2018

Some Aggregation Operators For Bipolar-Valued Hesitant Fuzzy Information

Florentin Smarandache
University of New Mexico, smarand@unm.edu

Tahir Mahmood

Kifayat Ullah

Qaisar Khan

Journal of Fundamental and Applied Sciences

ISSN 1112-9867

Available online at http://www.jfas.info

Research Article

Special Issue

Some Aggregation Operators For Bipolar-Valued Hesitant Fuzzy Information

Tahir Mahmood, Kifayat Ullah, Qaisar Khan
Department of Mathematics, International Islamic University
Islamabad, Pakistan
tahirbakhat@yahoo.com, kifayat555@gmail.com, qaisarkhan421@gmail.com

Florentin Smarandache
Department of Mathematics, University of New Mexico
Gallup, NM, USA
fsmarandache@gmail.com

Published online: 5 March 2018

***Abstract*—In this article we define some aggregation operators for bipolar-valued hesitant fuzzy sets. These operations include bipolar-valued hesitant fuzzy ordered weighted averaging (BPVHFOWA) operator, bipolar-valued hesitant fuzzy ordered weighted geometric (BPVHFOWG) operator and their generalized forms. We also define hybrid aggregation operators and their generalized forms and solved a decision-making problem on these operation.**

Keywords—Bipolar-valued hesitant fuzzy sets (BPVHFSs), bipolar-valued hesitant fuzzy elements (BPVHFEs), BPVHFOWA operator, BPVHFOWG operator, BPVHFHA operator, BPVHFHG operator, score function and decision making (DM).

I. INTRODUCTION

At any level of our life decision making plays an essential role. It is a very famous research field now days. Everyone needs to take decision about the selection of best choice at any stage of his life [1, 19, 20]. Using ordinary mathematical techniques, we are not able to solve DM problems .

To deal with problems related to different kind of uncertainties, L. A. Zadeh [39] in 1965 initiated the concept of fuzzy sets (FSs). After his idea of FSs, researchers started to think about different extensions of FSs and some advanced forms of FSs have been established. Some of these extensions are interval-valued fuzzy set (IVFS) [5], intuitionistic fuzzy set (IFS) [1], hesitant fuzzy set (HFS) [22] and bipolar-valued fuzzy set (BVFS) [15] are some well known sets. Later on these new extensions of FSs have been extensively used in decision making [5], [16], [27] [28].

As different advanced forms of FSs came one after another, scientist started to merge two kinds of fuzzy information in a single set. The idea was quite useful and some very interesting extensions of FSs have been defined. These extensions include intuitionistic hesitant fuzzy sets (IHFSs), inter-valued hesitant fuzzy sets (IVHFSs) and bipolar-valued hesitant fuzzy sets (BVHFSs). The idea of merging different kind of fuzzy sets was quite useful and very shortly some new advanced forms of FSs have been established which are inter-valued intuitionistic hesitant fuzzy sets (IVIHFSs), cubic hesitant fuzzy sets (CHFSs) and bipolar-valued hesitant fuzzy sets (BPVHFSs).

Tahir M [25] introduced BPVHFSs, a new extension of FSs and a combination of HFSs and BVFSs. BPVHFSs have affiliation functions (membership function) in terms of set of some values. The positive affiliation function is a set having values in the interval [0, 1] which conveys the satisfaction extent of an element belong to the given set. While the negative affiliation function is a set of some values in [-1,0] which conveys the negative or counter satisfaction degree of an element belong to given set. Tahir M [25] defines some basic operations for BPVHFSs and proved some interesting results. He also defines aggregation operators for BPVHFSs and then used these operators in DM.

In our article, we apply some order on previously defined bipolar-valued hesitant fuzzy weighted averaging and weighted geometric operators by defining bipolar-valued hesitant fuzzy ordered weighted averaging and bipolar-valued hesitant fuzzy ordered weighted geometric operators along with their generalized operators. We also defined some hybrid aggregation operators on BPVHFSs along with their generalized forms. Finally, we did solve a DM problem using these newly defined aggregation operators and get very useful results.

This article consists of 4 sections with section one as introduction. In section two we recall the definition of BPVHFSs, their properties and some aggregation operators of BPVHFSs. Section three contain BPVHFOWA operators, BPVHFOWG operators, BPVHFHA operators and BPVHFHG operators. We also solve some examples on these defined operations. In the last section, we solve a DM problem using the defined operations in section three. Finally, we finish our article by adding a conclusion to it.

