2 + x_3^2,$$

hence also the conic $\mathcal{C} = \{C = 0\}$ in $\mathbb{CP}^2$. This conic contains two sets of orbits of size four: two of the points fixed projectively by each of the three-fold T_k. This is a manifestation of the pairs of *antipodal tetrahedra* found in the icosahedron, that is, tetrahedra obtained from one another by negating all coordinates. The third fixed point is one of the elements of the set A above. With $\rho = e^{2\pi i/3}$ the respective points are

$$\begin{aligned} v_1 &= [\rho,\ \rho^2,\ 1], & v_{\bar 1} &= [\rho^2,\ \rho,\ 1],\\ v_2 &= [-\rho,\ -\rho^2,\ 1], & v_{\bar 2} &= [-\rho^2,\ -\rho,\ 1],\\ v_3 &= [-\rho,\ \rho^2,\ 1], & v_{\bar 3} &= [-\rho^2,\ \rho,\ 1],\\ v_4 &= [\rho,\ -\rho^2,\ 1], & v_{\bar 4} &= [\rho^2,\ -\rho,\ 1]. \end{aligned}$$

The barred notation $v_{\bar a}$ derives from Fricke, being suggested by an antiholomorphic relationship between the two systems of tetrahedra. Indeed, in the x coordinates chosen above, the conjugation map $x \to \bar{x}$ exchanges a tetrahedron and its antipode.

The order-four transformation

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \rho^2 \\ 0 & -\rho & 0 \end{pmatrix}$$

cyclically permutes the v_a but not the $v_{\bar{a}}$, while

$$\bar{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \rho \\ 0 & -\rho^2 & 0 \end{pmatrix}$$

cyclically permutes the $v_{\bar{a}}$ but not the v_a. Also, $Q^2 = \bar{Q}^2 = Z_1$. Since $\mathrm{PGL}_3(\mathbb{C})$ is four-times transitive, Q and $\bar{Q}$ are the unique such projective transformations. Accordingly, the groups $\mathcal{O} = \langle \mathcal{T}, Q\rangle$ and $\bar{\mathcal{O}} = \langle \mathcal{T}, \bar{Q}\rangle$ are octahedral, that is, isomorphic to $\mathcal{S}_4$.

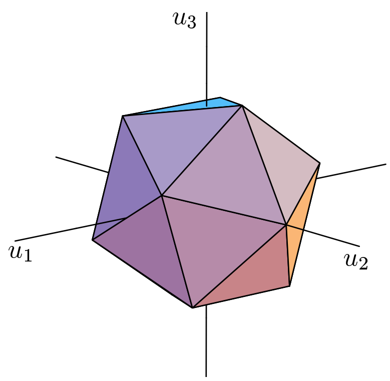

FIGURE 2. The icosahedron in icosahedral coordinates.

Extending $\mathcal{T}$ to an icosahedral group $\mathcal{I}$ requires a projective transformation P of order five that preserves $\mathcal{C}$ and point-wise fixes a pair of antipodal icosahedral vertices. One way of producing such a P is to turn the icosahedron of Figure 1 as in Figure 2, so that a pair of antipodal vertices corresponds to the point $[0,1,0]$, i.e., to the affine points $(0, \pm 1, 0)$. In these *icosahedral coordinates* $\{u_1, u_2, u_3\}$ the desired transformation of order five is

$$P_u = \begin{pmatrix} \cos\frac{2\pi}{5} & 0 & -\sin\frac{2\pi}{5} \\ 0 & 1 & 0 \\ \sin\frac{2\pi}{5} & 0 & \cos\frac{2\pi}{5} \end{pmatrix}.$$

The change of basis from octahedral to icosahedral coordinates is

$$u = Ax = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where

$$c = \sqrt{\frac{5+\sqrt{5}}{10}}, \quad s = \sqrt{\frac{5-\sqrt{5}}{10}}.$$

Thus, in icosahedral coordinates, the conic form preserved by $\mathcal{I}$ is

$$C(u) = u_1^2 + u_2^2 + u_3^2,$$

while P has the expression

$$P_x = A^{-1} P_u A = \frac{1}{2}\begin{pmatrix} 1 & \tau^{-1} & -\tau \\ \tau^{-1} & \tau & 1 \\ \tau & -1 & \tau^{-1} \end{pmatrix},$$

with

$$\tau = \frac{1+\sqrt{5}}{2}.$$

Finally, $\mathcal{I} = \langle \mathcal{T}, P\rangle$. This produces two Valentiner groups distinguished by chirality:

$$\mathcal{V} = \langle \mathcal{I}, Q\rangle, \quad \bar{\mathcal{V}} = \langle \mathcal{I}, \bar{Q}\rangle.$$

Our terms "octahedral coordinates" and "icosahedral coordinates" are from Fricke [1926, pp. 263 ff.]. His octahedral generators are nearly those above. In his icosahedral coordinates, the conic form is

$$C_{\text{Fricke}}(z) = z_1 z_3 + z_2^2.$$

The change of coordinates B that yields $C(Bz) = C_{\text{Fricke}}(z)$ is

$$B = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 1 & 0 & -i \end{pmatrix}.$$

Set $\varepsilon = e^{2\pi i/5}$ and $Z = Z_2$. The generators $Z_u = A Z_x A^{-1}$ and $P_u = A P_x A^{-1}$ become Fricke's icosahedral generators T and S [1926, p. 263]:

$$Z_z = B Z_u B^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} -\frac{1+\sqrt{5}}{2} & 2 & \frac{-1+\sqrt{5}}{2} \\ 1 & 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & 2 & -\frac{1+\sqrt{5}}{2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{pmatrix} \varepsilon^2+\varepsilon^3 & 2 & \varepsilon+\varepsilon^4 \\ 1 & 1 & 1 \\ \varepsilon+\varepsilon^4 & 2 & \varepsilon^2+\varepsilon^3 \end{pmatrix}$$

and

$$P_z = B P_u B^{-1} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon^4 \end{pmatrix}.$$

2C. Valentiner Geometry

Icosahedral conics. Since a ternary icosahedral group has an orthogonal representation on $\mathbb{R}^3$, its complexification stabilizes the quadratic form

$$C(x) = x_1^2 + x_2^2 + x_3^2$$

over $\mathbb{C}$, hence also the conic $\mathcal{C} = \{C(x) = 0\}$ in $\mathbb{CP}^2$. For reasons that will become clear, the two systems of six icosahedral groups $\mathcal{I}_{\bar{k}}$ and $\mathcal{I}_k$ in $\mathcal{V}$ receive the designations "barred" and "unbarred". Corresponding to each of the twelve icosahedral subgroups of $\mathcal{V}$ is a quadratic form $C_{\bar{a}}$ or C_a and conic $\mathcal{C}_{\bar{a}}$ or $\mathcal{C}_a$ that $\mathcal{I}_{\bar{a}}$ or $\mathcal{I}_a$ preserves. Thereby, each such conic possesses the structure of an icosahedron.

Call the form above $C_{\bar{1}}$. The action of $\mathcal{V}$ produces the five remaining forms. With $\eta = (3 + \sqrt{15}\, i)/4$, we have

$$\begin{aligned}
C_{\bar{1}}(x) &= x_1^2+x_2^2+x_3^2,\\
C_{\bar{2}}(x) &= C_{\bar{1}}(Q^{-1}x) = x_1^2+\rho^2 x_2^2+\rho x_3^2,\\
C_{\bar{3}}(x) &= C_{\bar{2}}(P^{-4}x)\\
&= \eta\big(\tfrac{\eta}{3}(x_1^2+\rho x_2^2+\rho^2 x_3^2)+(\rho^2 x_1x_2+\rho x_1x_3-x_2x_3)\big),\\
C_{\bar{4}}(x) &= C_{\bar{2}}(P^{-3}x)\\
&= \eta\big(\tfrac{\eta}{3}(\rho x_1^2+\rho^2 x_2^2+x_3^2)+(-x_1x_2+\rho^2 x_1x_3+\rho x_2x_3)\big),\\
C_{\bar{5}}(x) &= C_{\bar{2}}(P^{-2}x)\\
&= \eta\big(\tfrac{\eta}{3}(\rho x_1^2+\rho^2 x_2^2+x_3^2)-(x_1x_2+\rho^2 x_1x_3+\rho x_2x_3)\big),\\
C_{\bar{6}}(x) &= C_{\bar{2}}(P^{-1}x)\\
&= \eta\big(\tfrac{\eta}{3}(x_1^2+\rho x_2^2+\rho^2 x_3^2)+(\rho^2 x_1x_2-\rho x_1x_3+x_2x_3)\big).
\end{aligned}$$

The action of $\mathcal{V}$ is given as follows (where, as before, $Z = Z_2$):

$$\begin{aligned}
&P: C_{\bar{1}} \leftrightarrow C_{\bar{1}},\ C_{\bar{2}} \to C_{\bar{6}} \to C_{\bar{5}} \to C_{\bar{4}} \to C_{\bar{3}} \to C_{\bar{2}};\\
&Z: C_{\bar{1}} \leftrightarrow C_{\bar{1}},\ C_{\bar{2}} \leftrightarrow C_{\bar{2}},\ C_{\bar{3}} \leftrightarrow \rho^2 C_{\bar{4}},\ C_{\bar{5}} \leftrightarrow \rho C_{\bar{6}};\\
&Q: C_{\bar{1}} \leftrightarrow C_{\bar{2}},\ C_{\bar{3}} \to C_{\bar{6}} \to \rho^2 C_{\bar{5}} \to \rho^2 C_{\bar{4}} \to C_{\bar{3}}.
\end{aligned}$$

Direct calculation yields this result:

Proposition 2.1. *The quadratic forms $C_{\bar{m}}$ are linearly independent and so span the six dimensional space of ternary quadratic forms. In particular, an unbarred conic form C_a is a linear combination of the $C_{\bar{m}}$ that is invariant under the icosahedral group $\mathcal{I}_a$.*

To be specific, the "5-cycle" P belongs to $\mathcal{I}_{\bar{1}}$ and to an unbarred icosahedral group — say $\mathcal{I}_3$ — so that the indexing agrees with that of Fricke. (Recall that the intersection of two nonconjugate $\mathcal{A}_5$ subgroups of $\mathcal{A}_6$ is a $\mathcal{D}_5$.) Relative or *projective* invariance under P requires C_3 to take the form

$$C_3 = \alpha C_{\bar{1}} + C_{\bar{2}} + C_{\bar{3}} + C_{\bar{4}} + C_{\bar{5}} + C_{\bar{6}}.$$

To determine the constant, apply to the $C_{\bar{m}}$ an element T of $\mathcal{I}_3$ that does not belong to $\mathcal{I}_{\bar{1}}$, say one of the 20 elements of order three in $\mathcal{I}_3$. Recalling the association between $\mathcal{I}_3$ and the permutations of the six pairs of antipodal icosahedral vertices labeled according to the action of P, such a transformation corresponds to a double 3-cycle of the form $(abc)(def)$. Specifically, the permutation $(\overline{164})(\overline{235})$ used below corresponds to an element of $\mathcal{I}_3$.

The action of the generators on the *conics* $\mathcal{C}_{\bar{m}}$ is given by the permutation of the indices:

$$\begin{aligned}
P &: (\overline{26543}),\\
Z &: (\overline{34})(\overline{56}),\\
Q &: (\overline{12})(\overline{3654}).
\end{aligned}$$

Computation in $\mathcal{A}_6$ yields the correspondence

$$T = QP^2QPQ^3 : (\overline{164})(\overline{235}).$$

Moreover, the action on the conic forms is

$$T: C_{\bar{1}} \to C_{\bar{6}} \to \rho C_{\bar{4}} \to C_{\bar{1}},\ C_{\bar{2}} \to \rho C_{\bar{3}} \to \rho^2 C_{\bar{5}} \to C_{\bar{2}},$$

so that

$$C_3(T^{-1}x) = \rho^2\, C_{\bar{1}} + \rho(C_{\bar{2}} + C_{\bar{3}} + C_{\bar{4}} + C_{\bar{5}}) + \alpha C_{\bar{6}}.$$

The projective invariance under T of the conic $\mathcal{C}_3 = \{C_3 = 0\}$ requires $\alpha = \rho$. Accordingly,

$$\begin{aligned}
C_3(T^{-1}x) &= \rho^2\, C_{\bar{1}} + \rho(C_{\bar{2}} + C_{\bar{3}} + C_{\bar{4}} + C_{\bar{5}} + C_{\bar{6}})\\
&= \rho C_3(x).
\end{aligned}$$

Just as the barred forms stem from $C_{\bar{1}}$, the remaining unbarred conic forms arise from C_3 (again, our indices are chosen to agree with Fricke's labels):

$$\begin{aligned}
C_1(x) &= C_2(P^{-1}x)\\
&= C_{\bar{1}}+\rho C_{\bar{2}}+C_{\bar{3}}+\rho^2 C_{\bar{4}}+C_{\bar{5}}+\rho C_{\bar{6}}\\
&= -\eta\big(\rho^2 x_1^2+\tfrac{4}{3}\eta^2 x_2^2+\rho x_3^2+2\rho(\rho-1)x_1x_3\big),\\
C_2(x) &= C_3(Q^{-1}x)\\
&= C_{\bar{1}}+\rho C_{\bar{2}}+\rho C_{\bar{3}}+C_{\bar{4}}+\rho^2 C_{\bar{5}}+C_{\bar{6}}\\
&= -\eta\big(\rho^2 x_1^2+\tfrac{4}{3}\eta^2 x_2^2+\rho x_3^2-2\rho(\rho-1)x_1x_3\big),
\end{aligned}$$

$$\begin{aligned}C_3(x) &= \rho C_{\bar 1}+C_{\bar 2}+C_{\bar 3}+C_{\bar 4}+C_{\bar 5}+C_{\bar 6}\\ &= -\eta\rho\big(\rho x_1^2+\rho^2 x_2^2+\tfrac43\eta^2x_3^2+2\rho(\rho-1)x_1x_2\big),\\ C_4(x) &= C_2(P^{-3}x)\\ &= C_{\bar 1}+\rho^2 C_{\bar 2}+C_{\bar 3}+\rho C_{\bar 4}+\rho C_{\bar 5}+C_{\bar 6}\\ &= -\eta\big(\rho x_1^2+\rho^2 x_2^2+\tfrac43\eta^2x_3^2-2\rho(\rho-1)x_1x_2\big),\\ C_5(x) &= C_2(P^{-2}x)\\ &= C_{\bar 1}+C_{\bar 2}+\rho^2 C_{\bar 3}+C_{\bar 4}+\rho C_{\bar 5}+\rho C_{\bar 6}\\ &= -\eta\big(\tfrac43\eta^2x_1^2+\rho x_2^2+\rho^2 x_3^2+2\rho(\rho-1)x_2x_3\big),\\ C_6(x) &= C_2(P^{-4}x)\\ &= C_{\bar 1}+C_{\bar 2}+\rho C_{\bar 3}+\rho C_{\bar 4}+C_{\bar 5}+\rho^2 C_{\bar 6}\\ &= -\eta\big(\tfrac43\eta^2x_1^2+\rho x_2^2+\rho^2 x_3^2-2\rho(\rho-1)x_2x_3\big).\end{aligned}$$

Application of $\mathcal{V}$ yields

$$\begin{aligned}&P: C_3\leftrightarrow C_3,\ C_1\to C_5\to C_4\to C_6\to C_2\to C_1,\\ &Z: C_1\leftrightarrow C_1,\ C_2\leftrightarrow C_2,\ C_3\leftrightarrow \rho C_4,\ C_5\leftrightarrow C_6,\\ &Q: C_5\leftrightarrow C_6,\ C_1\to C_3\to C_2\to \rho C_4\to C_1.\end{aligned}$$

Antiholomorphic symmetry. The one-dimensional icosahedral group $\mathcal{G}_{60}$ acts on two sets of five tetrahedra each of which corresponds to a quadruple of points in $\mathbb{CP}^1$. However, no element of the group sends the tetrahedra of one set to those of the other. Such an exchange occurs by means of anti-holomorphic maps of degree one. Of these, 15 correspond to reflections through the 15 great circles of reflective icosahedral symmetry; the remaining 45 are the various "odd" compositions of the 15 basic reflections — e.g., the antipodal map. Extending the holomorphic $\mathcal{G}_{60}$ by such an "anti-involution" produces the group $\bar{\mathcal{G}}_{120}$ of all 120 symmetries of the icosahedron. The 15 icosahedral reflections generate this extended group while their even products result in $\mathcal{G}_{60}$. In coordinates where one of the great circles corresponds to the real axis, the associated anti-involution is complex conjugation: in homogeneous coordinates,

$$[x_1,x_2]\to[\bar x_1,\bar x_2].$$

The Valentiner analogues of the tetrahedra are the two systems of conics. Are there ternary anti-involutions that exchange the barred and unbarred conics? If so, can they take the form

$$[x_1,x_2,x_3]\to[\bar x_1,\bar x_2,\bar x_3]\,?$$

Fricke answered both questions affirmatively [1926, pp. 270–271, 286–289]. (See below for a combinatorial geometric computation of this additional symmetry.) In the current octahedral coordinates, this *bar-unbar map* is

$$\mathrm{bub}[x_1,x_2,x_3]=[\rho^2\bar x_1-\rho\bar x_3,\,-\rho(\rho+\tau)\bar x_2,\,-\rho\bar x_1-\bar x_3].$$

The action on the conic forms is:

$$\begin{aligned}C_1(\mathrm{bub}(x)) &= \alpha\rho^2 C_{\bar 1}(x),\\ C_2(\mathrm{bub}(x)) &= \alpha\rho C_{\bar 2}(x),\\ C_3(\mathrm{bub}(x)) &= \alpha C_{\bar 3}(x),\\ C_4(\mathrm{bub}(x)) &= \alpha C_{\bar 4}(x),\\ C_5(\mathrm{bub}(x)) &= \alpha\rho^2 C_{\bar 5}(x),\\ C_6(\mathrm{bub}(x)) &= \alpha\rho C_{\bar 6}(x),\end{aligned}$$

where $\alpha=\frac12(3+\sqrt{15}\,i)$. (The match between $C_{\bar a}$ and C_a is no accident. Fricke used this map to dub the unbarred conics.)

Proposition 2.2. *The group* $\bar{\mathcal{V}}_{2\cdot 360}=\langle\mathcal{V},\mathrm{bub}\rangle$ *is a degree two extension of* $\mathcal{V}$.

Proof. For $T\in\mathcal{V}$, the composition $T'=\mathrm{bub}\circ T\circ\mathrm{bub}$ is a projective transformation that permutes the conics within a system. Therefore it belongs to $\mathcal{V}$. □

Concerning the form of a bub map, there are coordinate systems in which its expression *is* conjugation of each coordinate. While interesting in their own right, such coordinates also yield computational benefits. Some of the Valentiner structure suggests a means of achieving this diagonalization. I will take up the topic once the relevant framework is in place.

Special orbits. Some of the special icosahedral points on a conic $\mathcal{C}_{\bar a}$ occur at the intersections of $\mathcal{C}_{\bar a}$ and the other 11 conics.

Fact 2.3. *Within a system*, $\mathcal{C}_{\bar a}$ *meets each* $\mathcal{C}_{\bar m}$ (*where* $\bar m\neq\bar a$) *in four tetrahedral points*; *this gives the* 20 *face-centers on* $\mathcal{C}_{\bar a}$.

The overall result is a 60 point $\mathcal{V}$-orbit $\mathcal{O}_{\overline{60}}$. Similarly, the unbarred intersections yield $\mathcal{O}_{60}$. Alternatively, each member of $\mathcal{O}_{\overline{60}}$ (or $\mathcal{O}_{60}$) is a point fixed by one of the 20 barred (or unbarred) cyclic subgroups of order three in $\mathcal{V}$. (In $\mathcal{A}_6$, the barred-unbarred splitting manifests itself in the two structurally distinct sets in order three, namely (abc) and $(abc)(def)$.)

Fact 2.4. *Across systems the intersection of $\mathcal{C}_{\bar{a}}$ with the $\mathcal{C}_b$ gives six pairs of antipodal icosahedral vertices $\{p_{\bar{a}b_1}, p_{\bar{a}b_2}\}$.*

These total to $72 = 6{\cdot}12$ points each of which is fixed under one of 36 order-five cyclic groups $\langle P_{\bar{a}b}\rangle$. Since an icosahedral group is transitive on its vertices—indeed, some element $\mathcal{I}_{\bar{a}}$ of order two exchanges $p_{\bar{a}b_1}$ and $p_{\bar{a}b_2}$, the 72 points

$$\mathcal{O}_{72} = \{p_{\bar{a}b_1} | a, b = 1, \ldots, 6\} \cup \{p_{\bar{a}b_2} | a, b = 1, \ldots, 6\}$$

form a $\mathcal{V}$-orbit. A Valentiner exchange of $p_{\bar{a}b_1}$ and $p_{\bar{a}b_2}$ also transposes the lines $\mathcal{L}_{\bar{a}b_2}$ and $\mathcal{L}_{\bar{a}b_1}$ tangent to $\mathcal{C}_a$ and $\mathcal{C}_{\bar{b}}$ at $p_{\bar{a}b_1}$ and $p_{\bar{a}b_2}$. (The labeling of these lines in the subsubscript is arbitrary and is done so as to agree with the natural cases in which a point does not reside on its associated line.) Hence, the intersection of these lines belongs to a 36-point orbit, as show in Figure 3. We will call them "36-points"; in general, we will refer to special points and lines in terms of the size of their orbits. Each 36-point $p_{\bar{a}b}$ corresponds to the 36-line $\mathcal{L}_{\bar{a}b} = \{L_{\bar{a}b} = 0\}$ passing through $p_{\bar{a}b_1}$ and $p_{\bar{a}b_2}$. Furthermore, a dihedral $\mathcal{D}_5$ stabilizes the triangle $\{p_{\bar{a}b_1}, p_{\bar{a}b_2}, p_{\bar{a}b}\}$:

$$\begin{aligned}\mathcal{D}_{\bar{a}b} &= \mathrm{Stab}\{p_{\bar{a}b}\} = \mathrm{Stab}\{p_{\bar{a}b_1}, p_{\bar{a}b_2}\} \\ &= \langle P_{\bar{a}b}, Z_{\overline{ac}bd}\rangle \simeq \mathcal{D}_5.\end{aligned}$$

An explanation of the indices attached to the element $Z_{\overline{ac}bd}$ of order two occurs below.

