



Solving Zhou's Chaotic System Using Euler's Method

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Abstract : An algorithm for Euler's method is used to obtain an approximation for the initial-value problem for ordinary differential equation. This method is then employing to Zhou's chaotic system. This system is a three-dimensional autonomous system according to the numerical simulation as well as the theoretical analysis. We use the C++ software to solve this system and MATLAB to plot the solutions and the results are given for different number of iterations.

Keywords : Chaos; Euler's method; initial value problem.

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1 Introduction

One of the simplest approaches to obtain the numerical solution of a differential equation is the method of Euler [4]. This method is not an efficient method and seldom used, but it is easy to analyze and many of the ideas to solve the differential equations are in it. Chapra and Canale [3] stated that the Euler's method has the following general form

$$\text{New value} = \text{old value} + \text{slope} \times \text{step size}$$

or in the mathematical form,

$$y(t_{i+1}) = y(t_i) + \phi h \tag{1.1}$$

where ϕ is the slope which is used to extrapolate from an old value $y(t_i)$ to a new value $y(t_{i+1})$ over a distance h .

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In principle, Euler's method uses the forward-difference formula approximation of $y'(t)$. That is

$$y' = \frac{dy}{dt} \approx \frac{y(t_{i+1}) - y(t_i)}{\Delta t} \quad (1.2)$$

where Δt is the step size and is equal to $t_{i+1} - t_i$ [2]. Euler's method is used to obtain an approximation for the initial-value problem for ordinary differential equation [1]

$$\frac{dy}{dt} = f(t, y) \quad a \leq t \leq b. \quad (1.3)$$

Euler's method can be defined as follow:

$$\begin{aligned} \Delta t &= \frac{b - a}{N} = t_{i+1} - t_i \\ t_i &= a + i\Delta t \end{aligned}$$

for each $i = 0, 1, \dots, N$.

To derive Euler's method, we use the Taylor's series expansion in which the third term is called the remainder term,

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2}y''(\xi_i),$$

for some number ξ_i in (t_{i+1}, t_i) . Since $\Delta t = t_{i+1} - t_i$, therefore

$$y(t_{i+1}) = y(t_i) + (\Delta t)y'(t_i) + \frac{(\Delta t)^2}{2}y''(\xi_i),$$

and since $y(t)$ satisfy (1.3),

$$y(t_{i+1}) = y(t_i) + (\Delta t)f(t_i, y(t_i)) + \frac{(\Delta t)^2}{2}y''(\xi_i). \quad (1.4)$$

Euler's method constructs $w \approx y(t_i)$, for each $i = 1, 2, \dots, N$, by deleting the remainder term and we only use the first two terms in (1.4). Therefore, the Euler's method is

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + \Delta t f(t_i, w_i), \end{aligned} \quad (1.5)$$

for each $i = 0, 1, \dots, N - 1$.

In this study, we will apply this Euler's method to solve the ordinary differential equation of Zhou's system [5]. Zhou's system is a dynamical system with the presence of chaos phenomena.

2 The Algorithm

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha, \quad (2.1)$$

at $(N + 1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b ; integer N ; initial condition α

OUTPUT approximation w to y at the $(N + 1)$ values of t

Step 1 Set $\Delta t = (b - a)/N$;

$t = a$;

$w = \alpha$;

OUTPUT (t, w) .

Step 2 For $i = 1, 2, \dots, N$ do Steps 3, 4.

Step 3 Set $w = w + \Delta t f(t, w)$; (compute w_i)
 $t = a + i\Delta t$ (compute t_i)

Step 4 OUTPUT (t, w)

Step 5 STOP.

3 The Mathematical Model

In this section, we will discuss Zhou's chaotic system. This system is a three-dimensional autonomous system according to the numerical simulation as well as the theoretical analysis. The chaotic attractor of this Zhou's system also has the butterfly-shaped which is same as the Lorenz attractor, but both them are not topologically equivalent. Figure 1 to Figure 4 shows the Zhou's system from different views. This system is found to be unstable for all three equilibrium points. The equilibrium points are at the origin, E^+ and $E^+[5]$.