II. PRELIMINARIES

This section consists of the definition of BPVHFS and some aggregation operators on BPVHFSs. We also add some properties of BPVHFSs to this section and we recall the concept of score function for BPVHFSs.

doi: http://dx.doi.org/10.4314/jfas.v10i4s.85

A. Definition 1: [25]

For any set $\mathfrak{X}$, the BPVHFS $\mathfrak{B}$ on some domain of $\mathfrak{X}$ is denoted and defined by:

$$\mathfrak{B} = \{(\text{ϗ}, (\text{Ѣ}^{+}(\text{ϗ}), \text{Ѣ}^{-}(\text{ϗ}))) : \text{ϗ} \in \mathfrak{X}\}$$

where $\text{Ѣ}^{+} : \mathfrak{X} \to [0,1]$ is a finite set of few distinct values in the interval [O, 1]. It conveys the satisfaction extent of "ϗ" corresponding to BPVHFS $\mathfrak{B}$ and $\text{Ѣ}^{-} : \mathfrak{X} \to [-1,0]$is a finite set of few distinct values in the interval [-1, O]. It conveys the implicit counter or negative property of "ϗ" corresponding to BPVHFS $\mathfrak{B}$.

Here $\text{Ѣ} = \{\text{Ѣ}^{+}(\text{ϗ}), \text{Ѣ}^{-}(\text{ϗ})\}$ is a BPVHFE. The set of all BPVHFEs is denoted by Φ.

Consider two BPVHFSs:

$$\mathfrak{A} = \{(\text{ϗ}, (\text{Ѣ}^{+}{}_{\mathfrak{A}}(\text{ϗ}), (\text{Ѣ}^{-}{}_{\mathfrak{A}}(\text{ϗ})) : \text{ϗ} \in \mathfrak{X}\}$$

$$\mathfrak{B} = \{(\text{ϗ}, (\text{Ѣ}^{+}{}_{\mathfrak{B}}(\text{ϗ}), (\text{Ѣ}^{-}{}_{\mathfrak{B}}(\text{ϗ})) : \text{ϗ} \in \mathfrak{X}\}$$

The set operations for BPVHFSs are defined as:

$$\mathfrak{A} \cup \mathfrak{B} = \{\text{ҕ} \in \text{Ѣ}_{\mathfrak{A}}(\text{ϗ}) \cup \text{Ѣ}_{\mathfrak{B}}(\text{ϗ}) : (\text{Ѣ}^{+}{}_{\mathfrak{A}} \cup \text{Ѣ}^{+}{}_{\mathfrak{B}})(\text{ϗ}), (\text{Ѣ}^{-}{}_{\mathfrak{A}} \cup \text{Ѣ}^{-}{}_{\mathfrak{B}})(\text{ϗ})\}$$

$$\mathfrak{A} \cap \mathfrak{B} = \{\text{ҕ} \in \text{Ѣ}_{\mathfrak{A}}(\text{ϗ}) \cup \text{Ѣ}_{\mathfrak{B}}(\text{ϗ}) : (\text{Ѣ}^{+}{}_{\mathfrak{A}} \cap \text{Ѣ}^{+}{}_{\mathfrak{B}})(\text{ϗ}), (\text{Ѣ}^{-}{}_{\mathfrak{A}} \cap \text{Ѣ}^{-}{}_{\mathfrak{B}})(\text{ϗ})\}$$

$$(\mathfrak{A})^{c} = \{(\text{ϗ}, (\text{Ѣ}^{+}{}_{\mathfrak{A}}(\text{ϗ}))^{c}, (\text{Ѣ}^{-}{}_{\mathfrak{A}}(\text{ϗ}))^{c}) : \text{ϗ} \in \mathfrak{X}\}$$

$$(\text{Ѣ}^{+}{}_{\mathfrak{A}}(\text{ϗ}))^{c} = \{1 - \text{ҕ} : \text{ҕ} \in \text{Ѣ}^{+}{}_{\mathfrak{A}}(\text{ϗ})\}$$

$$(\text{Ѣ}^{-}{}_{\mathfrak{A}}(\text{ϗ}))^{c} = \{-1 - \text{ҕ} : \text{ҕ} \in \text{Ѣ}^{-}{}_{\mathfrak{A}}(\text{ϗ})\}$$

$$(\mathfrak{A} \oplus \mathfrak{B})(\text{ϗ}) = \{\text{ҕ}_1 + \text{ҕ}_2 - \text{ҕ}_1\text{ҕ}_2 : \text{ҕ}_1 \in \text{Ѣ}^{+}{}_{\mathfrak{A}}(\text{ϗ}), \text{ҕ}_2 \in \text{Ѣ}^{+}{}_{\mathfrak{B}}(\text{ϗ}), -(\text{ҕ}_1\text{ҕ}_2) : \text{ҕ}_1 \in \text{Ѣ}^{-}{}_{\mathfrak{A}}(\text{ϗ}), \text{ҕ}_2 \in \text{Ѣ}^{-}{}_{\mathfrak{B}}(\text{ϗ})\}$$

$$(\mathfrak{A} \otimes \mathfrak{B})(\text{ϗ}) = \{\text{ҕ}_1\text{ҕ}_2 : \text{ҕ}_1 \in \text{Ѣ}^{+}{}_{\mathfrak{A}}(\text{ϗ}), \text{ҕ}_2 \in \text{Ѣ}^{+}{}_{\mathfrak{B}}(\text{ϗ}), -(-\text{ҕ}_1 - \text{ҕ}_2 - \text{ҕ}_1\text{ҕ}_2) : \text{ҕ}_1 \in \text{Ѣ}^{-}{}_{\mathfrak{A}}(\text{ϗ}), \text{ҕ}_2 \in \text{Ѣ}^{-}{}_{\mathfrak{B}}(\text{ϗ})\}$$