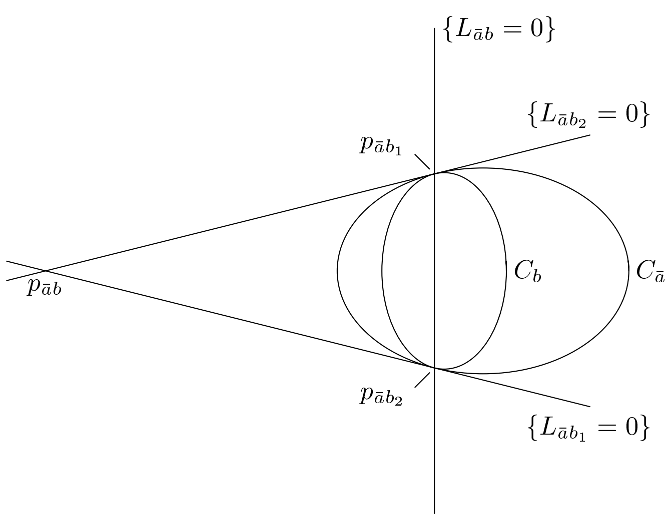

FIGURE 3. The triangle of one 36-point and two 72-points.

As for other special orbits, each of the 45 involutions Z in $\mathcal{V}$ is conjugate to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In $\mathbb{CP}^2$ each such Z fixes a point p_Z and point-wise fixes a line $\mathcal{L}_Z$. Furthermore, Z is the square of an element Q of order four. The three fixed points of Q consist of p_Z and two points $(p_Q)_1$ and $(p_Q)_2$ on $\mathcal{L}_Z$. Each of the latter points have a $\mathbb{Z}/4$ stabilizer and so belong to a 90-point orbit $\mathcal{O}_{90}$. The points p_Z, having $\mathcal{D}_4$ stabilizers, give an orbit $\mathcal{O}_{45}$. Since Z acts trivially on $\mathcal{L}_Z$, the $\mathcal{D}_4$ action restricted to $\mathcal{L}_Z$ reduces to that of a Klein-four group. Finally, the generic points *on* a "45-line" lie in four-point orbits and, overall, provide $\mathcal{V}$-orbits of size 180.

The configuration of 45 lines and points. The intersections of 45-lines yield special orbits of size less than 180. Furthermore, the number of lines meeting at such a site p corresponds to the number of involutions in the stabilizer of p. Being $\mathcal{D}_5$-stable, a 36-point lies on five of the 45-lines. Similarly, four of the 45-lines meet at a 45-point while three concur at each of the 60 and $\overline{60}$-points. This accounts for all intersections of the 45-lines:

$$36\tbinom{5}{2} + 45\tbinom{4}{2} + 60\tbinom{3}{2} + 60\tbinom{3}{2} = \tbinom{45}{2}.$$

On a 45-line there are four points of each type so that the corresponding clusters of lines complete the remaining set of 44 lines:

$$4\Big(\underset{36-\text{points}}{4} + \underset{45-\text{points}}{3} + \underset{60-\text{points}}{2} + \underset{\overline{60}-\text{points}}{2}\Big) = 44.$$

Furthermore, only one involution fixes a 90-point. Hence, such points lie on just one of the 45-lines. Since a 72-point has a $\mathbb{Z}/5$ stabilizer, $\mathcal{O}_{72}$ acquires its exceptional status as the only special orbit that is not a subset of the 45-lines.

Valentiner-speak. Given that the combinatorial relationships among special objects derive from those at the level of conics, the special orbits should admit description in a conic-based terminology. We start with the observation that, by the Valentiner duality between special points and lines, whatever holds for such points also holds for the associated lines. Thus, I will usually supress reference to one or the other.

Already evident is the natural designation of 36, 72-points in terms of one barred and one unbarred conic:

$$\{p_{\bar{a}b_1}, p_{\bar{a}b_2}\} = \mathcal{C}_{\bar{a}} \cap \mathcal{C}_b$$

and $p_{\bar{a}b}$ is the pole of $\{p_{\bar{a}b_1}, p_{\bar{a}b_2}\}$ with respect to $\mathcal{C}_{\bar{a}}$ and $\mathcal{C}_b$. Turning to the 45-points, a given involution Z belongs to two tetrahedral subgroups of the barred and unbarred icosahedral systems alike. If Z preserves conics $\mathcal{C}_{\bar{a}}$, $\mathcal{C}_{\bar{b}}$, $\mathcal{C}_c$, and $\mathcal{C}_d$, then it does so uniquely (see below). Accordingly, $Z_{\overline{ab}cd}$, $p_{\overline{ab}cd}$, and $\mathcal{L}_{\overline{ab}cd}$ denote the transformation, corresponding point and line respectively. Alternatively, the 45-line $\mathcal{L}_{\overline{ab}cd}$ contains the 36-points $p_{\bar{a}c}$, $p_{\bar{a}d}$, $p_{\bar{b}c}$, $p_{\bar{b}d}$ or by duality, the 45-point $p_{\overline{ab}cd}$ lies at the intersection of the 36-lines $\mathcal{L}_{\bar{a}c}$, $\mathcal{L}_{\bar{a}d}$, $\mathcal{L}_{\bar{b}c}$, $\mathcal{L}_{\bar{b}d}$. Furthermore, $\mathcal{L}_{\overline{ab}cd}$ is an icosahedral axis of order two for $\mathcal{C}_{\bar{a}}$, $\mathcal{C}_{\bar{b}}$, $\mathcal{C}_c$, and $\mathcal{C}_d$, while the intersections

$$\mathcal{L}_{\overline{ab}cd} \cap \mathcal{C}_{\bar{a}},\ \mathcal{L}_{\overline{ab}cd} \cap \mathcal{C}_{\bar{b}},\ \mathcal{L}_{\overline{ab}cd} \cap \mathcal{C}_c,\ \mathcal{L}_{\overline{ab}cd} \cap \mathcal{C}_d$$

correspond to the antipodal pairs of edge midpoints. Of course, the labels for the pair of 90-points — each of which $Z_{\overline{ab}cd}$ fixes — should be $p_{\overline{ab}cd_1}$ and $p_{\overline{ab}cd_2}$.

	12	13	14	15	16	23	24	25	26	34	35	36	45	46	56
$\overline{12}$	●									●					●
$\overline{13}$			●						●		●				
$\overline{14}$		●						●						●	
$\overline{15}$					●	●							●		
$\overline{16}$				●			●					●			
$\overline{23}$				●		●								●	
$\overline{24}$					●		●				●				
$\overline{25}$			●					●				●			
$\overline{26}$		●							●				●		
$\overline{34}$	●											●	●		
$\overline{35}$		●					●								●
$\overline{36}$					●			●		●					
$\overline{45}$				●					●	●					
$\overline{46}$			●			●									●
$\overline{56}$	●										●			●	

FIGURE 4. Combinatorial scheme for the Valentiner group.

Now, given a 45-line $\mathcal{L}_{\overline{ab}cd}$, which four of the 45-points belong to it? Since there are $15 = \binom{6}{2}$ choices each for the prefix $\overline{ab}$ and suffix cd, a 15×15 array with 45 distinguished entries depicts the configuration of 45-points and lines: see Figure 4. Being $\mathcal{V}$-equivalent, the rows and columns each contain three marked entries. Associated with each of the 15 barred rows $\overline{ab}$ and its triple of 45-things indexed by $\{\overline{ab}cd, \overline{ab}ef, \overline{ab}gh\}$, where

$$\{c, d, e, f, g, h\} = \{1, \dots, 6\},$$

is a tetrahedral group

$$\mathcal{T}_{\overline{ab}} = \mathcal{I}_{\bar{a}} \cap \mathcal{I}_{\bar{b}}$$

whose three involutions are $Z_{\overline{ab}cd}$, $Z_{\overline{ab}ef}$, $Z_{\overline{ab}gh}$. The analogous state of affairs obtains for the unbarred columns where the tetrahedral group

$$\mathcal{T}_{cd} = \mathcal{I}_c \ \cap \ \mathcal{I}_d$$

contains involutions $Z_{\overline{ab}cd}$, $Z_{\overline{ij}cd}$, $Z_{\overline{k\ell}cd}$. Each of these 15 tetrahedral groups extends to an octahedral subgroup of $\mathcal{V}$,

$$\mathcal{O}_{\overline{ab}} = \mathrm{Stab}\{p_{\overline{ab}cd}, p_{\overline{ab}ef}, p_{\overline{ab}gh}\},$$
$$\mathcal{O}_{cd} = \mathrm{Stab}\{p_{\overline{ab}cd}, p_{\overline{ij}cd}, p_{\overline{kl}cd}\}.$$

Hence, the stabilizer of $p_{\overline{ab}cd}$ is the intersection of octahedral groups

$$\mathcal{O}_{\overline{ab}} \cap \mathcal{O}_{cd} = \mathrm{Stab}\{p_{\overline{ab}cd}\} = \mathrm{Stab}\{\mathcal{L}_{\overline{ab}cd}\} \simeq \mathcal{D}_4.$$

Furthermore, the involution $Z_{\overline{ab}cd}$ associates canonically with the pair of barred and unbarred tetrahedral groups

$$\mathcal{T}_{\overline{ab}} \cap \mathcal{T}_{cd} = \langle Z_{\overline{ab}cd} \rangle \simeq \mathbb{Z}/2.$$

This 45-array, a graphical version of which appears in [Wiman 1895, p. 542], encodes a wealth of combinatorial geometry including an answer to the query of the preceding paragraph. At a 45-point $p_{\overline{ab}cd}$ there are four concurrent 45-lines whose references have the form $\mathcal{L}_{\overline{ab}ef}$, $\mathcal{L}_{\overline{ab}gh}$, $\mathcal{L}_{\overline{mn}cd}$, and $\mathcal{L}_{\overline{rs}cd}$. To find these lines read along the $\overline{ab}$ row and the cd column. By way of example,

$$p_{\overline{12}34} \ \in \ \mathcal{L}_{\overline{12}12} \ \cap \ \mathcal{L}_{\overline{12}56} \ \cap \ \mathcal{L}_{\overline{36}34} \ \cap \ \mathcal{L}_{\overline{45}34}.$$

Duality gives

$$\{p_{\overline{12}12}, p_{\overline{12}56}, p_{\overline{36}34}, p_{\overline{45}34}\} \subset \mathcal{L}_{\overline{12}34}.$$

The $\mathcal{D}_5$ stabilizer of a 36-point $p_{\bar{a}b}$ contains five involutions whose indices have a prefix $\bar{a}$ and a suffix b. For $p_{\bar{3}5}$ this gives $\overline{1}335$, $\overline{2}315$, $\overline{3}445$, $\overline{3}556$, $\overline{3}625$. Hence,

$$p_{\bar{3}5} \ \in \ \mathcal{L}_{\overline{13}35} \cap \mathcal{L}_{\overline{23}15} \cap \mathcal{L}_{\overline{34}45} \cap \mathcal{L}_{\overline{35}56} \cap \mathcal{L}_{\overline{36}25}$$

and

$$\{p_{\overline{13}35}, p_{\overline{23}15}, p_{\overline{34}45}, p_{\overline{35}56}, p_{\overline{36}25}\} \subset \mathcal{L}_{\bar{3}5}.$$

The array also supplies a connection to $\mathcal{A}_6$. For instance, the involution $Z_{\overline{1}234}$ fixes $\mathcal{L}_{\overline{1}234}$ pointwise while preserving $\mathcal{L}_{\overline{1}212}$, $\mathcal{L}_{\overline{1}256}$, $\mathcal{L}_{\overline{3}634}$, and $\mathcal{L}_{\overline{4}534}$ as sets. Accordingly, it permutes the conics by

$$\mathcal{C}_{\overline{3}} \leftrightarrow \mathcal{C}_{\overline{6}}, \quad \mathcal{C}_{\overline{4}} \leftrightarrow \mathcal{C}_{\overline{5}}, \quad \mathcal{C}_1 \leftrightarrow \mathcal{C}_2, \quad \mathcal{C}_5 \leftrightarrow \mathcal{C}_6.$$

Finally, which three involutions fix a $\overline{60}$, 60-point p? (The stabilizer of p is a $\mathcal{D}_3$.) Unlike the other special orbits, $\mathcal{O}_{\overline{60}}$ and $\mathcal{O}_{60}$ have a bias toward one or the other system of conics. Recalling that p is an icosahedral face-center for two conics, say $\mathcal{C}_{\overline{a}}$ and $\mathcal{C}_{\overline{b}}$, an involution Z that fixes p cannot preserve the two conics individually; such an action would have order three. Hence, Z exchanges the conics and lacks a prefix $\overline{ab}$. For $\overline{ab} = \overline{25}$ the array indicates six such involutions:

$$\begin{aligned} Z_{\overline{1}314} &: (\overline{25})(\overline{46}), & Z_{\overline{4}614} &: (\overline{25})(\overline{13}), \\ Z_{\overline{3}625} &: (\overline{25})(\overline{14}), & Z_{\overline{1}425} &: (\overline{25})(\overline{36}), \\ Z_{\overline{3}436} &: (\overline{25})(\overline{16}), & Z_{\overline{1}636} &: (\overline{25})(\overline{34}). \end{aligned}$$

The six associated lines pass through the four points in $\mathcal{C}_{\overline{2}} \cap \mathcal{C}_{\overline{5}}$ as edges of the tetrahedron whose stabilizer is $\mathcal{T}_{\overline{25}}$. They naturally fall into four sets of three lines; among these triples a given index in $\{\overline{1}, \overline{3}, \overline{4}, \overline{6}\}$ appears twice. The intersection of the three lines occurs at the three-fold $\overline{60}$-points (see Figure 5), thereby suggesting appropriate names:

$$\begin{aligned} \mathcal{L}_{\overline{1}314} \cap \mathcal{L}_{\overline{1}425} \cap \mathcal{L}_{\overline{1}636} &= \{p_{\overline{1*346}}\} \\ \mathcal{L}_{\overline{1}314} \cap \mathcal{L}_{\overline{3}436} \cap \mathcal{L}_{\overline{3}625} &= \{p_{\overline{3*146}}\} \\ \mathcal{L}_{\overline{1}425} \cap \mathcal{L}_{\overline{3}436} \cap \mathcal{L}_{\overline{4}614} &= \{p_{\overline{4*136}}\} \\ \mathcal{L}_{\overline{1}636} \cap \mathcal{L}_{\overline{3}625} \cap \mathcal{L}_{\overline{4}615} &= \{p_{\overline{6*134}}\}. \end{aligned}$$

Similarly, I will call the 60-points p_{a*bcd}.

To finish off the description of the special Valentiner points on a 45-line: Which four of the $\overline{60}$, 60-points lie on $\mathcal{L}_{\overline{ab}cd}$? Since $\mathcal{L}_{\overline{ab}cd}$ contains 60-points whose indices satisfy $c{*}dst$ and $d{*}cxy$, the matter comes down to finding values of s, t, x, y that are "Valentiner consistent." This means that they fill out the scheme

$$\begin{aligned} p_{c*dst} &\in \mathcal{L}_{\overline{\cdot\cdot}cd} \cap \mathcal{L}_{\overline{\cdot\cdot}cs} \cap \mathcal{L}_{\overline{\cdot\cdot}ct}, \\ p_{c*duv} &\in \mathcal{L}_{\overline{\cdot\cdot}cd} \cap \mathcal{L}_{\overline{\cdot\cdot}cu} \cap \mathcal{L}_{\overline{\cdot\cdot}cv}, \\ p_{d*cxy} &\in \mathcal{L}_{\overline{\cdot\cdot}cd} \cap \mathcal{L}_{\overline{\cdot\cdot}dx} \cap \mathcal{L}_{\overline{\cdot\cdot}dy}, \\ p_{d*czw} &\in \mathcal{L}_{\overline{\cdot\cdot}cd} \cap \mathcal{L}_{\overline{\cdot\cdot}dz} \cap \mathcal{L}_{\overline{\cdot\cdot}dw}, \end{aligned}$$

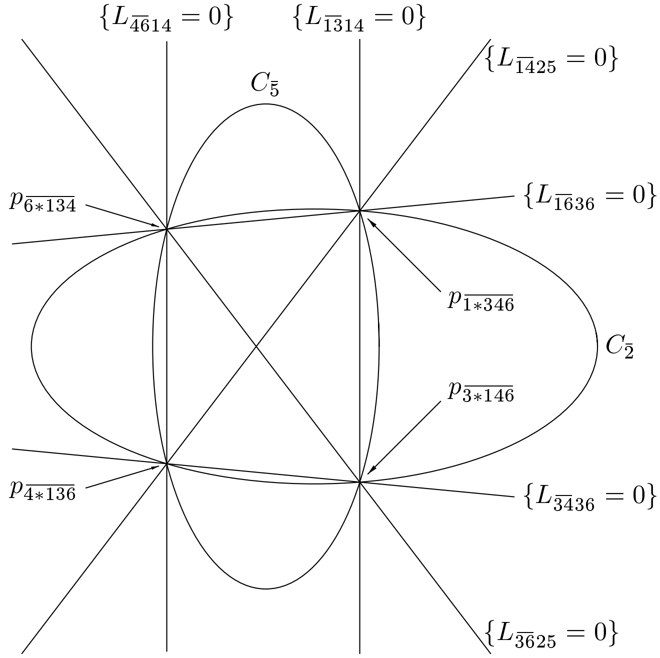

FIGURE 5. Intersecting conics within a system: a tetrahedral configuration.

where each triple of prefixes exhausts $\{\overline{1}, \ldots, \overline{6}\}$ and

$$\{s, t, u, v\} = \{x, y, z, w\} = \{1, \ldots, 6\} - \{c, d\}.$$

For $\mathcal{L}_{\overline{1}234}$,

$$\begin{aligned} p_{3*456} &\in \mathcal{L}_{\overline{1}234} \cap \mathcal{L}_{\overline{5}635} \cap \mathcal{L}_{\overline{3}436}, \\ p_{3*124} &\in \mathcal{L}_{\overline{1}234} \cap \mathcal{L}_{\overline{3}513} \cap \mathcal{L}_{\overline{4}623}, \\ p_{4*356} &\in \mathcal{L}_{\overline{1}234} \cap \mathcal{L}_{\overline{3}445} \cap \mathcal{L}_{\overline{5}646}, \\ p_{4*123} &\in \mathcal{L}_{\overline{1}234} \cap \mathcal{L}_{\overline{3}524} \cap \mathcal{L}_{\overline{4}614} \end{aligned}$$

and

$$\begin{aligned} p_{\overline{1*236}} &\in \mathcal{L}_{\overline{1}234} \cap \mathcal{L}_{\overline{1}326} \cap \mathcal{L}_{\overline{1}615}, \\ p_{\overline{1*245}} &\in \mathcal{L}_{\overline{1}234} \cap \mathcal{L}_{\overline{1}425} \cap \mathcal{L}_{\overline{1}516}, \\ p_{\overline{2*136}} &\in \mathcal{L}_{\overline{1}234} \cap \mathcal{L}_{\overline{2}315} \cap \mathcal{L}_{\overline{2}626}, \\ p_{\overline{2*145}} &\in \mathcal{L}_{\overline{1}234} \cap \mathcal{L}_{\overline{2}416} \cap \mathcal{L}_{\overline{2}525}. \end{aligned}$$

Collected below are the various data for $\mathcal{L}_{\overline{1}234}$.

Orbit	Special points on $\mathcal{L}_{\overline{1}234}$	Multiplicity of 45-lines
$\mathcal{O}_{36}$	$p_{\overline{1}3}$ $p_{\overline{1}4}$ $p_{\overline{2}3}$ $p_{\overline{2}4}$	$\binom{5}{2} = 10$
$\mathcal{O}_{45}$	$p_{\overline{1}212}$ $p_{\overline{1}256}$ $p_{\overline{3}634}$ $p_{\overline{4}534}$	$\binom{4}{2} = 6$
$\mathcal{O}_{\overline{60}}$	$p_{\overline{1*236}}$ $p_{\overline{2*136}}$ $p_{\overline{1*245}}$ $p_{\overline{2*145}}$	$\binom{3}{2} = 3$
$\mathcal{O}_{60}$	p_{3*124} p_{4*123} p_{3*456} p_{4*356}	$\binom{3}{2} = 3$
$\mathcal{O}_{90}$	$p_{\overline{1}234_1}$ $p_{\overline{1}234_2}$	0

For subsequent reference, we summarize the geometric terminology.

point/line	orbit	stabilizer
$p_{\bar{a}b}/\mathcal{L}_{\bar{a}b}$	$\mathcal{O}_{36}$	$\mathcal{D}_5$
$\{p_{\bar{a}b_1}, p_{\bar{a}b_2}\}/\{\mathcal{L}_{\bar{a}b_1}, \mathcal{L}_{\bar{a}b_2}\}$	$\mathcal{O}_{72}$	$\mathbb{Z}/5$
$p_{\overline{a}\bar{b}cd}/\mathcal{L}_{\overline{a}\bar{b}cd}$	$\mathcal{O}_{45}$	$\mathcal{D}_4$
$\{p_{\overline{abcd}_1}, p_{\overline{abcd}_2}\}/\{\mathcal{L}_{\overline{abcd}_1}, \mathcal{L}_{\overline{abcd}_2}\}$	$\mathcal{O}_{90}$	$\mathbb{Z}/4$
$p_{\overline{a*bcd}}/\mathcal{L}_{\overline{a*bcd}}$	$\mathcal{O}_{\overline{60}}$	$\mathcal{D}_3$
$p_{a*bcd}/\mathcal{L}_{a*bcd}$	$\mathcal{O}_{60}$	$\mathcal{D}_3$

2D. Computing a diagonal bub involution

One way to approach the matter of an antiholomorphic symmetry that exchanges systems of conics is to look for points that such a symmetry should fix. Given three such points a, b, c in coordinates y where

$$a = [1, 0, 0], \quad b = [0, 1, 0], \quad c = [0, 0, 1],$$

the associated bub map would have the diagonal form

$$\text{bub}[y_1,\, y_2,\, y_3] = [\alpha\,\overline{y_1},\, \beta\,\overline{y_2},\, \overline{y_3}].$$

Fixing a fourth point determines appropriate values for the inhomogeneous parameters α, β.