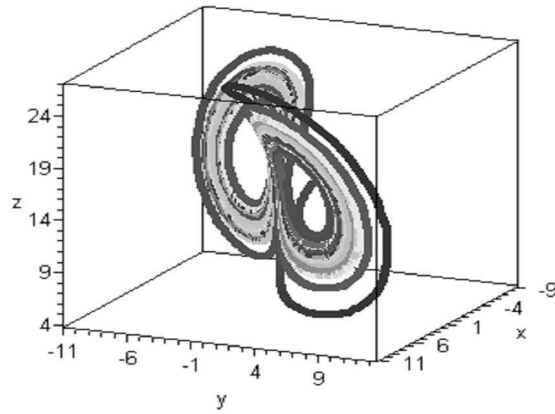


Figure 1: xyz phase plane of Zhou's attractor

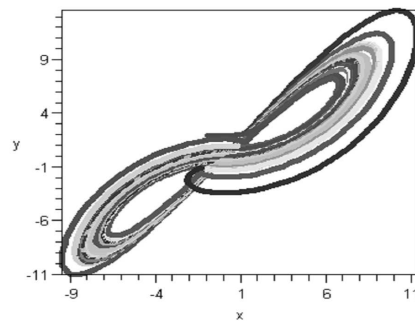


Figure 2: x-y phase plane of Zhou's attractor

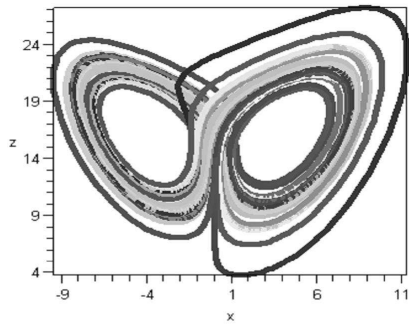


Figure 3: x-z phase plane of Zhou's attractor

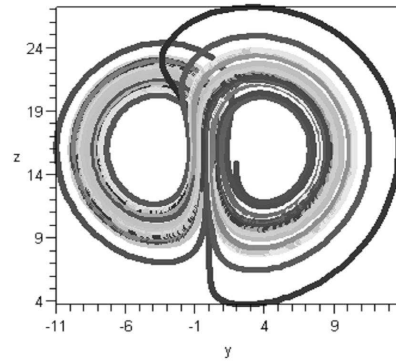


Figure 4: y-z phase plane of Zhou's attractor

The equations for Zhou's system are given by

$$\begin{aligned}\dot{x} &= a(y - x) \\ \dot{y} &= bx - xz \\ \dot{z} &= xy + cz\end{aligned}\tag{3.1}$$

where a, b and c are constants. Here, a dot denotes a derivative with respect to time, so that system (3.1) can be written as

$$\begin{aligned}\dot{x} &= \frac{dx}{dt} = a(y - x) \\ \dot{y} &= \frac{dy}{dt} = bx - xz \\ \dot{z} &= \frac{dz}{dt} = xy + cz.\end{aligned}\tag{3.2}$$

This system is chaotic when $a = 10$, $b = 16$ and $c = -1$. To determine the equilibrium points for this new system, we must find the equilibria by using Jacobian matrix. To find the equilibria of system (3.1), let

$$\begin{aligned}a(y - x) &= 0 \\ bx - xz &= 0 \\ xy + cz &= 0\end{aligned}\tag{3.3}$$

From (3.3), there exist three equilibria,

$$O(0, 0, 0), \quad E^+(x_1, y_1, z_1), \quad E^-(x_2, y_2, z_2)$$

where

$$\begin{aligned}x_1 &= y_1 = \sqrt{-bc}, & z_1 &= b, \\ x_2 &= y_2 = -\sqrt{-bc} & z_2 &= b.\end{aligned}$$

In addition, some basic dynamical properties of Zhou's attractor are; nonlinearity, symmetry and invariance, dissipativity and the existence of chaos which are determined from the Lyapunov exponent.

4 C++ Code

In this part, we show the programming code for Euler's method to find the solution of Zhou's system. This program is set up here for three variables Zhou's system but is easily modified for any other system of equations.

```
#include<iostream>
```

```

#include<fstream>
#include<iomanip>
using namespace std;
int main()
{
    double a, b, c, A, B, N, delta_t, x[10000], y[10000], z[10000];
    ifstream infile("Data2.DAT");
    infile>>a>>b>>c;
    infile>>x[0]>>y[0]>>z[0];
    infile>>A>>B>>N;
    delta_t=0.001;
    ofstream outfile("ZhouSolution.OUT");

    outfile<<"This is a programming by using Euler's method."<<endl;
    for(int t=1; t<=N; t++)
    {
        x[t]=x[t-1]+(a*y[t-1]-a*x[t-1])*delta_t;
        y[t]=y[t-1]+(b*x[t-1]-x[t-1]*z[t-1])*delta_t;
        z[t]=z[t-1]+(x[t-1]*y[t-1]+c*z[t-1])*delta_t;
    }
    for(t=0; t<=N; t++)
    {
        outfile<<setw(3)<<t<<fixed<<showpoint<<setprecision(9)<<setw(20)
            <<x[t]<<setprecision(9)<<setw(20)<<y[t]<<setprecision(9)<<
            setw(20)<<z[t]<<endl;
    }
    outfile.close();
    return 0;
}

```

5 Numerical Results

In this section, we apply Euler method for solving the Zhou's system. Given that the equations of Zhou's system were as follows:

$$\begin{aligned}
 \dot{x} &= ay(t) - ax(t) \\
 \dot{y} &= bx(t) - x(t)z(t) \\
 \dot{z} &= x(t)y(t) + cz(t).
 \end{aligned}
 \tag{5.1}$$

Using the Euler's formula, system (5.1) becomes:

$$\begin{aligned}x_{n+1} &= x_n + (ay_n - ax_n)\Delta t \\y_{n+1} &= y_n + (bx_n - x_nz_n)\Delta t \\z_{n+1} &= z_n + (x_ny_n + cz_n)\Delta t.\end{aligned}$$

The values of x_{n+1} , y_{n+1} and z_{n+1} will be used to find the values of x , y and z when the value of t increased. This process will be iterated for $N - 1$ times. Therefore, at last, we will obtain the approximation value for the solution of ordinary differential equation of Zhou's system. We employ the C++ software to solve the Euler's method for this system.