But, which points should such a map fix? Moreover, how many such anti-involutions should there be?

A heuristic for bub-symmetry. The basic Valentiner object involving a mixture of barred and unbarred conics is the $\mathcal{D}_5$ structure consisting of a pair of conics $\{\mathcal{C}_{\bar{a}}, \mathcal{C}_b\}$ that intersect in the pair of 72-points $\{p_{\bar{a}b_1}, p_{\bar{a}b_2}\}$. A bub map that exchanges these two conics, must preserve the set $\{p_{\bar{a}b_1}, p_{\bar{a}b_2}\}$ as well as the associated 36-point $p_{\bar{a}b}$. To put some flesh on the skeletal configuration of Figure 3, consider two icosahedra that share a five-fold axis and, about this axis, are one-tenth of a revolution away from each other; see Figure 6. The poles where the axis passes through the icosahedra correspond to the pair of five-fold points $p_{\bar{a}b_{1,2}}$. A reflection through the equatorial plane preserves this arrangement while exchanging the icosahedra and the poles. The icosahedra also transpose under reflection through five planes that include the polar axis. In these cases, the two poles are fixed. This model hints that for

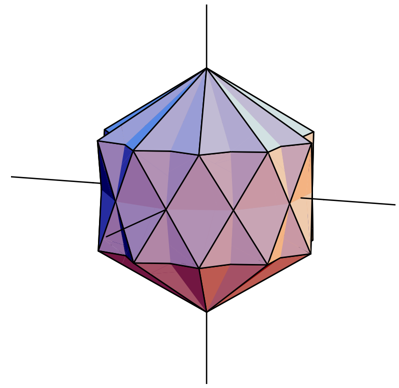

FIGURE 6. Intersecting icosahedra. The union of a barred and unbarred conic has a $\mathcal{D}_5$ structure represented by two icosahedra that meet at a pair of antipodal vertices and are turned away from one another by an angle of $\pi/5$. The reflection through the equatorial plane exchanges the icosahedra and so suggests that for each pair $\{\mathcal{C}_{\bar{a}}, \mathcal{C}_b\}$ there is a primary bub involution. Also transposing the icosahedra are secondary reflections through five vertical planes. These correspond to primary reflections for five other pairs of conics.

each pair $\mathcal{C}_{\bar{a}}$ and $\mathcal{C}_b$, there is a distinguished bub-involution and five of a secondary nature. This makes for a total of 36 maps $\text{bub}_{\bar{a}b}$.

For the primary reflection relative to the pair

$$\{\mathcal{C}_{\bar{2}}, \mathcal{C}_2\},$$

this heuristic demands that $p_{\bar{2}2_1}$ and $p_{\bar{2}2_2}$ exchange while $p_{\bar{2}2}$ remains fixed. What other points should "$\text{bub}_{\bar{2}2}$" fix? Since five other bub maps switch $\mathcal{C}_{\bar{2}}$ and $\mathcal{C}_2$, symmetry requires that $\text{bub}_{\bar{2}2}$ provide a secondary reflection for each barred-unbarred pair of icosahedra associated with the $\bar{2}2$ configuration. This being so, $\text{bub}_{\bar{2}2}$ fixes the corresponding poles of 72-points. Accordingly, the correspondence between the five remaining barred and unbarred conics determines these five pairs of points. Now, the five pairs of nonpolar antipodal vertices on the $\mathcal{C}_{\bar{2}}$ icosahedron correspond to the points of intersection of $\mathcal{C}_{\bar{2}}$ with the five conics $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$, and $\mathcal{C}_6$ while the nonpolar vertices on the $\mathcal{C}_2$ icosahedron correspond to the intersections of $\mathcal{C}_2$ with the five conics $\mathcal{C}_{\bar{1}}$, $\mathcal{C}_{\bar{3}}$, $\mathcal{C}_{\bar{4}}$, $\mathcal{C}_{\bar{5}}$, and $\mathcal{C}_{\bar{6}}$. Let these sets of five pairs correspond to vertices of two pentagons that are one-tenth of a revolution away from each other with antipodal pairs of points being $\text{bub}_{\bar{2}2}$ symmetric (see Figure 7 and imagine looking down, from above a pole, on the intersecting icosahedra of Figure 6). The $\mathcal{D}_5$

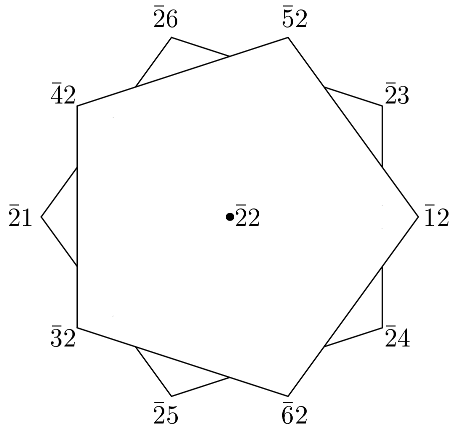

FIGURE 7. A heuristic for bub-symmetry. Regard each of the five pairs of antipodal vertices on the $\mathcal{D}_5$ union of conics $\mathcal{C}_{\bar{2}}$ and $\mathcal{C}_2$ as a vertex of one of two pentagons whose arrangement corresponds to that of the remaining icosahedra. The primary $\mathrm{bub}_{\bar{2}2}$ reflection interchanges the pentagons as well as antipodal vertices. The secondary reflections are $\{\mathrm{bub}_{\bar{a}a} : a \neq 2\}$ and transpose the vertices $\bar{2}a$ and $\bar{a}2$ respectively.

action $\mathcal{D}_{\bar{2}2} = \mathrm{Stab}\{\mathcal{C}_{\bar{2}}, \mathcal{C}_2\}$ determines the specific arrangement.

One of the elements of order five that belongs to $\mathcal{D}_{\bar{2}2}$ is

$$P_{\bar{2}2} = QPQ^{-1}.$$

The associated five-cyclings of conics are $(\overline{15436})$ and (15436). The matching of the $\bar{2}k$ and $\bar{m}2$ vertices depends upon the five involutions that stabilize $\mathcal{C}_{\bar{2}}$ and $\mathcal{C}_2$. For example, the generator $Z_{\bar{1}\bar{2}12}$ associates $\bar{1}$ with 1, while the 5-cycles above determine the remaining matches of $\bar{a}$ with a. This information is also readily available in the 45-array. The five entries that involve both $\bar{2}$ and 2 are $\overline{2a}2a$. Indeed, the array's symmetry about the diagonal $\overline{ab}ab$ is a combinatorial manifestation of $\mathrm{bub}_{\bar{2}2}$.

Special coordinates and the bub-$\mathbb{RP}^2$. Following the clue provided by the above heuristic, make a parametrized change of icosahedral to octahedral coordinates, of the form

$$A = \left(\; a p_{\bar{1}1_1}^T \;\middle|\; b p_{\bar{1}1_2}^T \;\middle|\; p_{\bar{1}1}^T \; \right)$$
$$= \left(a \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ -\sqrt{\frac{5-\sqrt{5}}{2}}\, i \\ 1 \end{pmatrix} \middle| \; b \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ \sqrt{\frac{5-\sqrt{5}}{2}}\, i \\ 1 \end{pmatrix} \middle| \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 0 \\ 1 \end{pmatrix} \right).$$

In these icosahedral coordinates y, the $\bar{1}1$ triangle is

$$p_{\bar{1}1_1} = [1,0,0], \quad p_{\bar{1}1_2} = [0,1,0], \quad p_{\bar{1}1} = [0,0,1].$$

The candidate $\mathrm{bub}_{\bar{2}2}$ map $K(y) = \bar{y}$ fixes each of these. In octahedral coordinates the $\bar{2}2$ triangle consists of

$$p_{\bar{2}2_1} = \left[\tfrac{1+\sqrt{5}}{2}\,\rho^2,\ \sqrt{\tfrac{5+\sqrt{5}}{2}}\, i\rho,\ 1\right]$$
$$p_{\bar{2}2_2} = \left[\tfrac{1+\sqrt{5}}{2}\,\rho^2,\ -\sqrt{\tfrac{5+\sqrt{5}}{2}}\, i\rho,\ 1\right]$$
$$p_{\bar{2}2} = \left[\tfrac{1-\sqrt{5}}{2}\,\rho^2,\ 0,\ 1\right].$$

The hope here is that, when transformed to y coordinates, some choice of a, b results in

$$K(A^{-1} p_{\bar{2}2_1}) = p_{\bar{2}2_2},$$
$$K(A^{-1} p_{\bar{2}2_2}) = p_{\bar{2}2_1},$$
$$K(A^{-1} p_{\bar{2}2}) = p_{\bar{2}2}.$$

Satisfying these conditions are the values

$$a = b = \frac{\sqrt{3} - \sqrt{5}\, i}{8}.$$

The change of coordinates becomes

$$A = \begin{pmatrix} \frac{(1-\sqrt{5})\,(3-\sqrt{5}\,i)}{4\sqrt{2}} & \frac{(1-\sqrt{5})\,(3-\sqrt{5}\,i)}{4\sqrt{2}} & \frac{1+\sqrt{5}}{2} \\ \frac{\sqrt{5-\sqrt{5}}\,(3-\sqrt{5}\,i)}{4} & \frac{\sqrt{5-\sqrt{5}}\,(3-\sqrt{5}\,i)}{4} & 0 \\ \frac{3-\sqrt{5}\,i}{2\sqrt{2}} & \frac{3-\sqrt{5}\,i}{2\sqrt{2}} & 1 \end{pmatrix}.$$

As for the conic forms, they satisfy the desired condition:

$$C_{\bar{k}}(y) = \overline{C_k(\bar{y})}.$$

A further change of coordinates given by a real diagonal matrix leaves $\mathrm{bub}_{\bar{2}2}(y) = \bar{y}$ undisturbed. In y coordinates the one-point orbit $p_{\bar{2}2}$ under the $\mathcal{D}_5$ for $\bar{2}2$ is

$$[1,\ 1,\ \sqrt{6}/\tau],$$

where as before $\tau = \frac{1}{2}(1+\sqrt{5})$. For a final simplification, the additional coordinate change

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{6}/\tau \end{pmatrix}$$

arranges for this point to be $[1,1,1]$. In these adjusted y coordinates, the $\bar{1}1$ and $\bar{2}2$ triangles are

$$\begin{aligned} p_{\bar{1}1_1} &= [1,0,0], & p_{\bar{2}2_1} &= [3,\ 2\eta^2,\ -\eta], \\ p_{\bar{1}1_2} &= [0,1,0], & p_{\bar{2}2_2} &= [3,\ 2\,\bar{\eta}^2,\ -\bar{\eta}], \\ p_{\bar{1}1} &= [0,0,1], & p_{\bar{2}2} &= [1,1,1], \end{aligned}$$

where as before $\eta = \frac{1}{4}(3+\sqrt{15}\,i)$; in these $\mathrm{bub}_{\bar{2}2}$ coordinates, the normalized conic forms for $\bar{1}$ and 1 are

$$C_{\bar{1}}(y) = \left(\tfrac{2}{3}\,\bar{\eta}\right)^2 y_1y_2 + y_3^2$$
$$C_1(y) = \left(\tfrac{2}{3}\,\eta\right)^2 y_1y_2 + y_3^2.$$

The unwieldy expressions for the remaining forms are not recorded here.

Since $\mathrm{bub}_{\bar{2}2}$ restricts to the identity on the $\mathbb{RP}^2$ given by

$$\mathcal{R}_{\bar{2}2} = \{[t_1,t_2,t_3] : t_k \in \mathbb{R}\},$$

symmetry provides for such a fixed set $\mathcal{R}_{\bar{a}b}$ for each of the 36 maps $\mathrm{bub}_{\bar{a}b}$. Figure 8 provides a geometric interpretation of these $\mathbb{RP}^2$s. One consequence of the extra symmetry is that $\overline{\mathcal{V}}_{2\cdot 360}$-invariant forms and $\overline{\mathcal{V}}_{2\cdot 360}$-equivariant maps are, when expressed in special bub-coordinates, given by polynomials with real and even, in special cases, rational coefficients. This is discussed in a later section on basic invariants; see page 221. We remark that although Wiman [1895, pp. 548–550] and Fricke [1926, pp. 286–289] mention these coordinates, they seem not to have made much use of them.

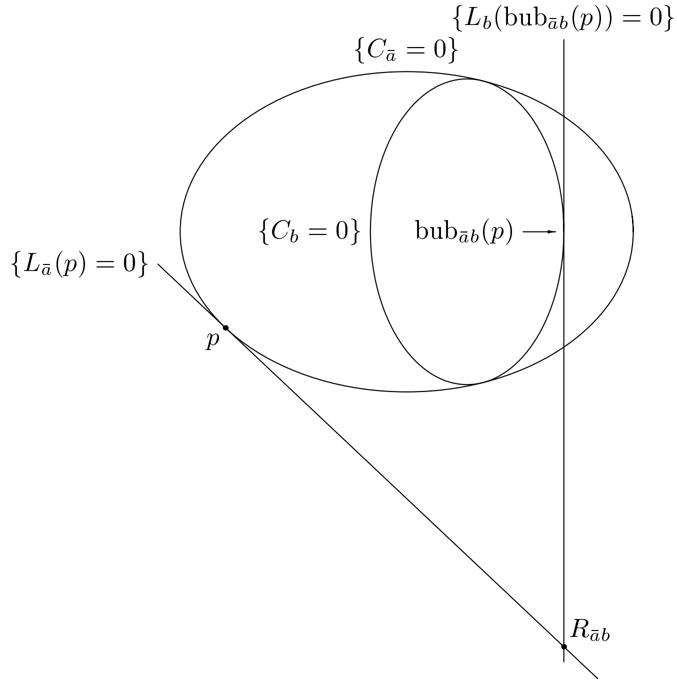

FIGURE 8. A geometric interpretation of bub-maps. Since $\mathrm{bub}_{\bar{a}b}$ interchanges the pair of conics $\mathcal{C}_{\bar{a}}$ and $\mathcal{C}_b$ as well as the lines $\mathcal{L}_{\bar{a}}(p) = \{L_{\bar{a}}(p)=0\}$ and $\mathcal{L}_b(\mathrm{bub}_{\bar{a}b}(p)) = \{L_b(p)=0\}$, tangent to $\mathcal{C}_{\bar{a}}$ at p and to $\mathcal{C}_b$ at $\mathrm{bub}_{\bar{a}b}$(p), it also fixes the $\mathbb{RP}^2$'s worth of points $\mathcal{R}_{\bar{a}b} = \{\mathcal{L}_{\bar{a}}(p)\cap\mathcal{L}_b(\mathrm{bub}_{\bar{a}b}(p)) : p\in\mathcal{C}_{\bar{a}}\}$.

2E. Invariant Structure

For a group action on a vector space the Molien series provides one of the basic tools of classical invariant theory. Given a finite group $\mathcal{G}$ acting faithfully on $\mathbb{C}^n$, the dimension of the space $\mathbb{C}[x]^{\mathcal{G}}_m$ of invariant homogeneous polynomials of degree m appears as the coefficient of the mth degree term in the *Molien series* for $\mathcal{G}$:

$$M(\mathbb{C}[x]^{\mathcal{G}}) = \sum_{m=0}^{\infty}\left(\dim\mathbb{C}[x]^{\mathcal{G}}_m\right)t^m.$$

In the Valentiner case the space is $\mathbb{C}^3$ while the group is a 1-to-3 lift of $\mathcal{V}$ to a subgroup $\mathcal{V}_{3\cdot 360}$ of SU_3. As a result of the character $\langle\rho\rangle$ that appears under $\mathcal{V}$'s action on the icosahedral conic forms, this lift of $\mathcal{V}$ to a linear group has minimal order. (A lift of a projective group $\mathcal{G}$ to a linear group $\mathcal{G}'$ has *minimal order* if any other lift $\mathcal{H}$ of $\mathcal{G}$ satisfies $|\mathcal{H}|\geq|\mathcal{G}'|$. See [Fricke 1926, pp. 267–268] for details.) A further consequence of minimality is that the Molien series for the $\mathcal{V}_{3\cdot 360}$ gives complete information concerning the invariants of the projective group $\mathcal{V}$.

Proposition 2.5. *The invariants of $\mathcal{V}$ and $\mathcal{V}_{3\cdot 360}$ are in one-to-one correspondence.*

Proof. Trivially, a $\mathcal{V}_{3\cdot 360}$-invariant gives a $\mathcal{V}$-invariant. Conversely, let $F(x)$ be a $\mathcal{V}$-invariant with

$$F(T^{-1}x) = \alpha(T)\,F(x)$$

for $T\in\mathcal{V}_{3\cdot 360}$. The kernel of the multiplicative character

$$\alpha : \mathcal{V}_{3\cdot 360} \longrightarrow \mathbb{C}-\{0\}$$

is the normal subgroup $\mathrm{Stab}(F)\subset\mathcal{V}_{3\cdot 360}$ that stabilizes F. Since the projective image $[\mathrm{Stab}(F)]\simeq\mathrm{Stab}(F)/\langle\rho\rangle$ is normal in the simple group $\mathcal{V}\simeq\mathcal{V}_{3\cdot 360}/\langle\rho\rangle$, $[\mathrm{Stab}(F)]$ is either trivial or $\mathcal{V}$. In the former case, $\mathrm{Stab}(F)$ would be either trivial or $\langle\rho\rangle$ so that $\mathcal{V}_{3\cdot 360}/\mathrm{Stab}(F)$ would be nonabelian. Thus, $[\mathrm{Stab}(F)]=\mathcal{V}$. Since $\mathcal{V}_{3\cdot 360}$ includes no subgroup of order 360, $\mathrm{Stab}(F)=\mathcal{V}_{3\cdot 360}$ and F is $\mathcal{V}_{3\cdot 360}$-invariant. □

Also, $\mathcal{V}$ lifts 1-to-6 to a so-called unitary reflection group $\mathcal{V}_{6\cdot 360}$ generated by 45 involutions on $\mathbb{C}^3$; see

[Shephard and Todd 1954, pp. 278, 287]. The elements of $\mathcal{V}_{6\cdot 360}$ satisfy $\det T = \pm 1$ while those of $\mathcal{V}_{3\cdot 360}$ satisfy $\det T = 1$.

Molien's theorem and its application to $\mathcal{V}$. By projecting $\mathbb{C}[x]_m$ onto $\mathbb{C}[x]_m^{\mathcal{G}}$, one arrives at a generating function for the Molien series.

Theorem 2.6. *For a finite group action $\mathcal{G}$ on $\mathbb{C}^n$,*

$$M(\mathcal{G}) = \frac{1}{|\mathcal{G}|} \sum_{C_T \subset \mathcal{G}} \frac{|C_T|}{\det(I - tT^{-1})},$$

where C_T are conjugacy classes.

For the proof, see [Benson 1993, pp. 21–22].

Proposition 2.7. *For the Valentiner groups $\mathcal{V}_{3\cdot 360}$ and $\mathcal{V}_{6\cdot 360}$, the Molien series are given by*

$$\begin{aligned} M(\mathcal{V}_{3\cdot 360}) &= \frac{1 + t^{45}}{(1-t^6)(1-t^{12})(1-t^{30})} \\ &= 1 + t^6 + 2t^{12} + 2t^{18} + \cdots + t^{45} + \cdots, \\ M(\mathcal{V}_{6\cdot 360}) &= \frac{1}{(1-t^6)(1-t^{12})(1-t^{30})} \\ &= 1 + t^6 + 2t^{12} + 2t^{18} + \cdots. \end{aligned}$$

Proof. With $k = 0, 1, 2$, the matrices

$$\pm\rho^k I,\ \pm\rho^k P,\ \pm\rho^k Z,\ \pm\rho^k Q,\ \pm\rho^k PZ,\ \pm\rho^k T$$

represent distinct conjugacy classes in $\mathcal{V}_{6\cdot 360}$. For $\mathcal{V}_{3\cdot 360}$ the three matrices of each type corresponding to the $+\rho^k$ do the job. Substitution into the formula of Molien's theorem produces the indicated generating functions. □

The basic invariants themselves. We know from the theory of complex reflection groups that there are three algebraically independent *basic* forms that generate the ring of $\mathcal{V}_{6\cdot 360}$-invariants. The generating function for the Molien series indicates that these occur in degrees 6, 12, and 30. Techniques of classical invariant theory provide for the computation of the forms in degrees 12 and 30 from that of degree 6. But, how does the latter arise? Although $\mathcal{V}_{6\cdot 360}$ permutes the *conics*, its action on the conic forms is not *simple*—a nontrivial character appears. However, the cubes of the forms do receive simple treatment by $\mathcal{V}_{6\cdot 360}$. Hence, summing the cubes of either system of conic forms and normalizing the coefficients yields a $\mathcal{V}_{6\cdot 360}$-invariant:

$$\begin{aligned} F(x) &= \alpha \sum_{m=1}^{6} C_{\bar{m}}(x)^3 = \alpha \sum_{m=1}^{6} C_m(x)^3 \\ &= x_1^6 + x_2^6 + x_3^6 + 3\bigl(5 - \sqrt{15}\, i\bigr) x_1^2 x_2^2 x_3^2 \\ &\quad + \tfrac{3}{4}\bigl(2\sqrt{5} - (5 - \sqrt{5})\,\rho\bigr)(x_1^4 x_2^2 + x_2^4 x_3^2 + x_1^2 x_3^4) \\ &\quad - \tfrac{3}{4}\bigl(2\sqrt{5} + (5 + \sqrt{5})\,\rho^2\bigr)(x_1^4 x_3^2 + x_1^2 x_2^4 + x_2^2 x_3^4). \end{aligned}$$

By uniqueness, F is also $\overline{\mathcal{V}}_{2\cdot 360}$-invariant. Expressed and normalized in $\mathrm{bub}_{\bar{2}2}$ coordinates,

$$\begin{aligned} F(y) = {} & 10\, y_1^3 y_2^3 + 9\, y_1^5 y_3 + 9\, y_2^3 y_3 \\ & - 45\, y_1^2 y_2^2 y_3^2 - 135\, y_1 y_2 y_3^4 + 27\, y_3^6. \end{aligned}$$

The form Φ of degree 12 arises from the determinant of the Hessian $H(F)$ of F:

$$\begin{aligned} \Phi(y) &= \alpha_\Phi \left| H(F(y)) \right| \\ &= 6\, y_1^{11} y_2 - 38\, y_1^6 y_2^6 + 6\, y_1 y_2^{11} + 90\, y_1^8 y_2^3 y_3 \\ &\quad + 90\, y_1^3 y_2^8 y_3 - 9\, y_1^{10} y_3^2 - 468\, y_1^5 y_2^5 y_3^2 - 9\, y_2^{10} y_3^2 \\ &\quad + 1080\, y_1^7 y_2^2 y_3^3 + 1080\, y_1^2 y_2^7 y_3^3 + 3375\, y_1^4 y_2^4 y_3^4 \\ &\quad - 324\, y_1^6 y_2 y_3^5 - 324\, y_1 y_2^6 y_3^5 - 1080\, y_1^3 y_2^3 y_3^6 \\ &\quad + 2916\, y_1^5 y_3^7 + 2916\, y_2^5 y_3^7 + 1215\, y_1^2 y_2^2 y_3^8 \\ &\quad + 4374\, y_1 y_2 y_3^{10} + 729\, y_3^{12}. \end{aligned}$$

Similarly, the form Ψ of degree 30 arises from the determinant of the "bordered Hessian" $BH(F, \Phi)$ of F and Φ:

$$\begin{aligned} \Psi(y) &= \alpha_\Phi \left| BH(F(y), \Phi(y)) \right| \\ &= \alpha_\Phi \left| \begin{array}{ccc|c} & & & F_{y_1} \\ & H(\Phi(y)) & & F_{y_2} \\ & & & F_{y_3} \\ \hline F_{y_1} & F_{y_2} & F_{y_3} & 0 \end{array} \right| \\ &= 3\, y_1^{30} + \cdots + 3\, y_2^{30} + \cdots + 57395628\, y_3^{30}. \end{aligned}$$

The constants $\alpha_\Phi = -1/20250$ and $\alpha_\Psi = 1/24300$ remove the highest common factor among the coefficients.

Finally, the product of the 45 linear forms that correspond to the generating involutions is a relative $\mathcal{V}_{6\cdot 360}$-invariant but an absolute $\mathcal{V}_{3\cdot 360}$-invariant, and hence a projective $\mathcal{V}$-invariant. As a specific instance of a general result [Shephard and Todd 1954,

p. 283], this degree-45 form is given by the Jacobian determinant

$$\begin{aligned} X(y) &= \alpha_X \left|J(F(y),\Phi(y),\Psi(y))\right| \\ &= \alpha_X \begin{vmatrix} F_{y_1} & F_{y_2} & F_{y_3} \\ \Phi_{y_1} & \Phi_{y_2} & \Phi_{y_3} \\ \Psi_{y_1} & \Psi_{y_2} & \Psi_{y_3} \end{vmatrix} \\ &= \beta \prod L_{\overline{abcd}}(y) \\ &= y_1^{45} + \cdots - y_2^{45} + \cdots + 3570467226624\, y_2^5 y_3^{40}, \end{aligned}$$

where $\alpha_X = -1/4860$ and β is a constant. Being $\mathcal{V}_{6\cdot 360}$-invariant, X^2 is a polynomial in F, Φ, Ψ:

$$\begin{aligned} 3^9 X^2 = {} & 4F^{13}\Phi + 80F^{11}\Phi^2 + 816F^9\Phi^3 \\ & + 4376F^7\Phi^4 + 13084F^5\Phi^5 + 12312F^3\Phi^6 \\ & + 5616F\Phi^7 + 18F^{10}\Psi + 198F^8\Phi\Psi \\ & + 954F^6\Phi^2\Psi - 198F^4\Phi^3\Psi - 5508F^2\Phi^4\Psi \\ & - 1944\Phi^5\Psi - 162F^5\Psi^2 - 1944F^3\Phi\Psi^2 \\ & - 1458F\Phi^2\Psi^2 + 729\Psi^3. \end{aligned} \tag{2–1}$$

$\mathcal{V}$-symmetric maps and the sextic. The system of invariants provides a foundation on which to construct mappings of $\mathbb{C}^3$ or $\mathbb{CP}^2$ that are symmetric or *equivariant* under the action of $\mathcal{V}_{3\cdot 360}$ or $\mathcal{V}$. Algebraically, this means that the map commutes with the action. Given such a map that also possesses reliable dynamics, the sixth-degree equation has an iterative solution.

3. RATIONAL MAPS WITH VALENTINER SYMMETRY

An iterative solution to the sextic utilizes a parametrized family of dynamical systems having $\mathcal{A}_6$ symmetry. In practice, a given sixth-degree polynomial p with galois group $\mathcal{A}_6$ specifies a projective transformation

$$S_p : \mathbb{CP}^2 \to \mathbb{CP}^2$$

and thereby hooks up to a rational map

$$S_p^{-1} \circ f \circ S_p$$

that has $\mathcal{A}_6$ symmetry. Accordingly, the fixed map f is the centerpiece of a sextic-solving algorithm.

3A. Finding Equivariant Maps

A linear group $\mathcal{G}$ acts on the exterior algebra $\Lambda(\mathbb{C}^n)$ by

$$(T(\alpha))(x) = \alpha(T^{-1}x),$$

where $T \in \mathcal{G}$, α is a 1-form, and $x \in \mathbb{C}^n$. As in the case of $\mathcal{V}$-invariant polynomials, $\mathcal{V}$-invariant p-forms associate one-to-one with $\mathcal{V}_{3\cdot 360}$-invariant p-forms. Hence, the search for symmetric maps can take place within the regime of the linear action.

Define a map f to be *relatively $\mathcal{G}$-equivariant* if, for all $T \in \mathcal{G}$, we have $Tf = \chi_T f \circ T$ (for an appropriate choice of χ_T). If the character χ_T is trivial, f is *absolutely equivariant*. As we look for equivariants, the $\mathcal{V}_{3\cdot 360}$ action on $\Lambda(\mathbb{C}^3)$ provides guidance, thanks to a correspondence between $\mathcal{V}_{3\cdot 360}$-equivariants and $\mathcal{V}_{3\cdot 360}$-invariant 2-forms. Set

$$dX_2 = (dx_2 \wedge dx_3,\ dx_3 \wedge dx_1,\ dx_1 \wedge dx_2)$$

and let $\cdot$ signify a formal dot product.

Proposition 3.1. *For a given finite action $\mathcal{G} \subset \mathrm{U}_3$ and a $\mathcal{G}$-invariant 2-form*

$$\begin{aligned} \varphi(x) = f(x)\cdot dX_2 = {} & f_1(x)\, dx_2 \wedge dx_3 \\ & + f_2(x)\, dx_3 \wedge dx_1 + f_3(x)\, dx_1 \wedge dx_2, \end{aligned}$$

the map $f = (f_1, f_2, f_3)$ is relatively $\mathcal{G}$-equivariant. If $\mathcal{G} \subset \mathrm{SU}_3$, the equivariance of f is absolute.

Proof. Given $T \in \mathcal{G}$, set $T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$ and

$$T^{-1} = |T|^{-1} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}.$$

We wish to show that $Tf(T^{-1}x)$ is a multiple of $f(x)$. Let (e_1, e_2, e_3) be the canonical basis of $\mathbb{C}^3$, and evaluate the forms $Tf(T^{-1}x)\cdot dX_2$ and $f(x)\cdot dX_2$ on (e_2, e_3). By definition and the invariance of φ,

$$\begin{aligned} & (f(x)\cdot dX_2)(e_2,e_3) \\ & \quad = \varphi(x)(e_2,e_3) = (T(\varphi))(x)(e_2,e_3) = \varphi(T^{-1}x)(e_2,e_3) \\ & \quad = |T|^{-2}\big(f_1(T^{-1}(x))(C_{22}C_{33} - C_{23}C_{32}) \\ & \qquad\qquad + f_2(T^{-1}(x))(C_{32}C_{13} - C_{33}C_{12}) \\ & \qquad\qquad + f_3(T^{-1}(x))(C_{12}C_{23} - C_{13}C_{22})\big) \\ & \quad = |T|^{-3}\big(f_1(T^{-1}(x))t_{11} + f_2(T^{-1}(x))t_{21} \\ & \qquad\qquad + f_3(T^{-1}(x))t_{31}\big). \end{aligned}$$

(Here $C_{22}C_{33} - C_{23}C_{32}$ comes from $d\big(\sum_i C_{2i}x_i\big) \wedge d\big(\sum_i C_{3i}x_i\big)$ applied to (e_2, e_3), and so on.) The preceding expression, apart from the factor $|T|^{-3}$, is exactly the result of evaluating $(Tf(T^{-1}x)) \cdot dX_2$

on (e_2, e_3). The same equality holds for other pairs of basis vectors, showing that $Tf(T^{-1}x) = |T|^3 f(x)$, and f is relatively equivariant. If $T \in \mathrm{SU}_3$, we have $|T| = 1$ and absolute equivariance occurs. □

Conversely, an absolute equivariant corresponds to a relatively invariant 2-form, with absolute invariance holding in case $\mathcal{G} \subset \mathrm{SU}_3$.

For invariant exterior forms, there is a 2-variable "exterior Molien series"

$$M(\Lambda^{\mathcal{G}}) = \sum_{p=0}^{n} \biggl(\sum_{m=0}^{\infty} \Bigl(\dim \Lambda_p \left(\mathbb{C}^n\right)_m^{\mathcal{G}} \Bigr) t^m \biggr) s^p,$$

in which the variables s and t index respectively the rank of the form and the polynomial degree, and the $\Lambda_p(\mathbb{C}^n)_m^{\mathcal{G}}$ are the $\mathcal{G}$-invariant homogeneous p-forms of degree m; see [Benson 1993, p. 62] or [Smith 1995, pp. 265 ff.]. Projection of $\Lambda_p(\mathbb{C}^n)_m$ onto $\Lambda_p(\mathbb{C}^n)_m^{\mathcal{G}}$ yields the analogue to Molien's theorem.

Theorem 3.2. *The exterior Molien series for a finite group action $\mathcal{G}$ is given by the generating function*

$$M(\Lambda^{\mathcal{G}}) = \frac{1}{|\mathcal{G}|} \sum_{\mathcal{C}_T \subset \mathcal{G}} |\mathcal{C}_T| \, \frac{\det\left(I + s\,T^{-1}\right)}{\det\left(I - t\,T^{-1}\right)},$$

where the $\mathcal{C}_T$ are conjugacy classes.

As in the case of invariants, this has the following consequence:

Proposition 3.3. *For $\mathcal{V}_{3\cdot360}$, the exterior Molien series $M(\Lambda^{\mathcal{V}_{3\cdot360}})$ is given by*

$$\frac{\begin{aligned}1{+}t^{45} + (t^5{+}t^{11}{+}t^{20}{+}t^{26}{+}t^{29}{+}t^{44})\,s \\ + (t{+}t^{16}{+}t^{19}{+}t^{25}{+}t^{34}{+}t^{40})\,s^2 + (1{+}t^{45})\,s^3\end{aligned}}{(1{-}t^6)(1{-}t^{12})(1{-}t^{30})}$$

$$\begin{aligned}&= 1{+}t^6{+}2t^{12}{+}2t^{18}{+}3t^{24}{+}4t^{30}{+}\cdots \\ &\quad + (t^5{+}2t^{11}{+}3t^{17}{+}t^{20}{+}4t^{23}{+}2t^{26}{+}6t^{29}{+}\cdots)\,s \\ &\quad + (t{+}t^7{+}2t^{13}{+}t^{16}{+}3t^{19}{+}t^{22}{+}5t^{25}{+}2t^{28}{+}\cdots)\,s^2 \\ &\quad + (1{+}t^6{+}2t^{12}{+}2t^{18}{+}3t^{24}{+}4t^{30}{+}\ldots)\,s^3.\end{aligned}$$

3B. A Query on Finite Reflection Groups

For a reflection group $\mathcal{G}$ that acts on $\mathbb{C}^n$ the n basic invariant 0-forms are algebraically independent [Shephard and Todd 1954, pp. 282 ff.]. Multiplication of an invariant p-form α of degree ℓ by an invariant 0-form F of degree m promotes α to an invariant p-form $F\alpha$ of degree $\ell + m$. In the series $M(\Lambda^{\mathcal{H}})$ for a subgroup $\mathcal{H} \subset \mathcal{G}$ the contribution of the free algebra generated by the basic 0-forms disappears upon division of $M(\Lambda^{\mathcal{H}})$ by $M(\Lambda_0^{\mathcal{G}}) = M(\mathbb{C}[x]^{\mathcal{G}})$. The resulting *polynomial* in two variables displays the degrees of the generating $\mathcal{H}$-invariant forms. In the cases of the 0, n-forms, which have identical series, what remains are the terms corresponding to nonconstant polynomials that are $\mathcal{H}$-invariant but not $\mathcal{G}$-invariant.

Proposition 3.4. *For the Valentiner group, the "exterior Molien quotient" is*

$$\begin{aligned}&M(\Lambda^{\mathcal{V}_{3\cdot360}})/M(\Lambda_0^{\mathcal{V}_{6\cdot360}}) \\ &\qquad = (1{+}t^{45}) + (t^5{+}t^{11}{+}t^{20}{+}t^{26}{+}t^{29}{+}t^{44})s \\ &\qquad\quad + (t{+}t^{16}{+}t^{19}{+}t^{25}{+}t^{34}{+}t^{40})s^2{+}(1{+}t^{45})s^3.\end{aligned}$$

Notice the duality in degree 45 between 0 and 3-forms:

$$\begin{array}{lcc} s^0: & 1 = t^0 & t^{45} \\ s^3: & t^{45} & 1 = t^0 \end{array}$$

and between 1 and 2-forms:

$$\begin{array}{lcccccc} s^1: & t^5 & t^{11} & t^{20} & t^{26} & t^{29} & t^{44} \\ s^2: & t^{40} & t^{34} & t^{25} & t^{19} & t^{16} & t. \end{array}$$

By uniqueness, up to scalar multiplication, of the 3-form X vol associated with the 45 complex planes of reflection, the exterior product of "dual" forms must yield a multiple of this form.

This duality between invariant p and $(n{-}p)$-forms also appears in the *ternary* icosahedral group $\mathcal{I}_{60}$. The generating 0-forms for the full reflection group $\mathcal{I}_{2\cdot60}$ have degrees 2, 6, and 10 [Shephard and Todd 1954, p. 301, line 23 of table]. From the discussion of the Valentiner conics the invariant of degree 2 is familiar, while those of degrees 6 and 10 are products of linear forms that $\mathcal{I}_{60}$ preserves. Here the duality occurs in degree 15 which is the number of reflection-planes for the icosahedron:

$$\begin{aligned}&M(\Lambda^{\mathcal{I}_{60}})/M(\Lambda_0^{\mathcal{I}_{2\cdot60}}) \\ &\qquad = (1{+}t^{15}) + (t{+}t^5{+}t^6{+}t^9{+}t^{10}{+}t^{14})s \\ &\qquad\quad + (t{+}t^5{+}t^6{+}t^9{+}t^{10}{+}t^{14})s^2 + (1{+}t^{15})s^3.\end{aligned}$$

Which finite reflection groups have series with this property? Is this duality connected to that described by Orlik and Terao [1992, p. 286]?

On $\mathcal{V}$-equivariance and special orbits. Suppose a is fixed by an element $T \in \mathcal{G}$. Since a $\mathcal{G}$-equivariant map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ satisfies

$$f(a) = f(Ta) = Tf(a),$$

T also fixes $f(a)$. Hence, special orbits map to special orbits. For the Valentiner action, the points fixed by an involution are a 45-point and its line while the 3, 4, 5-fold fixed points come in triples. Table 1 summarizes the matter. Thus, under a $\mathcal{V}$-equivariant f, a 45-line $\mathcal{L}_{\overline{abcd}}$ maps either to itself or to its point $p_{\overline{abcd}}$. In the former case, f preserves the pair of 90-points $\{p_{\overline{abcd}_1}, p_{\overline{abcd}_2}\}$. Since $\mathcal{O}_{45}$ cannot map to $\mathcal{O}_{90}$, f must fix the 45-points. (The possibility of f's being projectively undefined at $p_{\overline{abcd}}$ exists; see below for a case study.) Concerning the 36-72 triples the matter stands just as in the case of the 45-90 points so that f either fixes or exchanges the 72-points $p_{\bar{a}b_1}$, $p_{\bar{a}b_2}$ and fixes the 36-points $p_{\bar{a}b}$. What about a triple of 60 points? Symmetry forces f to permute the three points. Since the Valentiner group does not distinguish between 60-points, an equivariant action must fix the orbit pointwise.

Order	Number and Type	Notation
2-fold	1 45-point, 1 45-line	$\{p_{\overline{abcd}}, \mathcal{L}_{\overline{abcd}}\}$
3-fold	3 60-points	$\{p_{a*def}, p_{b*def}, p_{c*def}\}$
3-fold	3 $\overline{60}$-points	$\{p_{\overline{ade*f}}, p_{\overline{b*def}}, p_{\overline{c*def}}\}$
4-fold	1 45-point, 2 90-points	$\{p_{\overline{abcd}}, p_{\overline{abcd}_1}, p_{\overline{abcd}_2}\}$
5-fold	1 36-point, 2 72-points	$\{p_{\bar{a}b,} p_{\bar{a}b_1}, p_{\bar{a}b_2}\}$.

TABLE 1. Fixed points of $\mathcal{V}$

3C. The Degree 16 Map

Returning to the exterior Molien series for $\mathcal{V}_{3\cdot 360}$, the coefficient of $t^m s^2$ gives the dimension of the space of degree m equivariants. The series in t begins

$$t + t^7 + 2t^{13} + t^{16} + \cdots.$$

The first term t is due to the identity map, while t^7 occurs through promotion of the identity to the degree-7 map $F\,\mathrm{id}$. (Note that although the identity *map* is always absolutely equivariant, its 2-form counterpart $x_1\,dx_2 \wedge dx_3 + x_2\,dx_3 \wedge dx_1 + x_3\,dx_1 \wedge dx_2$ will not be absolutely invariant if the group is not in SU_3.) Two dimensions worth of invariants in degree 12 account for the $2t^{13}$ term. The occurrence in degree 16 of the first nontrivial equivariant finds explanation in exterior algebra. Since exterior differentiation and multiplication preserve invariance, the 2-form $dF \wedge d\Phi$ is invariant, and hence corresponds to an equivariant map whose coordinate functions are given by the coefficients of the 2-form basis

$$\{dx_2 \wedge dx_3,\ dx_3 \wedge dx_1,\ dx_1 \wedge dx_2\}.$$

Proposition 3.5. *Up to scalar multiplication, the only $\mathcal{V}_{3\cdot 360}$-invariant 2-form of degree 16 is*

$$dF(x) \wedge d\Phi(x) = (\nabla F(x) \times \nabla \Phi(x)) \cdot dX_2.$$

Consequently, the unique degree 16 $\mathcal{V}$-equivariant is

$$\psi_{16}(x) = \nabla F(x) \times \nabla \Phi(x).$$

Here ∇ is a formal gradient $\nabla F = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3}\right)$ and $\times$ is the cross-product. Geometrically, ψ associates a point $y \in \mathbb{CP}^2$ with the intersection of the pair of lines

$$\psi(y) = \{\nabla F(y) \cdot x = 0\} \cap \{\nabla \Phi(y) \cdot x = 0\}.$$

Dynamics of the 16-map. Given a generic point a that lies on *just one* 45-line $\mathcal{L}_{\overline{abcd}}$ the lines

$$\{\nabla F(a) \cdot x = 0\} \quad \text{and} \quad \{\nabla \Phi(a) \cdot x = 0\}$$

pass through the point $p_{\overline{abcd}}$. Thus, ψ collapses $\mathcal{L}_{\overline{abcd}}$ to its companion point. Taking $p_{\overline{abcd}} = [1, 0, 0]$ and $\mathcal{L}_{\overline{abcd}} = \{x_1 = 0\}$,

$$\psi(a) = [\psi_1(a), 0, 0]$$

where $\psi_1(a)$ has degree 16 in the homogeneous coordinates $a = [0, a_2, a_3]$. The 16 roots of $\psi_1(a)$ correspond to the 16 points where $\mathcal{L}_{\overline{abcd}}$ intersects the 44 remaining 45-lines. These occur at the 36, 45, 60, $\overline{60}$-points of which there are four each on $\mathcal{L}_{\overline{abcd}}$. The blowing-down of the 45-lines forces the blowing-up of their intersections: To which 45-point could the intersection go?