We set $a = 10$, $b = 10$ and $c = -1$ with the initial values $(-1, 2, 15)$. The output of C++ gives the values of x , y and z when the value of t increased. These values represent the points of (x, y, z) for $N - 1$ iterations. Next, we use the MATLAB software to plot these points and the results are given in Figure 5 to Figure 8 with different number of iterations used.

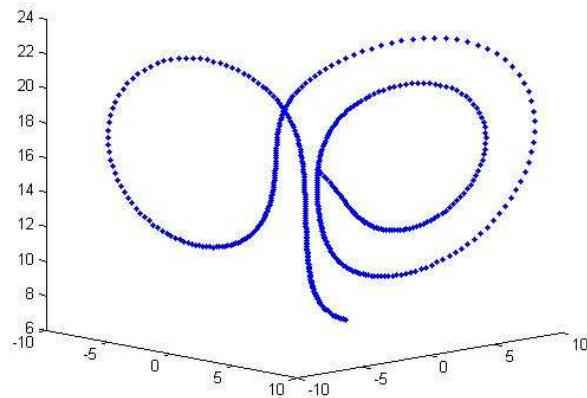


Figure 5: Zhou's attractor with 500 iterations

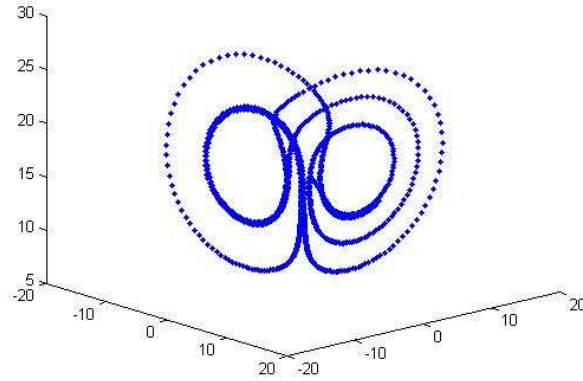


Figure 6: Zhou's attractor with 1000 iterations

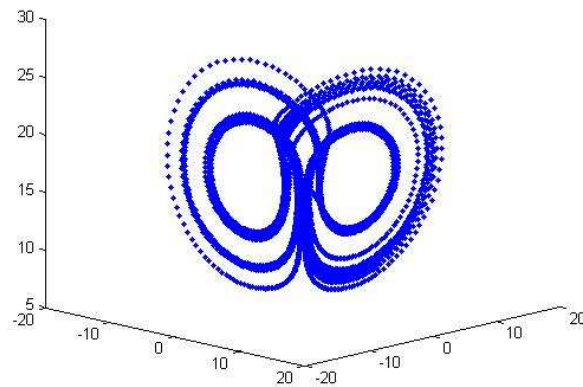


Figure 7: Zhou's attractor with 5000 iterations

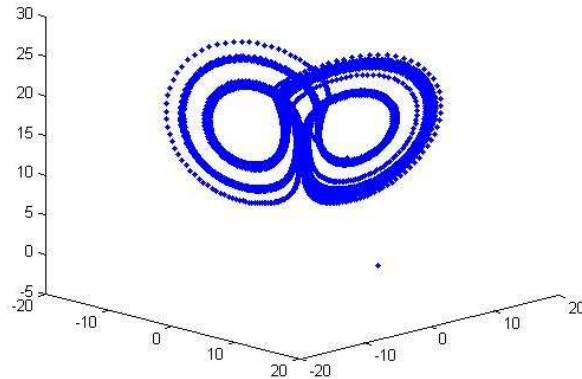


Figure 8: Zhou's attractor with 10000 iterations

The figures above are given for 500, 1000, 5000 and 10000 iterations. By using different number of iterations, we can see how the attractor is designed and moves.

6 Conclusions

An algorithm for Euler's method is used to obtain an approximation for the initial-value problem for ordinary differential equation. In principle, this method uses the forward-difference formula approximation of $y'(t)$ and Taylor's series expansion. This method is then employing to Zhou's chaotic system and it gives the values of x_{n+1} , y_{n+1} and z_{n+1} . These values will be used to approximate the points (x, y, z) the solution of Zhou's chaotic system. We use the C++ software to solve this system and MATLAB to plot the solutions. The results are given for 500, 1000, 5000 and 10000 iterations. By using different number of iterations, we can see how the attractor is designed and moves.

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