Their collapsing behavior makes ψ critical on the 45-lines. Since the Jacobian determinant $|J_\psi|$ has degree $3\,(16 - 1) = 45$, $\{X = 0\}$ is exactly the critical set. Thus, the 45-lines are superattracting. But, in approaching $\mathcal{L}_{\overline{abcd}}$ a trajectory inevitably gets carried near the point $p_{\overline{abcd}}$ which is blowing up onto $\mathcal{L}_{\overline{abcd}}$; this means that the $\mathbb{CP}^1$ of directions through $p_{\overline{abcd}}$ maps, by the agency of the Jacobian transformation J_ψ, to points on $\mathcal{L}_{\overline{abcd}}$ (see below). Conversely, J_ψ associates a point on $\mathcal{L}_{\overline{abcd}}$ with a direction through $p_{\overline{a}bcd}$. The next two iterations of ψ first return the trajectory to the vicinity of $p_{\overline{a}bcd}$ and then send it back to $\mathcal{L}_{\overline{a}bcd}$ *somewhere.*

Observation of trajectories that start near $p_{\overline{a}bcd}$ or $\mathcal{L}_{\overline{a}bcd}$ reveals a rapid attraction on every other iteration. What about the 3 and 5-fold intersections of 45-lines? Are they attracting? While these points do indeed blow-up, their "image" under ψ is not a curve that blows-down to the point; such is the sole propriety of the 45-points. Hence, ψ draws a generic point near a 60, $\overline{60}$, 36-point into one of three or five point-line cycles.

This "every-other" dynamics poses the following problem: which every-other iterate do we watch? For some initial postions, the even iterates converge to a 45-point while the trajectory spends the odd times around the corresponding 45-line. For others, the process is reversed. Moreover, experiment indicates that the dynamics at the 45-line eventually settles down; trajectories end up at one of the 45-points on the line. Hence, the iteration outputs a pair of 45-points each of which lies on the line of the other. But, there are, for each 45-point, four possible pairs of this sort so that taking every other iterate amounts to a neglect of information.

The dynamics of ψ appears to come down to what takes place on the critical 45-lines. Given a point x on a 45-line $\mathcal{L}_{\overline{a}bcd}$, the derivative J_ψ associates with x a $\mathbb{CP}^1$ through $p_{\overline{a}bcd}$, namely, the image $\mathcal{L}_\psi(x)$ of $J_\psi(x)$. In turn, $J_\psi(p_{\overline{a}bcd})$ sends the line $\mathcal{L}_\psi(x)$ to the point $(J_\psi(p_{\overline{a}bcd}))\mathcal{L}_\psi(x)$ on $\mathcal{L}_{\overline{a}bcd}$. The degree-15 map

$$x \to (J_\psi(p_{\overline{a}bcd}))\mathcal{L}_\psi(x)$$

gives ψ on $\mathcal{L}_{\overline{a}bcd}$.

Figure 9 on page 237 shows a basins-of-attraction plot illustrating the dynamics on a 45-line. The union of the patches in a given color indicates a basin of attraction for one of the four attracting 45-points. The black dots contain points that the 45-points *might* not attract. Does this set have interior or positive measure? (For more details about the figure, see the Appendix.)

On a 45-line the map is not critically finite (see [Fornæss and Sibony 1994, pp. 223ff.] for this concept); in particular, its critical points are not periodic. Indeed, there might be a wandering critical point there, as suggested by Curt McMullen (private communication). Hence, establishing convergence almost everywhere would be difficult should the map even possess this property. Moreover, this behavior hardly reveals the geometric elegance whose prospective discovery motivates the present enterprise.

Unlike the one-dimensional case of the icosahedral group, for which the nontrivial equivariant of lowest degree provides an elegant dynamical system [Doyle and McMullen 1989, pp. 152–153] for the purposes of solving the quintic, the higher-dimensional Valentiner action fails to bear similar fruit. The failure occurs in spite of the 16-map's being obtained by a procedure analogous to that employed by Doyle and McMullen in producing the degree-11 icosahedral map:

$$\underline{\text{1 dimension}}$$

$$F\colon \mathcal{I}\text{-invariant of degree } 12$$

$$f_{11} = \begin{vmatrix} \hat{x} & \hat{y} \\ F_x & F_y \end{vmatrix}$$

$$\underline{\text{2 dimensions}}$$

$$F, \Phi\colon \mathcal{V}\text{-invariants of degrees } 6,\ 12$$

$$\psi_{16} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ F_x & F_y & F_z \\ \Phi_x & \Phi_y & \Phi_z \end{vmatrix}$$

where $\hat{x}, \hat{y}, \hat{z}$ represent unit coordinate vectors. But in one dimension *all* $\mathcal{I}$-symmetric maps arise as combinations of three others each of which are constructed in the manner of f_{11} but with different basic invariants standing in for F. A richer stock of equivariants inhabits the Valentiner waters. In contrast, the next higher degree offers promise as well as a bit of mystery.

3D. A Family of 19-Maps

The Molien series for Valentiner equivariants

$$t + t^7 + 2t^{13} + t^{16} + 3t^{19} \dots$$

specifies three dimensions worth of maps in degree 19 of which two are due to promotion of the identity by degree-18 invariants. Hence, there are, as the exterior Molien quotient (Section 3B) indicates, nontrivial $\mathcal{V}$-symmetric maps in degree 19. How do these arise? Since there is no apparent exterior algebraic means of producing such a map, the more practical matter of computing them takes priority. (Peter Doyle and Anne Shepler appear to have found a "differential" method of generating all invariant 2-forms. The geometric and dynamic consequences remain to be explored.)

19 = 64 – 45. Multiplication of a degree-19 equivariant f by X_{45} elevates f to the 14-dimensional space of 64-maps. There are 14 ways of promoting the maps ψ_{16}, φ_{34}, and f_{40} to degree 64:

(a) 7 dimensions of degree 48 invariants to promote $\psi_{16} = \nabla F \times \nabla \Phi$;
(b) 4 dimensions of degree 30 invariants to promote $\varphi_{34} = \nabla F \times \nabla \Psi$;
(c) 3 dimensions of degree 24 invariants to promote $f_{40} = \nabla \Phi \times \nabla \Psi$.

Proposition 3.6. *These* 14 *maps span the space of degree* 64 *equivariants.*

Proof. If not, then for some α_k not all 0,

$$\begin{aligned}(\alpha_1 F^8 + \cdots + \alpha_7 F\,\Phi\,\Psi)\, dF \wedge d\Phi \\ + (\alpha_8 F^5 + \cdots + \alpha_{11}\,\Psi)\, dF \wedge d\Psi \\ + (\alpha_{12} F^4 + \alpha_{13} F^2\,\Phi + \alpha_{14}\,\Phi^2)\, d\Phi \wedge d\Psi = 0.\end{aligned}$$

Since $dF \wedge d\Phi \wedge d\Psi = \beta\, X\,\mathrm{vol}$ for some constant $\beta \neq 0$, at least one of the following equalities hold:

$$\begin{aligned}(\alpha_1 F^8 + \cdots + \alpha_7 F\,\Phi\,\Psi)\beta X = 0,\\ (\alpha_8 F^5 + \cdots + \alpha_{11}\,\Psi)\beta X = 0,\\ (\alpha_{12} F^4 + \alpha_{13} F^2\,\Phi + \alpha_{14}\,\Phi^2)\beta X = 0.\end{aligned}$$

In at least one case, there is a nonzero α_k. Then the invariants F, Φ, Ψ are not algebraically independent, contrary to the theory of complex reflection groups [Shephard and Todd 1954, p. 282]. □

In this event, Xf is a combination of maps whose computation is straightforward.

Reasoning in the other direction, a 64-map

$$f_{64} = F_{48}\psi_{16} + F_{30}\varphi_{34} + F_{24} f_{40}$$

that "vanishes" on the 45-lines — i.e., each coordinate function of f_{64} vanishes — must have a factor of X. The quotient is a degree 19 equivariant

$$f_{19} = \frac{f_{64}}{X_{45}}.$$

Arranging for the vanishing of f_{64} on the 45-lines requires consideration of only one line; symmetry tends to the remaining 44. Forcing f_{64} to vanish at 12 independent points on a 45-line — independent in the sense that the 12 resulting linear conditions in the 14 undetermined coefficients of f_{64} are independent — we obtain a two-parameter family of 64-maps each member of which vanishes on $\{X = 0\}$. The two *inhomogeneous* parameters reflect the three dimensions (i.e., homogeneous parameters) of degree-19 $\mathcal{V}$-equivariants. In $\mathrm{bub}_{\bar{2}2}$ coordinates, setting these two parameters equal to 0 and normalizing the coefficients yields

$$\begin{aligned} f_{64}(y) = \big(10\,F(y)^6\,\Phi(y) + 100\,F(y)^4\,\Phi(y)^2 \\ + 45\,F(y)^2\,\Phi(y)^3 + 156\,\Phi(y)^4 \\ + 39\,F(y)^3\,\Psi(y) + 51\,F(y)\Phi(y)\,\Psi(y)\big)\,\psi(y) \\ - 27\,\Psi(y)\,\varphi(y) + 54\,\Phi(y)^2\,f(y). \end{aligned} \tag{3–1}$$

The two-parameter family of nontrivial 19-maps is then

$$g_{19}(y; a, b) = f_{19}(y) + \big(a\,F(y)^3 + b\,F(y)\,\Phi(y)\big)\,y. \tag{3–2}$$

Are any of these maps dynamically "special"? Indeed, what might it mean to be special in this sense?

Extended symmetry in degree 19. Since f_{19} is a nontrivial $\overline{\mathcal{V}}_{2\cdot 360}$-equivariant — note the integer coefficients in (3–1), each member of the two *real* parameter family

$$f_{19} + F\,(a\,B_{12} + \bar{a}\,U_{12})\,\mathrm{id}$$

is impartial towards the two systems of conics and so, enjoys the additional symmetry. Here $B_{12} = \prod C_{\bar{k}}$ and $U_{12} = \prod C_k$ are the degree-12 invariants given by the product of the respective six conic forms. To honor the doubled symmetry a member of this family must preserve each $\mathcal{R}_{\bar{a}b}$ that $\mathrm{bub}_{\bar{a}b}$ fixes point-wise.

3E. *The* 19-Map

The icosahedron again. An intriguing aspect of the 19-maps is the degree itself. Since 19 is one of the special equivariant numbers [Doyle and McMullen 1989, p. 166] for the binary icosahedral group, there arises the prospect of finding a $\mathcal{V}$-equivariant that restricts to self-mappings of the conics. By symmetry, an equivariant that fixes (setwise) a conic of a given system also fixes the other five. Might there be a map that preserves each of the 12 conics?

From [Doyle and McMullen 1989, p. 163] comes a geometric description of the canonical degree-19 icosahedral mapping of the round Riemann sphere:

stretch each face F over the 19 faces in the complement of the face antipodal to F while making a half-turn in order to place the three vertices and edges of F on the three antipodal vertices and edges. By symmetry, the 20 face-centers are fixed and repelling. Since the resulting map is critical only at the 12 period-2 vertices, it has reliable dynamics [Doyle and McMullen 1989, p. 156]. A conic-fixing $\mathcal{V}$-equivariant of degree 19 would, when restricted to a conic $\mathcal{C}_{\bar{a}}$ or $\mathcal{C}_b$, give the unique map in degree 19 with binary icosahedral symmetry. This would determine its effect on the special icosahedral orbits:

(1) fix the face-centers: $\overline{60}$, 60-points of the appropriate system;
(2) exchange antipodal vertices: pairs of 72-points;
(3) exchange antipodal midpoints of edges: intersections of conics $\mathcal{C}_{\bar{a}}$, $\mathcal{C}_b$ with a 45-line indexed by $\bar{a}$ and b.

General $\mathcal{V}$-equivariants satisfy condition (1). (We say "general" because, while some maps can be set to blow-up at the 60-points, such a circumstance is rare.) Since

$$\mathcal{O}_{72} = \{F = 0\} \cap \{\Phi = 0\},$$

the image of a 72-point under each g_{19} in (3–2) is the same as that under f_{19}. Propitiously, f_{19} exchanges pairs of 72-points. Finally, a 19-map cannot blow-down a 45-line; this would make it critical there — a condition precluded by the critical set's being an invariant of degree $54 = 3\,(19 - 1)$. Consequently, each 45-line must map to itself so that arranging for 3) costs one parameter for each system of conics. Fortunately, there are two parameters to spend and their expenditure purchases a canonical $\mathcal{V}$-equivariant h_{19} that maps each of the 12 conics onto itself. See below for an explicit expression.

Fact 3.7. *There is a unique degree-*19 *$\mathcal{V}$-equivariant h_{19} that preserves each of the* 12 *icosahedral conics.*

By favoring neither system of conics, h_{19} possesses bub-symmetry and so, self-maps each of the $\mathcal{R}_{\bar{a}b}$. Expressing the family of 19-maps by

$$g_{19} = h_{19} + F\,(a\,B_{12} + bU_{12})\,\mathrm{id}$$

makes evident the one-parameter collections that fix the barred $(b = 0)$ and the unbarred $(a = 0)$ conics.

Unlike the 16-map ψ_{16}, the 19-map h_{19} does not blow up somewhere.

Proposition 3.8. *The conic-preserving map h_{19} is holomorphic on $\mathbb{CP}^2$.*

Proof. By equivariance, the set of points on which h_{19} blows up is empty or a union of $\mathcal{V}$-orbits. Direct calculation shows that

$$h_{19}(p) \neq 0$$

for

$$p \in \mathcal{O}_{36} \cup \mathcal{O}_{45} \cup \mathcal{O}_{60} \cup \mathcal{O}_{\overline{60}} \cup \mathcal{O}_{90}.$$

The remaining possibilities are that $h_{19} = 0$ on a 180 or 360 point orbit.

First, take the case of a 180 point orbit and recall that each such point belongs to one 45-line. Also, let $h_{19} = [h_1, h_2, h_3]$. Since h_{19} preserves each 45-line $\mathcal{L}$, the only way that $h_1 = h_2 = h_3 = 0$ is for the coordinates of the restriction $h_{19}|_{\mathcal{L}}$ to have a common factor. In bub$_{\bar{2}2}$ coordinates we have $\mathcal{L}_{\bar{1}\bar{2}12} = \{y_1 - y_2 = 0\}$, so that

$$h_{19}|_{\mathcal{L}_{\bar{1}\bar{2}12}} = [f, f, g].$$

But the resultant of f and g does not vanish. Hence, f and g do not have a common factor.

Finally, suppose that $h_{19} = 0$ at a 360 point orbit and that [0,0,1] is a 36-point p_{36}. Since

$$\bigl|\{h_1 = 0\} \cap \{h_2 = 0\} \cap \{h_3 = 0\}\bigr| \leq 19 \cdot 19 = 361,$$

there is only one member of $h_{19}^{-1}(p_{36})$ in $\mathbb{CP}^2$. Of course, this holds for every 36-point of which there are four on a 45-line $\mathcal{L}$. Moreover, $h_{19}(p_{36}) = p_{36}$. Thus, the one-dimensional rational map $h_{19}|_{\mathcal{L}}$ has four exceptional points — a state of affairs that requires the map to be of degree one [Beardon 1991, p. 52]. Since the restricted map is not of degree one, h_{19} does not vanish at a 360 point orbit. (Indeed, no degree 19 equivariant can blow up a 360 point orbit.) □

Fact 3.9. *The conic-fixing equivariant has the* bub$_{\bar{2}2}$ *expression given at the top of the next page.*

Dynamical behavior. The discovery of h_{19} supplies the unique degree-19 $\mathcal{V}$-equivariant that self-maps, in addition to the 45-lines and the 36 bub-$\mathbb{RP}^2$s, the 12 conics. The dynamics *on* each conic is well-understood. A plot showing the basins of attraction appears in Figure 10 on page 237.

$$
\begin{aligned}
&h_{19}(y) = 1620\, F(y)^3 \cdot [y_1, y_2, y_3] + f_{19}(y) = \\
&\big[-3591 y_1^{15} y_2^4 - 5263 y_1^{10} y_2^9 + 9747 y_1^5 y_2^{14} - 81 y_2^{19} + 17955 y_1^{12} y_2^6 y_3 + 10260 y_1^7 y_2^{11} y_3 - 7695 y_1^2 y_2^{16} y_3 - 107730 y_1^{14} y_2^3 y_3^2 \\
&\quad - 74385 y_1^9 y_2^8 y_3^2 + 161595 y_1^4 y_2^{13} y_3^2 - 969570 y_1^{11} y_2^5 y_3^3 + 1292760 y_1^6 y_2^{10} y_3^3 - 46170 y_1 y_2^{15} y_3^3 - 2346975 y_1^8 y_2^7 y_3^4 \\
&\quad - 807975 y_1^3 y_2^{12} y_3^4 - 3587409 y_1^{10} y_2^4 y_3^5 + 10277442 y_1^5 y_2^9 y_3^5 + 13851 y_2^{14} y_3^5 - 969570 y_1^{12} y_2 y_3^6 - 3986010 y_1^7 y_2^6 y_3^6 \\
&\quad - 1939140 y_1^2 y_2^{11} y_3^6 - 5263380 y_1^9 y_2^3 y_3^7 - 28117530 y_1^4 y_2^8 y_3^7 + 831060 y_1^{11} y_3^8 + 2423925 y_1^6 y_2^5 y_3^8 + 4363065 y_1 y_2^{10} y_3^8 \\
&\quad - 24931800 y_1^8 y_2^2 y_3^9 + 43630650 y_1^3 y_2^7 y_3^9 - 31123197 y_1^5 y_2^4 y_3^{10} + 9598743 y_2^9 y_3^{10} + 14959080 y_1^7 y_2 y_3^{11} + 23269680 y_1^4 y_2^3 y_3^{12} \\
&\quad - 26178390 y_1^6 y_3^{13} + 52356780 y_1 y_2^5 y_3^{13} + 18698850 y_1^3 y_2^2 y_3^{14} + 20194758 y_2^4 y_3^{15} + 22438620 y_1^2 y_2 y_3^{16} + 7479540 y_1 y_3^{18}, \\[1ex]
&-81 y_1^{19} + 9747 y_1^{14} y_2^5 - 5263 y_1^9 y_2^{10} - 3591 y_1^4 y_2^{15} - 7695 y_1^{16} y_2^2 y_3 + 10260 y_1^{11} y_2^7 y_3 + 17955 y_1^6 y_2^{12} y_3 + 161595 y_1^{13} y_2^4 y_3^2 \\
&\quad - 74385 y_1^8 y_2^9 y_3^2 - 107730 y_1^3 y_2^{14} y_3^2 - 46170 y_1^{15} y_2 y_3^3 + 1292760 y_1^{10} y_2^6 y_3^3 - 969570 y_1^5 y_2^{11} y_3^3 - 807975 y_1^{12} y_2^3 y_3^4 \\
&\quad - 2346975 y_1^7 y_2^8 y_3^4 + 13851 y_1^{14} y_3^5 + 10277442 y_1^9 y_2^5 y_3^5 - 3587409 y_1^4 y_2^{10} y_3^5 - 1939140 y_1^{11} y_2^2 y_3^6 - 3986010 y_1^6 y_2^7 y_3^6 \\
&\quad - 969570 y_1 y_2^{12} y_3^6 - 28117530 y_1^8 y_2^4 y_3^7 - 5263380 y_1^3 y_2^9 y_3^7 + 4363065 y_1^{10} y_2 y_3^8 + 2423925 y_1^5 y_2^6 y_3^8 + 831060 y_2^{11} y_3^8 \\
&\quad + 43630650 y_1^7 y_2^3 y_3^9 - 24931800 y_1^2 y_2^8 y_3^9 + 9598743 y_1^9 y_3^{10} - 31123197 y_1^4 y_2^5 y_3^{10} + 14959080 y_1 y_2^7 y_3^{11} + 23269680 y_1^3 y_2^4 y_3^{12} \\
&\quad + 52356780 y_1^5 y_2 y_3^{13} - 26178390 y_2^6 y_3^{13} + 18698850 y_1^2 y_2^3 y_3^{14} + 20194758 y_1^4 y_3^{15} + 22438620 y_1 y_2^2 y_3^{16} + 7479540 y_2 y_3^{18}, \\[1ex]
&-1026 y_1^{17} y_2^2 - 3078 y_1^{12} y_2^7 - 3078 y_1^7 y_2^{12} - 1026 y_1^2 y_2^{17} - 5130 y_1^{14} y_2^4 y_3 + 113240 y_1^9 y_2^9 y_3 - 5130 y_1^4 y_2^{14} y_3 + 3078 y_1^{16} y_2 y_3^2 \\
&\quad - 272916 y_1^{11} y_2^6 y_3^2 - 272916 y_1^6 y_2^{11} y_3^2 + 3078 y_1 y_2^{16} y_3^2 + 215460 y_1^{13} y_2^3 y_3^3 + 687420 y_1^8 y_2^8 y_3^3 + 215460 y_1^3 y_2^{13} y_3^3 \\
&\quad - 4617 y_1^{15} y_3^4 + 937251 y_1^{10} y_2^5 y_3^4 + 937251 y_1^5 y_2^{10} y_3^4 + 4617 y_2^{15} y_3^4 + 290871 y_1^{12} y_2^2 y_3^5 + 4813992 y_1^7 y_2^7 y_3^5 + 290871 y_1^2 y_2^{12} y_3^5 \\
&\quad - 1454355 y_1^9 y_2^4 y_3^6 - 1454355 y_1^4 y_2^9 y_3^6 + 2520882 y_1^{11} y_2 y_3^7 + 8812314 y_1^6 y_2^6 y_3^7 + 2520882 y_1 y_2^{11} y_3^7 + 19876185 y_1^8 y_2^3 y_3^8 \\
&\quad - 19876185 y_1^3 y_2^8 y_3^8 - 2036097 y_1^{10} y_3^9 + 5623506 y_1^5 y_2^5 y_3^9 - 2036097 y_2^{10} y_3^9 + 5235678 y_1^7 y_2^2 y_3^{10} + 5235678 y_1^2 y_2^7 y_3^{10} \\
&\quad - 37813230 y_1^4 y_2^4 y_3^{11} - 2617839 y_1^6 y_2 y_3^{12} - 2617839 y_1 y_2^6 y_3^{12} - 2908710 y_1^3 y_2^3 y_3^{13} + 6357609 y_1^5 y_3^{14} + 6357609 y_2^5 y_3^{14} \\
&\quad - 5983632 y_1^2 y_2^2 y_3^{15} - 4487724 y_1 y_2 y_3^{17} - 1023516 y_3^{19}\big]
\end{aligned}
$$

The conic-fixing equivariant h_{19}.

Proposition 3.10. *Under h_{19}, the trajectory of almost any point on an icosahedral conic tends to an antipodal pair of the superattracting vertices.*

Moreover, the conics themselves are attracting.

Proposition 3.11. *The Jacobian $J_{h_{19}}$ has rank one at the superattracting 72-points. Thus, h_{19} attracts on a full $\mathbb{CP}^2$ neighborhood of such a point. Furthermore, the Fatou components of the restricted map $h_{19}|_{\mathcal{C}_{\bar{a}}}$ are the intersections with $\mathcal{C}_{\bar{a}}$ of Fatou components of the map on $\mathbb{CP}^2$.*

Is this attracting behavior of the conics pervasive in the measure-theoretic sense? What about the restricted dynamics on the 45-lines and $\mathbb{RP}^2$s?

Perhaps the place to begin is at a 72-point, say $p_{\bar{1}1_1}$, which lies at the hub of much Valentiner activity. Passing through $p_{\bar{1}1_1}$ are many special objects:

(1) the pair $\{\mathcal{C}_1, \mathcal{C}_{\bar{1}}\}$ of conics, which meet tangentially;

(2) the 36-line $\mathcal{L}_{\bar{1}1}$, which gives a 10-fold $\mathcal{D}_5$ axis about which $\mathcal{C}_1 \cup \mathcal{C}_{\bar{1}}$ turns;

(3) the 72-line $\mathcal{L}_{\bar{1}1_2}$, which is stable under the cyclic half of the $\mathcal{D}_5$ stabilizer of $\mathcal{L}_{\bar{1}1}$ and thereby tangent to $\mathcal{C}_{\bar{1}}$ and $\mathcal{C}_1$;

(4) the sixth-degree curve $\{F = 0\}$, which is tangent to $\mathcal{L}_{\bar{1}1}$;

(5) the twelfth-degree curve $\{\Phi = 0\}$, which is tangent to $\mathcal{L}_{\bar{1}1_2}$;

(6) the five $\mathbb{RP}^2$s $\{\mathcal{R}_{\bar{2}2}, \mathcal{R}_{\bar{3}4}, \mathcal{R}_{\bar{4}3}, \mathcal{R}_{\bar{5}6}, \mathcal{R}_{\bar{6}5}\}$, each of which intersect $\mathcal{C}_1$ and $\mathcal{C}_{\bar{1}}$ only at $p_{\bar{1}1_1}$ and $p_{\bar{1}1_2}$.

In addition, a 72-point situates itself at the intersection of two components $\{F_6 = 0\}$ and $\{G_{48} = 0\}$ of h_{19}'s critical set.

Fact 3.12. *The Jacobian determinant $|J_{h_{19}}| = F_6\, G_{48}$ where the invariant G_{48} distinguishes itself by lacking a Ψ term when decomposed into an expression in the basic invariants F, Φ, Ψ:*

$$G_{48}(y) = -13718\big(14\,F(y)^8 + 180\,F(y)^6\,\Phi(y) + 1701\,F(y)^4\,\Phi(y)^2 + 3402\,F(y)^2\,\Phi(y)^3 + 5103\,\Phi(y)^4\big).$$

Since G_{48} is a polynomial in F and Φ alone, its curve satisfies

$$\{F = 0\} \cap \{G_{48} = 0\} = \{F = 0\} \cap \{\Phi = 0\} \;=\; \mathcal{O}_{72}.$$

Furthermore, the special invariant structure of G_{48} has an alternative expression in terms of B_{12} and U_{12} alone. From the identities

$$B_{12}(y) = 2(5-\sqrt{15}i)\rho F(y)^2 - 2\sqrt{15}(1+\sqrt{15}i)\rho^2\Phi(y),$$
$$U_{12}(y) = 2(5+\sqrt{15}i)\rho^2 F(y)^2 - 2\sqrt{15}(1-\sqrt{15}i)\rho\Phi(y),$$

it follows that

$$\begin{aligned} G_{48}(y) = 2^3 3^{24} 5^8 19^3\big(&-6\,(3-\sqrt{15}i)\,\rho\,B_{12}(y)^4 \\ &+ 4\,(32+3\sqrt{15}i)\,\rho^2\,B_{12}(y)^3\,U_{12}(y) \\ &- 333\,B_{12}(y)^2\,U_{12}(y)^2 \\ &+ 4\,(32-3\sqrt{15}i)\,\rho\,B_{12}(y)\,U_{12}(y)^3 \\ &- 6\,(3+\sqrt{15}i)\,\rho^2\,U_{12}(y)^4\big). \end{aligned}$$

Consequently, the degree-48 component of the critical set meets the conics *only* at the 72-points:

$$\begin{aligned} \{G_{48} = 0\} \cap \{B_{12} = 0\} &= \{G_{48} = 0\} \cap \{U_{12} = 0\} \\ &= \{B_{12} = 0\} \cap \{U_{12} = 0\} \\ &= \mathcal{O}_{72}. \end{aligned}$$

Accounting for the multiplicity at these eighth-order intersections is the singularity of $\{G_{48} = 0\}$ at $\mathcal{O}_{72}$, a result that follows directly from the invariant decomposition.

$\mathbb{RP}^2$ dynamics. On each of the five bub-$\mathbb{RP}^2$s that are mutually tangent at $p_{\bar{1}1_1}$ and $p_{\bar{1}1_2}$, these 72-points are superattracting for the restricted maps

$$h_{19}|_{\mathcal{R}_{\bar{a}b}}, \quad \bar{a}b = \bar{2}2,\, \bar{3}4,\, \bar{4}3,\, \bar{5}6,\, \bar{6}5.$$

Are there attracting sites on $\mathcal{R}_{\bar{a}b}$ other than the five pairs of 72-points? Are there sets of positive measure or open sets on which $h_{19}|_{\mathcal{R}_{\bar{a}b}}$ fails to converge to a pair of 72-points? The experimental evidence strongly suggests that

(1) the 72-points are the only attractors;
(2) there is no region with thickness or positive measure that remains outside of their influence;
(3) the set of 45-lines $\{X = 0\}$ is repelling.

The Appendix exhibits basin plots of $h_{19}|_{\mathcal{R}_{\bar{2}2}}$ which, of course, is dynamically equivalent to each $h_{19}|_{\mathcal{R}_{\bar{a}b}}$. What significance does the $\mathbb{RP}^2$-dynamics hold for that on $\mathbb{CP}^2$? Extensive trials [Crass 1997a] in $\mathbb{CP}^2$ have not revealed behavior contrary to that observed on the $\mathbb{RP}^2$s.

Conjecture 3.13. *The only attracting periodic points for h_{19} are the elements of $\mathcal{O}_{72}$. Moreover, the union of the basins of attraction for $\mathcal{O}_{72}$ has full measure in $\mathbb{CP}^2$.*

The 45-lines present a problem in that they map to themselves but do not contain the 72-points. (Again, this is a feature peculiar to the 72-points. They form the only special $\mathcal{V}$-orbit that does not lie on the 45-lines.) The basin plot in Figure 11 (page 238) reveals repelling behavior along the $\mathbb{RP}^1$ where the $\mathbb{RP}^2$ meets one of the 45-lines left invariant by $\mathrm{bub}_{\bar{2}2}$.

Conjecture 3.14. *On the 45-lines h_{19} is repelling, and hence $\{X = 0\}$ resides in the Julia set $J_{h_{19}}$.*

Is $J_{h_{19}}$ the closure of the backward orbit of the 45-lines?

4. SOLVING THE SEXTIC

By means of various algebraic manipulations, a general sixth-degree polynomial reduces to a member of a two-parameter family of Valentiner resolvents. Such a reduction requires the extraction of square and cube roots. Furthermore, a certain set of sextics transforms into a special one-parameter collection of resolvents. These resolvents are especially suited for solution by an iterative algorithm that exploits Valentiner symmetry and symmetry-breaking.

4A. General Sixth-Degree Valentiner Resolvents

At the core of Klein's program for equation-solving is the "form problem" relative to a particular action of a given equation's symmetry group: for prescribed values $a_1, \dots, a_n$ of the generating invariants $F_1, \dots, F_n$ find a point p common to the inverse images $F_1^{-1}(a_1), \dots, F_n^{-1}(a_n)$. As with the quintic, solving the general sextic is tantamount to solving the corresponding form-problem; see [Fricke 1926, pp. 308–310; Coble 1911]. This circumstance has a projectively equivalent formulation in terms of rational functions in the basic invariants. In the Valentiner setting this concerns

$$Y_1 = \alpha \frac{\Phi}{F^2} \quad \text{and} \quad Y_2 = \beta \frac{\Psi}{F^5},$$

where α and β are chosen so that $Y_1 = 1$ and $Y_2 = 1$ at a 36-point. (In $\text{bub}_{\bar{2}2}$ coordinates $\alpha = 1$ and $\beta = \frac{1}{4}$.) Given values a_1 and a_2 of Y_1 and Y_2, the task is to find a point z in $\mathbb{CP}^2$ that belongs to the $\mathcal{V}$-orbit

$$Y_1^{-1}(a_1) \cap Y_2^{-1}(a_2).$$

Accordingly, the general 6-parameter sextic $p(x)$ reduces to a resolvent that depends on the two parameters Y_1 and Y_2. Such a reduction requires the extraction of a cube root [Fricke 1926, p. 285] in addition to the square root of p's discriminant. This cube root is a so-called "accessory irrationality" — its adjunction to the coefficient field does not reduce the galois group. The 1-to-3 correspondence between the projective and linear Valentiner groups $\mathcal{V}$ and $\mathcal{V}_{3\cdot 360}$ accounts for its appearance. In the one-dimensional icosahedral case, the projective group lifts 1-to-2 to a linear group, thereby producing the need for an accessory square root; see [Klein 1913, pp. 172–173].

As for the *derivation* of a two-parameter resolvent, the map

$$Y : \mathbb{CP}^2 - \{F(z) = 0\} \to (\mathbb{CP}^2 - \{F(z) = 0\})/\mathcal{V}$$

given by

$$Y(z) = \left[F(z)^3\,\Phi(z), \Psi(z), F(z)^5\right] = \left[Y_1(z), Y_2(z), 1\right]$$

provides the $\mathcal{V}$-quotient of $\mathbb{CP}^2 - \{F(z) = 0\}$ in that the fibers are $\mathcal{V}$-orbits. The exceptional status of the sixth-degree curve is due to its being the fiber above the single point $[0, 1, 0]$. Furthermore, under the icosahedral function

$$U_{\bar{1}}(z) = \frac{C_{\bar{1}}(z)^3}{F(z)},$$

a fiber $Y^{-1}[a_1, a_2, 1]$ maps to six points

$$\left\{U_{\bar{n}}(z) = C_{\bar{n}}(z)^3/F(z) : n = 1, \ldots, 6\right\},$$

where $z \in Y^{-1}[a_1, a_2, 1]$. The $U_{\bar{n}}(z)$ are the roots of the sixth-degree polynomial

$$R_z(u) = \prod_{n=1}^{6} (u - U_{\bar{n}}(z)).$$

As z varies in $\mathbb{CP}^2 - \{F(z) = 0\}$, $R_z(u)$ yields a family of sextic resolvents. Since $\mathcal{V}_{3\cdot 360}$ permutes the $C_{\bar{n}}(z)^3$ simply — no multiplicative character appears, R_z is $\mathcal{V}$-invariant in z, and hence so is each u-coefficient. Expressing the coefficients in terms of the basic invariants $F(z)$, $\Phi(z)$, $\Psi(z)$ and then converting to Y_1 and Y_2 yields, in $\text{bub}_{\bar{2}2}$ coordinates, the resolvents $R_Y(u) = R_{(Y_1,Y_2)}(u)$. With $\omega = \sqrt{15}\,i$, their expression is

$$\begin{aligned} R_Y(u) \\ &= u^6 + \tfrac{-5+\omega}{90}\, u^5 + \tfrac{11(1-\omega)-3(3+\omega)Y_1}{2^2 3^5 5^2}\, u^4 \\ &\quad + \tfrac{(100+57\omega)+9(30+\omega)Y_1}{3^9 5^4}\, u^3 \\ &\quad + \tfrac{-(152+17\omega)+18(-21+4\omega)Y_1+27(-4+\omega)Y_1^2}{2^2 3^{11} 5^5}\, u^2 \\ &\quad + \tfrac{(425+103\omega)+6(75+193\omega)Y_1+27(-25+33\omega)Y_1^2-7776\omega Y_2}{2^3 3^{14} 5^8}\, u \\ &\quad + \tfrac{-(5+3\omega)+9(15-7\omega)Y_1+81(25-\omega)Y_1^2+81(45+11\omega)Y_1^3}{2^4 3^{18} 5^8}. \end{aligned}$$

This makes explicit the fact that the solution of $R_Y(u)$ follows from inversion of Y.

For the unbarred functions

$$U_n(z) = \frac{C_n(z)^3}{F(z)},$$

one obtains the associated resolvents S_Y from R_Y by complex conjugation of the u-coefficients:

$$S_Y(u) = \overline{R_Y(\bar{u})}.$$

4B. Special Sixth-Degree Resolvents

For the resolvents R_Y the parameter space is an affine plane $[Y_1, Y_2, 1]$ that lifts to $\mathbb{CP}^2 - \{F(z) = 0\}$. There is a complementary set of resolvents parametrized by a $\mathbb{CP}^1$ that lifts to $\{F(z) = 0\}$.

The sixth-degree $\mathcal{V}$-invariant curve $\{F(z) = 0\}$ is a genus 10 surface that contains three special Valentiner orbits:

$$\begin{aligned} \mathcal{O}_{72} &= \{F(z) = 0\} \cap \{\Phi(z) = 0\}, \\ \mathcal{O}_{90} &= \{F(z) = 0\} \cap \{\Psi(z) = 0\}, \\ \mathcal{O}_{180} &= \{F(z) = 0\} \cap \{X(z) = 0\} - \mathcal{O}_{90}. \end{aligned}$$

Set

$$V(z) = \alpha \frac{\Phi(z)^5}{\Psi(z)^2},$$

constant α is chosen so that $V = 1$ at a 180-point. (In $\text{bub}_{\bar{2}2}$ coordinates, $\alpha = \frac{3}{8}$.) On $\{F(z) = 0\}$ this rational map gives the 2-4-5 quotient of $\{F(z)=0\}$ under $\mathcal{V}$:

$$V:\{F(z)=0\}\to\{F(z)=0\}/\mathcal{V}.$$

Furthermore, the icosahedral function

$$S_{\bar{1}}(z)=\frac{\Phi(z)^2}{\Psi(z)}C_{\bar{1}}(z)^3$$

divides $\{F(z)=0\}$ by the icosahedral subgroup $\mathcal{I}_{\bar{1}}$:

$$S_{\bar{1}}:\{F(z)=0\}\to\{F(z)=0\}/\mathcal{I}_{\bar{1}}.$$

A value $V_0\neq 0,1,\infty$ of V has an inverse image on $\{F(z)=0\}$ that consists of a $\mathcal{V}$-orbit of size 360 while the image of $V^{-1}(V_0)\cap\{F(z)=0\}$ under $S_{\bar{1}}$ is the set of six points

$$\big\{S_{\bar{n}}(z)=\Phi(z)^2C_{\bar{n}}(z)^3/\Psi(z):n=1,\dots,6\big\},$$

where $z\in V^{-1}(V_0)\cap\{F(z)=0\}$. The $S_{\bar{n}}(z)$ supply roots of a $\mathcal{V}$-parametrized family of sextic resolvents

$$T_z(s)=\prod_{n=1}^{6}(s-S_{\bar{n}}(z)).$$

As above, T_z is $\mathcal{V}$-invariant in z, and hence so is each s-coefficient. Expressing the coefficients in terms of the basic invariants $F(z)$, $\Phi(z)$, and $\Psi(z)$, restricting to $\{F(z)=0\}$, and converting to V gives the one-parameter resolvents

$$T_V(s)=s^6-\frac{-3+\sqrt{15}i}{2^5 3^3 5^2}V\,s^4-\frac{4+\sqrt{15}i}{2^8 3^6 5^5}V^2s^2 +\frac{\sqrt{15}i}{2^6 3^7 5^8}V^2s+\frac{45-11\sqrt{15}i}{2^{13}3^{11}5^8}V^3.$$

Again, with the unbarred functions

$$S_n(z)=\frac{\Phi(z)^2}{\Psi(z)}C_n(z)^3,$$

conjugation of the coefficients of T_V yields the unbarred resolvents.

4C. Parametrized Families of Valentiner Groups

The algorithms that solve given resolvents R_Y or T_V employ an iteration of a dynamical system $h_Y(w)$ or $h_V(w)$ that belongs to a family of maps parametrized by $Y=(Y_1,Y_2)$ or V, and each member of which is conjugate to $h_{19}(y)$. The first task is to parametrize by Y and V families of Valentiner groups. Each such group supports a conic-fixing 19-map the computation of which follows that of its conjugate, $h_{19}(y)$.

Invariant building-blocks. Success in finding a 19-map for arbitrary Y or V (with the exception of the singular values $Y(\{X(z)=0\})$ of Y and the values 0, 1, ∞ of V) requires provision only of basic invariant forms

$$F_Y,\Phi_Y,\Psi_Y,X_Y\quad\text{or}\quad F_V,\Phi_V,\Psi_V,X_V,$$

parametrized by Y and V. In turn, the latter three of each type depend on the single forms F_Y and F_V.

Much of this development amounts to keeping track of coordinates. The Valentiner actions $\mathcal{V}_z$ and $\mathcal{V}_y$ on the respective planes $\mathbb{CP}^2_z$ and $\mathbb{CP}^2_y$ are the same — the parameter z merely replaces y. Think of these as a parameter and reference space respectively.

To obtain a parametrized sixth-degree form in the general case, we proceed as follows:

(1) Compose $F(z)$ with a certain family of maps $\tau_z(w)$ each of which is $\mathcal{V}_z$-equivariant and linear in w. (The w-space $\mathbb{CP}^2_w$ is the iteration space.)
(2) Express the coefficients of the w monomials in terms of $F(z)$, $\Phi(z)$, and $\Psi(z)$.
(3) Convert these coefficients to expressions in Y_1 and Y_2 — as in the derivation of R_Y.

The special case requires more care.

(1) Compose $F(z)$ with a select family of maps $\sigma_z(w)$ each of which is $\mathcal{V}_z$-equivariant and linear in w.
(2) Restrict to $\{F(z)=0\}$ and express the coefficients of the w monomials in terms of $\Phi(z)$, $\Psi(z)$. (The choice of $\sigma_z(w)$ becomes significant at this stage; see below.)
(3) Divide through by any overall factors in Φ and Ψ to obtain a polynomial whose degree in z is a multiple of 60 — the degree of V — and then express the result in terms of V.

Sixth-degree forms in the two-parameter case. Consider the family of maps

$$\begin{aligned}y&=\tau_z(w)\\&=\big(F(z)^4z\big)w_1+\big(F(z)h_{19}(z)\big)w_2+k_{25}(z)w_3,\end{aligned}$$

which are of degree 25 in z and projective transformations in w. Here $k_{25}(z)$ is the equivariant whose expression in "Hermitian coordinates" is

$$k_{25}(z)=\overline{\nabla F(\overline{\nabla F(z)})}.$$

(We use the term "Hermitian" because each T in $\mathcal{V}_{3\cdot 360}$ satisfies $T\overline{T}^T = I$.)

With τ_z, one constructs a z-parametrized family of Valentiner groups $\mathcal{V}_w = \tau_z^{-1} V_y \tau_z$ each member of which acts on $\mathbb{CP}^2_w$. By construction, this family possesses an equivariance property: for $T \in \mathcal{V}_z, \mathcal{V}_y$,

$$\tau_{Tz}(w) = T\tau_z(w).$$

Hence,

$$F(\tau_{Tz}(w)) = F(\tau_z(w))$$

so that the w-coefficients of $F(\tau_z(w))$ are $\mathcal{V}_z$-invariant and thereby expressible in terms of the basic forms $F(z)$, $\Phi(z)$, $\Psi(z)$, and $X(z)$. However, since the degree *in* z of each w-coefficent is $6 \cdot 25 = 150$, an odd power of $X(z)$ cannot appear in the decomposition of these coefficients into polynomials in the basic invariants. Being $\mathcal{V}_{3\cdot 360}$-invariant, X^2 decomposes into a polynomial in F, Φ, and Ψ; see equation (2–1) on page 222. Thus, each coefficient is a combination of the forms of degrees 6, 12, and 30. After division by an appropriate power of $F(z)$ as well as a simplifying numerical factor α, the result is expressible in terms of Y_1 and Y_2:

$$F_Y(w) = \frac{F(\tau_z(w))}{\alpha F(z)^{25}}. \tag{4–1}$$

The coefficients of the Y monomials in a w-coefficient are solutions to a system of linear equations. The five-page explicit expression for this fundamental form can be found in [Crass 1997a; 1997b, pp. 64–68].

An important matter concerns the degeneration of $\tau_z(w)$ where the determinant $|\tau_z|$ vanishes. Taking z^T, $h_{19}(z)^T$, and $k_{25}(z)^T$ to be column vectors, we can write

$$\begin{aligned} |\tau_z| &= \begin{vmatrix} F(z)^4 z^T & F(z) h_{19}(z)^T & k_{25}(z)^T \end{vmatrix} \\ &= F(z)^5 \begin{vmatrix} z^T & h_{19}(z)^T & k_{25}(z)^T \end{vmatrix} \\ &= -1458 F(z)^5 X(z). \end{aligned}$$

The final equality follows by uniqueness of X as a degree-45 invariant and evaluation of $|\tau_z|$, $F(z)$, and $X(z)$ at a single point. Thus, the square of $|\tau_z|$ is expressible in $F(z)$, $\Phi(z)$, and $\Psi(z)$ alone. In terms of Y,

$$\begin{aligned} |\tau_z|^2 &= 1458^2 F^{10} X^2 \\ &= 432 F^{25} (Y_1 + 20Y_1^2 + 204Y_1^3 + 1094Y_1^4 \\ &\quad + 3271Y_1^5 + 3078Y_1^6 + 1404Y_1^7 + 18Y_2 \\ &\quad + 198Y_1Y_2 + 954Y_1^2Y_2 - 198Y_1^3Y_2 - 5508Y_1^4Y_2 \\ &\quad + 1944Y_1^5Y_2 - 648Y_2^2 - 7776Y_1Y_2^2 \\ &\quad - 5832Y_1^2Y_2^2 + 11664Y_2^3). \end{aligned} \tag{4–2}$$

The special case. Take the family of maps

$$\begin{aligned} \sigma_z(w) &= \big(72\Phi(z)^4 z\big)\, w_1 \\ &\qquad + \big(\Psi(z) h_{19}(z)\big)\, w_2 + \big(24\Phi(z)^2 k_{25}(z)\big)\, w_3, \end{aligned}$$

having z-degree 49 and w-degree one. The integer coefficients have been chosen so that, in bub$_{\bar{2}2}$-coordinates, the point

$$[w_1, w_2, w_3] = [1, 1, 1]$$

corresponds to the map

$$\nabla F(z) \times \nabla X(z)$$

associated with the 2-form $dF \wedge dX$. As in the general situation,

$$\sigma_{Tz}(w) = T\sigma_z(w)$$

so that

$$F(\sigma_{Tz}(w)) = F(\sigma_z(w)).$$

Thus the w-coefficients are $\mathcal{V}_z$-invariant and thereby expressible in terms of the basic forms $F(z)$, $\Phi(z)$, $\Psi(z)$, and $X(z)$. Since the degree in z of $F(\sigma_z(w))$ is $6 \cdot 49 = 294$, odd powers of $X(z)$ cannot take part in the basic invariant decomposition of these coefficients. Furthermore, restriction of the parameter space $\mathbb{CP}^2_z$ to $\{F(z) = 0\}$ yields coefficients in $\Phi(z)$ and $\Psi(z)$ alone. Finally, since $294 = 22 \cdot 12 + 30 = 2 \cdot 12 + 9 \cdot 30$, the restriction $F(\sigma_z(w))|_{\{F(z)=0\}}$ is divisible by $\Phi(z)^2 \Psi(z)$. Hence,

$$F(\sigma_z(w))|_{\{F(z)=0\}} = \eta\, \Phi(z)^2 \Psi(z)^9 F_V(w),$$

where η is a simplifying numerical factor and $F_V(w)$ is a polynomial that is degree four in V and degree six in w. The expression for $F_V(w)$ also appears in [Crass 1997b, pp. 68–69].

Since the parameter space gets restricted to

$$\{F(z) = 0\},$$

$|\sigma_z|$ should not vanish there. In fact, 49 is the lowest degree in which this fails to occur for three projectively distinct maps. Explicitly,

$$\begin{aligned}|\sigma_z| &= \begin{vmatrix} 72\,\Phi^4(z)z^T & \Psi(z)h_{19}(z)^T & 24\,\Phi^2(z)k_{25}(z)^T \end{vmatrix}\\ &= 24\cdot 72\,\Phi^4(z)\,\Psi(z)\,\Phi^2(z)\,\begin{vmatrix} z & h_{19}(z) & k_{25}(z)\end{vmatrix}\\ &= 2^7 3^9\,\Phi(z)^6\,\Psi(z)\,X(z).\end{aligned}$$

Furthermore, on $\{F(z)=0\}$, the expression (2–1) for X^2 reduces to

$$\begin{aligned}X^2 &= -\tfrac{1}{81}(8\Phi^5\Psi - 3\Psi^3) = -\frac{\Psi^3}{81}\left(8\frac{\Phi^5}{\Psi^2}-3\right)\\ &= -\frac{\Psi^3}{81}\left(8\frac{3V}{8}-3\right) = -\frac{\Psi^3}{27}(V-1).\end{aligned}$$

Consequently,

$$\begin{aligned}|\sigma_z|^2 &= -2^{14}3^{15}\Phi^{12}\Psi^5(V-1)\\ &= -2^{14}3^{15}\Phi^2\Psi^9\,\frac{\Phi^{10}}{\Psi^4}(V-1)\\ &= -2^{14}3^{15}\Phi^2\Psi^9\left(\frac{3V}{8}\right)^2(V-1)\\ &= -2^8 3^{17}\Phi^2\Psi^9 V^2(V-1).\end{aligned}$$

The remaining basic invariants. The forms of degrees 12, 30, and 45 arise from the sixth-degree invariant as before. However, a parametrized change of coordinates requires special handling. Under $y=Ax$, the Hessian H_x, the bordered Hessian BH_x, and the Jacobian J_x transform as

$$\begin{aligned}H_x(F(y)) &= A^T H_y(F(y))A,\\ BH_x(F(y),G(y)) &= \\ &\begin{pmatrix} A^T & 0\\ 0 & 1\end{pmatrix} BH_y(F(y),G(y)) \begin{pmatrix} A & 0\\ 0 & 1\end{pmatrix},\\ J_x(F(y),G(y),K(y)) &= J_y(F(y),G(y),K(y))A,\end{aligned}$$

where the subscripts x, y indicate the differentiation variable. As for transformation of the respective determinants:

$$\begin{aligned}\bigl|H_x(F(y))\bigr| &= |A|^2\bigl|H_y(F(y))\bigr|,\\ \bigl|BH_x(F(y),G(y))\bigr| &= |A|^2\bigl|BH_y(F(y),G(y))\bigr|,\\ \bigl|J_x(F(y),G(y),K(y))\bigr| &= |A|\,\bigl|J_y(F(y),G(y),K(y))\bigr|.\end{aligned}$$

For the parametrized change of coordinates $y=\tau_z(w)$, let

$$\begin{aligned}\Phi_Y(w) &= \alpha_\Phi\bigl|H_w(F_Y(w))\bigr|,\\ \Psi_Y(w) &= \alpha_\Psi\bigl|BH_w(F_Y(w),\Phi_Y(w))\bigr|,\\ X_Y(w) &= \alpha_X\bigl|J_w(F_Y(w),\Phi_Y(w),\Psi_Y(w))\bigr|,\end{aligned}$$

where the constants α_Φ, α_Ψ, α_X are thosed defined in Section 2E (page 221). Then we have

$$\begin{aligned}\Phi(y) &= \alpha_\Phi\bigl|H_y(F(y))\bigr|\\ &= \alpha_\Phi\bigl|H_y(F(\tau_z(w)))\bigr|\\ &= \alpha_\Phi|\tau_z|^{-2}\bigl|H_w(\alpha F(z)^{25}F_Y(w))\bigr|\\ &= \frac{\alpha_\Phi}{|\tau_z|^2}(\alpha F(z)^{25})^3\bigl|H_w(F_Y(w))\bigr|\\ &= \frac{(\alpha F(z)^{25})^3}{|\tau_z|^2}\Phi_Y(w),\end{aligned} \tag{4–3}$$

$$\begin{aligned}\Psi(y) &= \alpha_\Psi\bigl|BH_y(F(y),\Phi(y))\bigr|\\ &= \alpha_\Psi\bigl|BH_y(F(\tau_z(w)),\Phi(\tau_z(w)))\bigr|\\ &= \alpha_\Psi|\tau_z|^{-2}\biggl|BH_w\biggl(\alpha F(z)^{25}F_Y(w),\\ &\qquad\qquad \frac{(\alpha F(z)^{25})^3}{|\tau_z|^2}\Phi_Y(w)\biggr)\biggr|\\ &= \frac{(\alpha F(z)^{25})^8}{|\tau_z|^6}\Psi_Y(w),\end{aligned} \tag{4–4}$$

and

$$\begin{aligned}X(y) &= \alpha_X\bigl|J_y(F(y),\Phi(y),\Psi(y))\bigr|\\ &= \frac{(\alpha F(z)^{25})^{12}}{|\tau_z|^9}X_Y(w).\end{aligned} \tag{4–5}$$

With $y=\sigma_z(w)$, similar calculations lead to the one-parameter forms:

$$\begin{aligned}\Phi(y) &= \frac{(\eta\,\Phi(z)^2\,\Psi(z)^9)^3}{|\sigma_z|^2}\Phi_V(w)\\ \Psi(y) &= \frac{(\eta\,\Phi(z)^2\,\Psi(z)^9)^8}{|\sigma_z|^6}\Psi_V(w)\\ X(y) &= \frac{(\eta\,\Phi(z)^2\,\Psi(z)^9)^{12}}{|\sigma_z|^9}X_V(w).\end{aligned}$$

The 19-maps. With an invariant system in place for each (nonsingular) value of Y (or V), production of the degree-19 map that preserves all 12 of the conics proceeds as before:

(1) determine a 64-map $(f_{64})_Y(w)$ that vanishes at $\{X_Y(w)=0\}$;

(2) compute $(f_{19})_Y(w) = (f_{64})_Y(w)/X_Y(w)$;
(3) compute the conic-fixing map $h_Y(w)$.

In fact, the previous calculations in the coordinates $\{y_1, y_2, y_3\}$ provide a framework for those at hand. Into the y-expressions involving $F(y)$, $\Phi(y)$, $\Psi(y)$, and $X(y)$ as well as the maps $\psi_{16}(y)$, $\varphi_{34}(y)$, and $f_{40}(y)$ substitute the appropriate terms in Y and w, namely, $F_Y(w)$, $\Phi_Y(w)$, $\Psi_Y(w)$, $X_Y(w)$, $\psi_Y(w)$, $\varphi_Y(w)$, and $f_Y(w)$. The substitutions for the invariants appear in equation (4–1) on page 232 and in (4–3), (4–4), and (4–5) on the preceding page. Concerning maps, they transform as the coefficients of 2-forms. In terms of the cross-product, for $u = Ax$,

$$\begin{aligned}\nabla_x(F(u)) \times \nabla_x(G(u)) \\ &= A^T\nabla_u F(u) \times A^T\nabla_u G(u) \\ &= |A^T|((A^T)^{-1})^T\big(\nabla_u F(u) \times \nabla_u G(u)\big) \\ &= |A|\,A^{-1}\big(\nabla_u F(u) \times \nabla_u G(u)\big).\end{aligned}$$

Accordingly,

$$\begin{aligned}\psi(y) &= \nabla_y F(y) \times \nabla_y \Phi(y) \\ &= \frac{\tau_z}{|\tau_z|}\bigg(\nabla_w\big(\alpha F(z)^{25} F_Y(w)\big) \\ &\qquad\qquad \times\nabla_w\Big(\frac{(\alpha F(z)^{25})^3}{|\tau_z|^2}\,\Phi_Y(w)\Big)\bigg) \\ &= \frac{(\alpha F(z)^{25})^4}{|\tau_z|^3}\,\tau_z\nabla_w F_Y(w) \times \nabla_w\ \Phi_Y(w) \\ &= \frac{(\alpha F(z)^{25})^4}{|\tau_z|^3}\,\tau_z(\psi_Y(w)),\end{aligned}$$

$$\begin{aligned}\varphi(y) &= \nabla_y F(y) \times \nabla_y \Psi(y) \\ &= \frac{1}{|\tau_z|}\,\tau_z\bigg(\nabla_w\big(\alpha F(z)^{25} F_Y(w)\big) \\ &\qquad\qquad \times\nabla_w\Big(\frac{(\alpha F(z)^{25})^8}{|\tau_z|^6}\,\Psi_Y(w)\Big)\bigg) \\ &= \frac{(\alpha F(z)^{25})^9}{|\tau_z|^7}\,\tau_z(\varphi_Y(w)),\end{aligned}$$

and

$$\begin{aligned}f(y) &= \nabla_y \Phi(y) \times \nabla_y \Psi(y) \\ &= \frac{1}{|\tau_z|}\,\tau_z\bigg(\nabla_w\Big(\frac{(\alpha F(z)^{25})^3}{|\tau_z|^2}\,\Phi_Y(w)\Big) \\ &\qquad\qquad \times\nabla_w\Big(\frac{(\alpha F(z)^{25})^8}{|\tau_z|^6}\,\Psi_Y(w)\Big)\bigg) \\ &= \frac{(\alpha F(z)^{25})^{11}}{|\tau_z|^9}\,\tau_z(f_Y(w)).\end{aligned}$$

The one-parameter maps transform as follows:

$$\begin{aligned}\psi(y) &= \frac{(\eta\,\Phi(z)^2\,\Psi(z)^9)^4}{|\sigma_z|^3}\,\sigma_z(\psi_Y(w)) \\ \varphi(y) &= \frac{(\eta\,\Phi(z)^2\,\Psi(z)^9)^9}{|\sigma_z|^7}\,\sigma_z(\varphi_Y(w)) \\ f(y) &= \frac{(\eta\,\Phi(z)^2\,\Psi(z)^9)^{11}}{|\sigma_z|^9}\,\sigma_z(f_Y(w)).\end{aligned}$$

Making the substitutions into the expression for $f_{64}(y)$ yields a collection of maps $(f_{64})_Y(w)$ each of which is divisible by $X_Y(w)$. Finally, substitution into the formula for the canonical map $h_{19}(y)$ supplies the conic-fixing family $h_Y(w)$. The explicit calculations appear in [Crass 1997b, pp. 68–69].

General expressions for the canonical 19-map. Each w-coefficient in

$$h_z(w) = \tau_z^{-1}(h_{19}(\tau_z(w)))$$

is a polynomial in Y_1 and Y_2 with z-degree

$$\begin{aligned}\deg_z h_z(w) &\le 21\cdot 25 = 19\cdot 25 + 2\cdot 25 \\ &= 12\cdot 40 + 45 = 30\cdot 16 + 45.\end{aligned}$$

The factor $|\tau_z|^{-1}$ due to τ_z^{-1} does not affect the map on $\mathbb{CP}^2$, so we neglect it. After dividing away a factor of $X(z)$, the polynomial map $h_Y(w)$ satisfies

$$\deg_{Y_1} h_Y(w) \le 40 \qquad \deg_{Y_2} h_Y(w) \le 16.$$

Hence, one finds the coefficients of the w monomials by solving, for each term, 353 linear equations.

In the one-parameter case,

$$h_z(w) = \sigma_z^{-1}(h_{19}(\sigma_z(w)))$$

and

$$\deg_z h_z(w) \le 21\cdot 49 = 12\cdot 82 + 45 = 12\cdot 2 + 30\cdot 32 + 45.$$

Thus, on $\{F(z) = 0\}$, the map $h_z(w)$ is divisible by $\Phi(z)^2 X(z)$, so that

$$\deg_z\left(\frac{h_z(w)}{\Phi(z)^2 X(z)}\right) = 60\cdot 16, \quad \deg_V h_V(w) \le 16.$$

4D. Symmetry Lost, a Root Found

Under Conjecture 3.13, the trajectory $\{h_{19}^k(y_0)\}$ converges to a pair of 72-points for almost any $y_0 \in \mathbb{CP}^2_y$. Being conjugate to $h_{19}(y)$, the maps $h_Y(w)$ share this property for points in $\mathbb{CP}^2_w$. Breaking the

$\mathcal{A}_6$ symmetry of $R_Y(u) = 0$ qualifies $h_Y(w)$ for a role in root finding. An analogous treatment applies in the one-parameter case.

Root selection. Consider the rational function

$$\bar{J}_z(w) = \frac{\bar{\Gamma}_z(w)^3}{F(z)\,\Psi(\tau_z(w))},$$

where

$$\bar{\Gamma}_z(w) = \sum_{m=1}^{6} \prod_{n \neq m} C_{\bar{n}}(\tau_z(w)) C_{\bar{m}}(z).$$

At a 72-point pair $\{q_1, q_2\} = \tau_z^{-1}(\{p_{\bar{a}b_1}, p_{\bar{a}b_2}\})$ in w space, five of the six terms in $\bar{\Gamma}_z(w)$ vanish. The result is a "selection" of one of the six roots $U_{\bar{a}}(z)$ of $R_Y(u)$:

$$\begin{aligned}\bar{J}_z(\{q_1, q_2\}) &= \frac{\left(\prod_{n\neq a} C_{\bar{n}}(\{p_{\bar{a}b_1}, p_{\bar{a}b_2}\})\right)^3}{\Psi(\{p_{\bar{a}b_1}, p_{\bar{a}b_2}\})}\,\frac{C_{\bar{a}}(z)^3}{F(z)}\\ &= \frac{U_{\bar{a}}(z)}{\mu}.\end{aligned}$$

Here μ is the value of

$$\frac{\Psi(\{p_{\bar{a}b_1}, p_{\bar{a}b_2}\})}{\left(\prod_{n\neq a} C_{\bar{n}}(\{p_{\bar{a}b_1}, p_{\bar{a}b_2}\})\right)^3},$$

which is constant on $\mathcal{O}_{72}$:

$$\mu = \frac{6561\,(279 + 145\sqrt{15}\,i)}{2}.$$

In light of the $\mathcal{V}_z$-invariance of $\bar{\Gamma}_z(w)$, $\bar{J}_z$ also enjoys this property and so, presents a second face which expresses itself in Y and w. Since

$$\deg_z \bar{\Gamma}_z(w) = 252 = 10 \cdot 25 + 2 = 6 \cdot 42,$$

it transforms by

$$\bar{\Gamma}_Y(w) = \frac{\bar{\Gamma}_z(w)}{\beta\, F(z)^{42}}$$

where β is a simplifying factor for the coefficients over w. Each coefficient is a polynomial in Y_1 and Y_2 whose coefficients satisfy a system of linear equations. (See [Crass 1997a] for the lengthy expression of the polynomial $\bar{\Gamma}_Y(w)$.) Now, let T_Y be the polynomial in Y_1 and Y_2 that satisfies

$$|\tau_z|^2 = F(z)^{25}\, T_Y;$$

see (4–2) on page 232. Then, from (4–4),

$$\Psi(\tau_z(w)) = \frac{(\alpha\, F(z)^{25})^8}{(F(z)^{25} T_Y)^3}\,\Psi_Y(w) = \frac{\alpha^8\, F(z)^{125}}{T_Y^3}\,\Psi_Y(w).$$

Finally,

$$\bar{J}_Y(w) = \frac{\beta^3}{\alpha^8}\,\frac{(T_Y\,\bar{\Gamma}_Y(w))^3}{\Psi_Y(w)}.$$

In the one-parameter case, the above development goes through with $\bar{J}_z(w)$ and $\bar{\Gamma}_z(w)$ replaced by

$$\bar{K}_z(w) = \frac{\Phi(z)^2\,\Theta_z(w)^3}{\Psi(z)\,\Psi(\tau_z(w))}$$

and

$$\bar{\Theta}_z(w) = \sum_{m=1}^{6} \prod_{n\neq m} C_{\bar{n}}(\sigma_z(w)) C_{\bar{m}}(z).$$

First of all, there is the transformation of $\bar{\Theta}_z(w)$ under σ_z on $\{F(z) = 0\}$:

$$\bar{\Theta}_V(w) = \frac{\bar{\Theta}_z(w)|_{\{F(z)=0\}}}{\gamma\,\Phi(z)\,\Psi(z)^{16}}.$$

(See [Crass 1997a] for the expression of this polynomial.) Furthermore, restricting to $\{F(z) = 0\}$ yields, on the one hand, a root of $T_V(s)$

$$\bar{K}_z(\{q_1, q_2\}) = \frac{S_{\bar{a}}(z)}{\mu}$$

and, on the other,

$$\begin{aligned}\bar{K}_V(w) &= \frac{\Phi(z)^2\left(\gamma\,\Phi(z)\,\Psi(z)^{16}\,\bar{\Theta}_V(w)\right)^3|\sigma_z|^6}{(\eta\,\Phi(z)^2\Psi(z)^9)^8\,\Psi_V(w)}\\ &= -\frac{2^9 3^{50}\,\gamma^3\, V^5 (V-1)^3\,\bar{\Theta}_V(w)}{\eta^8\,\Psi_V(w)}.\end{aligned}$$

The algorithm. Now within reach are the ingredients required for preparation of a root-finding algorithm; see [Crass 1997a] for an implementation. To summarize the procedure:

(1) Select a value $A = (A_1, A_2)$ of $Y = (Y_1, Y_2)$ and, thereby, a sixth-degree resolvent $R_A(u)$. (For sake of description, let $z \in Y_1^{-1}(A_1) \cap Y_2^{-1}(A_2)$. The algorithm actually finds a root without explicitly inverting Y_1 or Y_2.)

(2) From an initial point $w_0 \in \mathbb{CP}^2_w$, iterate the map $h_A(w)$ to convergence:

$$h_A^n(w_0) \longrightarrow \{q_1, q_2\} \in (\mathcal{O}_{72})_w \subset \mathbb{CP}^2_w.$$

As output take the pair of approximate 72-points in $\mathbb{CP}^2_w$

$$\{p_1, p_2\} \approx \{q_1, q_2\} = \{\tau_z^{-1}(p_{\bar{a}b_1}), \tau_z^{-1}(p_{\bar{a}b_2})\}.$$

(3) Using p_1 or p_2, approximate a root of $R_A(u)$:

$$\begin{aligned} U_{\bar{a}}(z) &\approx \mu \bar{J}_A(p_1) \\ &= \frac{\mu\beta^3 (T_A \bar{\Gamma}_A(p_1))^3}{\alpha^8 \Psi_A(p_1)} \\ &= 2^{79}3^{94}(11+3\sqrt{15}i)\frac{T_A\bar{\Gamma}_A(p_1))^3}{\Psi_A(p_1)}. \end{aligned}$$

Performing the corresponding steps in the special case produces an approximate solution to the V-resolvent $T_{V_0}(s)$ given an initial choice V_0 of V:

$$\begin{aligned} S_{\bar{a}}(z) &\approx \mu \bar{K}_{V_0}(p_1) \\ &= -\frac{2^9 3^{50}\mu\gamma^3 {V_0}^5 (V_0-1)^3 \bar{\Theta}_{V_0}(p_1)}{\eta^8 \Psi_{V_0}(p_1)} \\ &= -2^5 5^{-96}(11+3\sqrt{15}i)\frac{{V_0}^5(V_0-1)^3\bar{\Theta}_{V_0}(p_1)}{\Psi_{V_0}(p_1)}. \end{aligned}$$

4E. Getting All the Roots

Because a pair of 72-points lies on *one* barred conic, the algorithm above determines just one of the six roots of the selected resolvent. This manifests the iteration's failure to break the galois symmetry completely; the stabilizer of a 72-point is a $\mathcal{D}_5$. From the coefficient field of the resolvent $R_A(u)$, the algorithm leads to an extended field K whose galois group $G(\Sigma/K)$ is $\mathcal{D}_5$. Of course, Σ is the splitting field for R_A.

Finding all six roots calls for a dynamical system that converges to 6-cycles in a 360 point orbit and a root selector function that gives equations in all the roots. This would likely complicate the associated formulas.

APPENDIX: SEEING IS BELIEVING

Figures 9–12 provide empirical dynamical information for some of the special maps discussed in the text. They were created using the program *Dynamics* [Nusse and Yorke 1994] running on a Silicon Graphics Indigo-2, and have a resolution of 720×720.

The first three images are the product of the BAS routine, which colors a grid-cell if the trajectory of the cell's center gets close enough to a specified attractor to guarantee ultimate attraction to it. The color depends upon the destination. If, in a specified number of iterations, the center's trajectory fails to converge to an attractor, the cell's color is black.

For Figure 9, which shows the basins of ψ_{16} restricted to a 45-line, the attractors are the 45-points, which lie in the large basins near the center. The figure suggests that most points converge to one of these attractors. Does this happen for almost every point? Or do the black specks contain sets of positive measure whose forward orbits fail to converge to one of the four attractive 45-points? The BAS algorithm checked 60 iterates before giving up on determining convergency.

Figure 10 shows the basins of the conic-fixing map h_{19} restricted to one of the invariant conics. Six attracting orbits are known, each consisting of a pair of 72-points. Each basin is the one-dimensional intersection of a two-dimensional basin in $\mathbb{CP}^2$. Does the backward orbit of these basins fill out $\mathbb{CP}^2$ in measure? (This is the second part of Conjecture 3.13.)

Figure 11 shows the same map, but this time restricted to one of the bub-$\mathbb{RP}^2$s — specifically $\mathcal{R}_{\bar{2}2}$, with the 1-point orbit $p_{\bar{2}2}$ at the origin and the 1-line orbit $\mathcal{L}_{\bar{2}2}$ at infinity. The chosen coordinates make evident the map's D_5 symmetry. Along the unit circle is the 10-point orbit of 72-points $p_{\bar{a}a_{1,2}}$, with $a = 1, 3, 4, 5, 6$. The five lines of reflective symmetry passing through $(0, 0)$ are affine lines in the five $\mathbb{RP}^1$ intersections with $\mathcal{R}_{\bar{2}2}$ of both the 45-lines $\mathcal{L}_{\overline{2a}2a}$ and the $\mathcal{R}_{\bar{a}a}$, for $a = 1, 3, 4, 5, 6$. The figure reveals repelling behavior along the $\mathbb{RP}^1$ where the $\mathbb{RP}^2$ meets one of the 45-lines left invariant by $\mathrm{bub}_{\bar{2}2}$. Only the five period-2 cycles of 72-points appear as attractors. No black specks are seen; their presence would compromise Conjecture 3.13. Several magnified views of this basin plot appear in [Crass 1997a].

Figure 12 is a partial plot using the BA algorithm of *Dynamics*, which colors whole trajectories of cells, thereby manifesting some aspects of the dynamics. (See [Nusse and Yorke 1994, pp. 269 ff.] for a more thorough description of BA).

Many of the points in the strip are mapped inside the hazy pentagon whose vertices lie on the 45-lines — the inner star is nearly filled. Around this pentagon is the outer star-like piece of the critical set shown in Figure 15. Futhermore, the pentagon seems to be the image of the inner pentagonal oval. Accordingly, the map folds the plane along the pentagon's edges just outside of which the 72-points make their presence seen in the dense streaks at

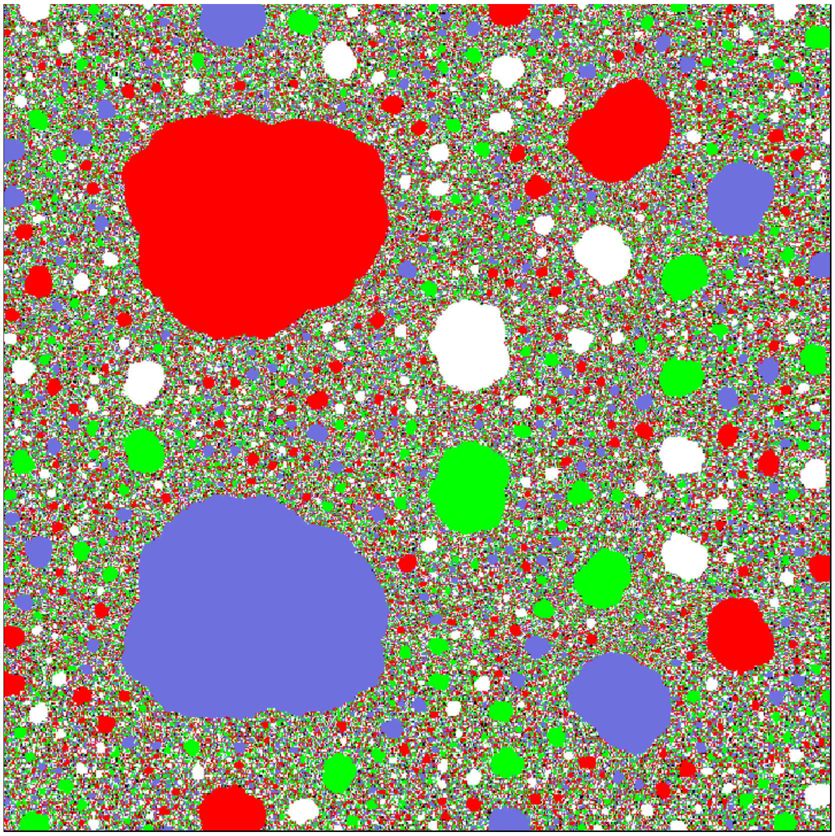

FIGURE 9. Basin of attraction of the degree-16 map ψ_{16} restricted to a 45-line; each color represents one of the four attracting 45-points. The black specks indicate points for which no attraction to one of these four points was detected after 60 iterations.

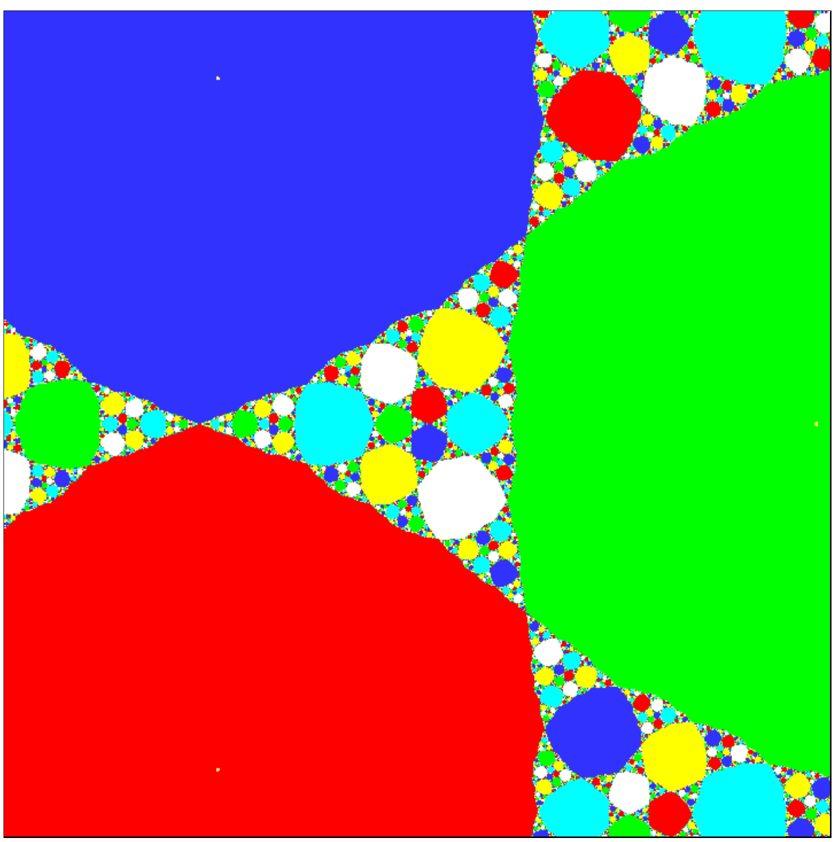

FIGURE 10. Dynamics of the degree-19 map with icosahedral symmetry on $\mathbb{C} \cup \{\infty\}$. Almost all points are attracted to a pair of antipodal vertices. Each of the six colors corresponds to such a pair; the three large basins each contain a vertex. For the conic-fixing h_{19}, the basin plot on each conic is conjugate to this one.

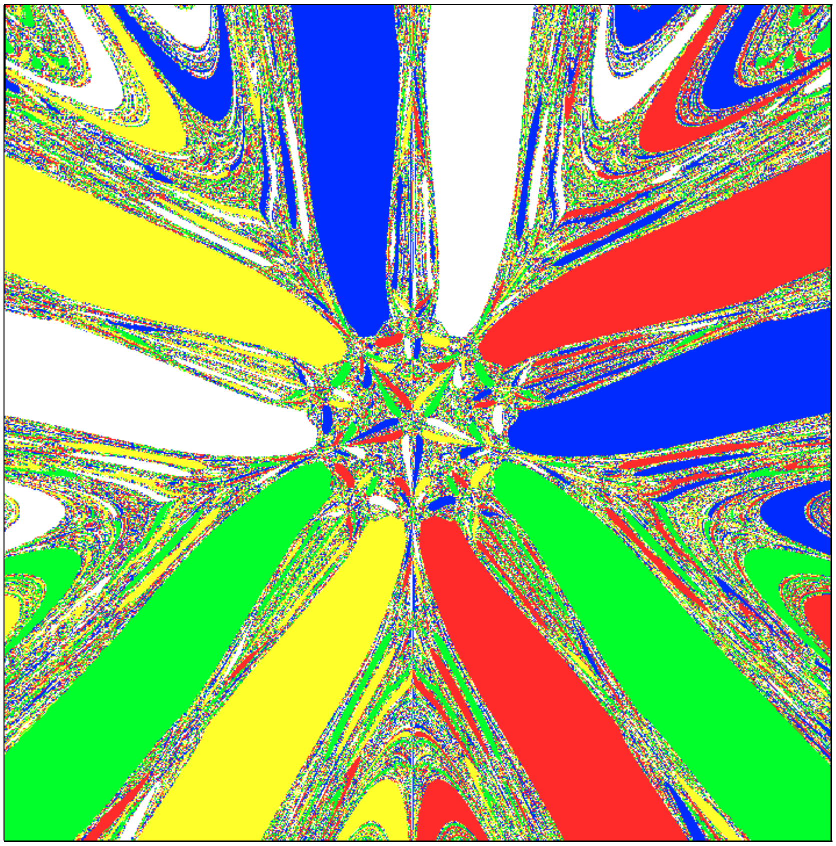

FIGURE 11. $\mathbb{RP}^2$ dynamics of h_{19}. The vertical and horizontal scales are roughly from -2 to 2. The large radial basins are immediate, i.e., each contains one of the 72-points and come in pairs as do the period-2 attractors. Notice the repelling behavior along the 45-lines and particularly at their intersection in the 36-point $p_{\bar{2}2}$.

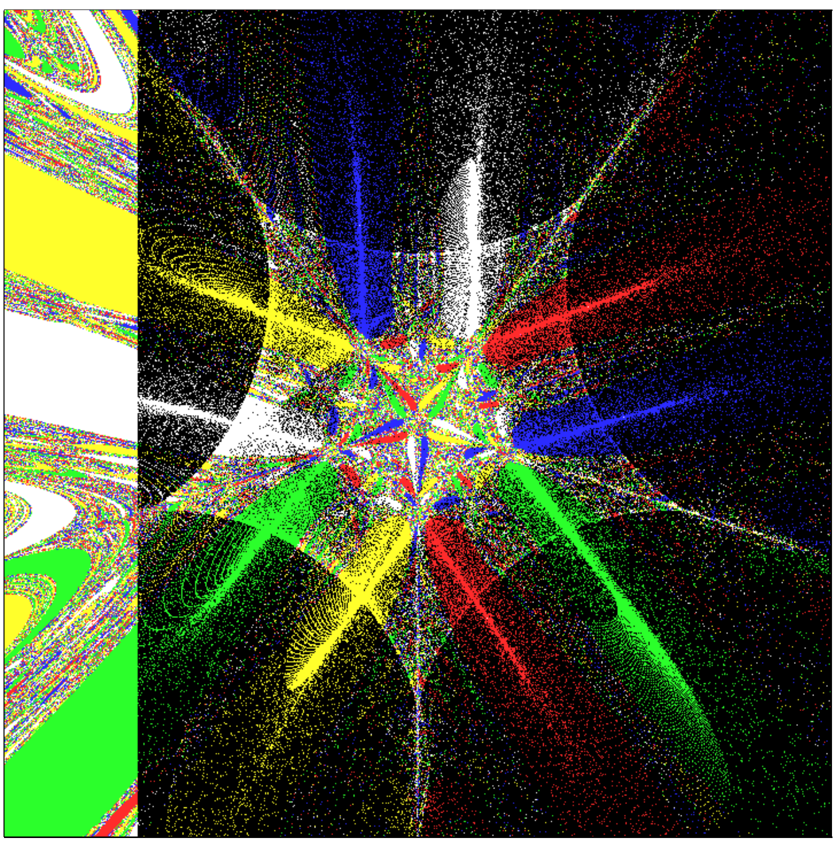

FIGURE 12. More $\mathbb{RP}^2$ dynamics of h_{19}. The vertical strip on the left is a copy of the same area in Figure 11; the rest of the figure shows trajectories, colored according to their destinations, of the points in the strip.

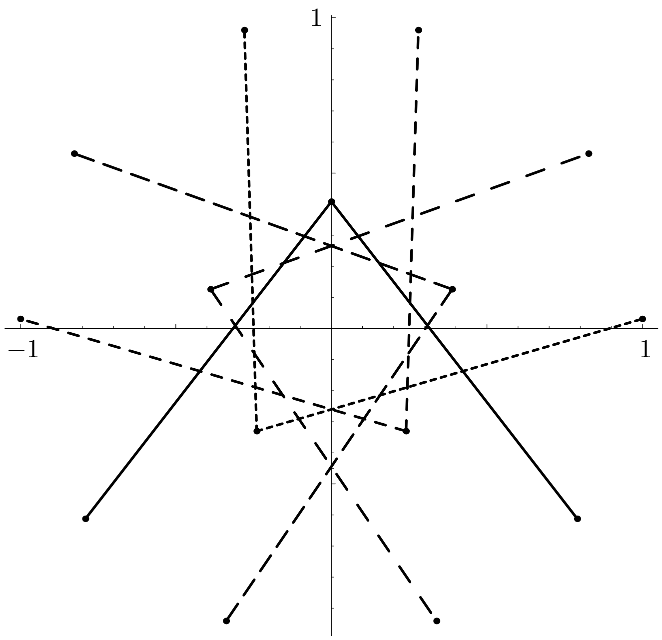

FIGURE 13. Configuration of 72-lines on a bub-$\mathbb{RP}^2$. Under bub$_{\bar{2}2}$, the five pairs of 72-lines $\{\mathcal{L}_{\bar{a}a_1}, \mathcal{L}_{\bar{a}a_2}\}$, for $a = 1, 3, 4, 5, 6$, map to themselves. Accordingly, each line meets $\mathcal{R}_{\bar{2}2}$ in an $\mathbb{RP}^1$. The picture shows their configuration in the affine plane of Figures 11 and 12. A given pair $\mathcal{L}_{\bar{a}a_1}$ and $\mathcal{L}_{\bar{a}a_2}$ passes through the 72-points $p_{\bar{a}a_2}$ and $p_{\bar{a}a_1}$ respectively; they intersect in the corresponding repelling and fixed 36-point $p_{\bar{a}a}$.

$p_{\bar{a}a_{1,2}}$. Compare this pattern of streaks to that of the 72-lines given in Figure 13. Since $\mathcal{C}_{\bar{a}}$ and $\mathcal{C}_a$ are tangent to $\mathcal{R}_{\bar{a}a}$ at $p_{\bar{a}a_{1,2}}$, the icosahedral 19-map opens up a triangular angle of $\pi/3$ to $4\pi/3$. Thus, the behavior at a 72-point consists of "fourth-powering". Figure 14 displays this local squeezing. (Figures 13–15 were produced using Mathematica.)

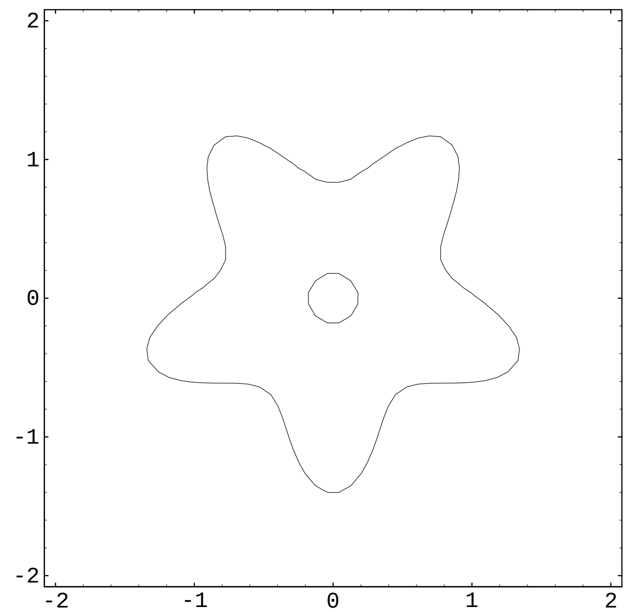

FIGURE 15. Contour plot of the critical set of h_{19}, the sixth-degree curve $\{F = 0\}$, on $\mathcal{R}_{\bar{2}2}$. At the 10 inflection points are the superattracting 72-points. Further computation suggests that $\{G_{48} = 0\}$ hits the $\mathbb{RP}^2$ in a discrete set. Might this set consist only of the singular 72-points?

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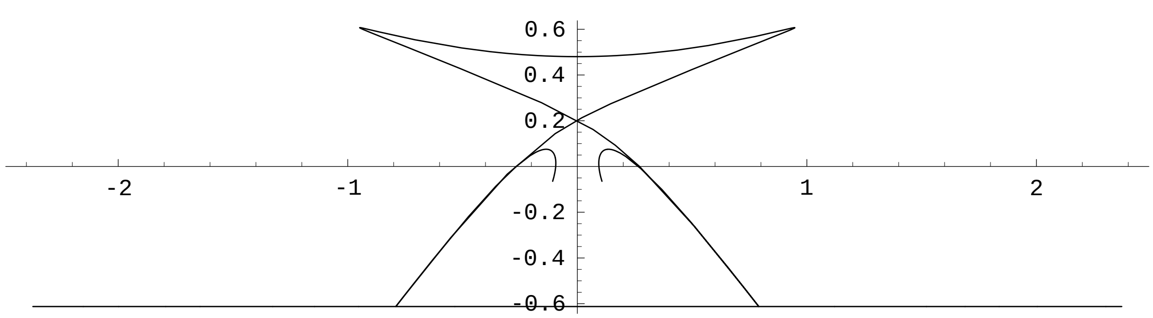

FIGURE 14. Image on a bub-$\mathbb{RP}^2$ of a 36-line under h_{19}. The lower horizontal line is the $\mathbb{RP}^1$ intersection of $\mathcal{R}_{\bar{2}2}$ and the 36-line associated with the pair of green 72-points from the basin plots. The curve is the image of the line under h_{19}. Sitting at the sharp cusps are the 72-points exchanged by the map. As indicated in the caption to Figure 12, the line folds over at these critical points. The upper two sharp turns are not critical values; they occur where the line passes through the yellow and red "streaks" that approximate 72-lines.

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