for any $\rho^{\cdot} > 0$

$$\rho^{\cdot}\mathfrak{A}(\text{ϗ}) = \{1 - (1 - \text{ҕ})^{\rho^{\cdot}} : \text{ҕ} \in \text{Ѣ}^{+}{}_{\mathfrak{A}}(\text{ϗ}), -(-\text{ҕ}^{\rho^{\cdot}}) : \text{ҕ} \in \text{Ѣ}^{-}{}_{\mathfrak{A}}(\text{ϗ})\}$$

$$\mathfrak{A}^{\rho^{\cdot}}(\text{ϗ}) = \{\text{ҕ}^{\rho^{\cdot}} : \text{ҕ} \in \text{Ѣ}^{+}{}_{\mathfrak{A}}(\text{ϗ}), -1 - \left(-(-(-1 - \text{ҕ}))^{\rho^{\cdot}}\right) : \text{ҕ} \in \text{Ѣ}^{-}{}_{\mathfrak{A}}(\text{ϗ})\}$$

B. Definition 2: [25]

Let $\text{Ѣ}_{i}\ (i = 1,2,3,4 \ldots n)$ be a set of BPVHFEs and let $\omega^{\cdot} = (\omega^{\cdot}_1, \omega^{\cdot}_2, \omega^{\cdot}_3, \omega^{\cdot}_4, \ldots \omega^{\cdot}_n)^{T}$ be the WV of $\text{Ѣ}_{i}\ (i = 1,2,3,4 \ldots n)$ with $\omega^{\cdot}_{i} \in [0,1]$ and $\sum_{i=1}^{n} \omega^{\cdot}_{i} = 1$, then

1. BPVHFWA Operator is a function $\Psi^{n} \to \Psi$ such that

$$BPVHFWA(\text{Ѣ}_1, \text{Ѣ}_2, \ldots \text{Ѣ}_n) = \bigoplus_{i=1}^{n} (\omega^{\cdot}_{i}\text{Ѣ}_{i})$$

$$= \left\{\left\{1 - \prod_{i=1}^{n} (1 - \text{ҕ}_{i})^{\omega^{\cdot}_{i}} : \text{ҕ}_{i} \in \text{Ѣ}_{i}^{+}\right\}, \left\{-\prod_{i=1}^{n} (-\text{ҕ}_{i})^{\omega^{\cdot}_{i}} : \text{ҕ}_{i} \in \text{Ѣ}_{i}^{-}\right\}\right\}$$

2. BPVHFWG Operator is a function $\Psi^{n} \to \Psi$ such that

$$BPVHFWG(\text{Ѣ}_1, \text{Ѣ}_2, \ldots \text{Ѣ}_n) = \bigotimes_{i=1}^{n} (\text{Ѣ}_{i})^{\omega^{\cdot}_{i}}$$

$$= \left\{\left\{\prod_{i=1}^{n} (\text{ҕ}_{i})^{\omega^{\cdot}_{i}} : \text{ҕ}_{i} \in \text{Ѣ}_{i}^{+}\right\}, \left\{-1 - \prod_{i=1}^{n} \left(-(-(-1 - \text{ҕ}_{i}))^{\omega^{\cdot}_{i}}\right) : \text{ҕ}_{i} \in \text{Ѣ}_{i}^{-}\right\}\right\}$$

3. A GBPVHFWA Operator is a function $\Psi^{n} \to \Psi$ such that

$$GBPVHFWA_{\rho^{\cdot}}(\text{Ѣ}_1, \text{Ѣ}_2, \ldots \text{Ѣ}_n) = \left(\bigoplus_{i=1}^{n} (\omega^{\cdot}_{i}\text{Ѣ}_{i}^{\rho^{\cdot}})\right)^{\frac{1}{\rho^{\cdot}}}$$

$$= \left\{\left\{\left(1 - \prod_{i=1}^{n} (1 - \text{ҕ}_{i}^{\rho^{\cdot}})^{\omega^{\cdot}_{i}}\right)^{\frac{1}{\rho^{\cdot}}} : \text{ҕ}_{i} \in \text{Ѣ}_{i}^{+}\right\}, \left\{-1 - \left(\prod_{i=1}^{n} -\left(-\left(-1\left(-\left((-\text{ҕ}_{i})^{\rho^{\cdot}}\right)^{\omega^{\cdot}_{i}}\right)\right)\right)\right)^{\frac{1}{\rho^{\cdot}}} : \text{ҕ}_{i} \in \text{Ѣ}_{i}^{-}\right\}\right\}$$

$$with\ \rho^{\cdot} > 0$$

4. A GBPVHFWG Operator is a function $\Psi^{n} \to \Psi$ such that

$$GBPVHFWG_{\rho^{\cdot}}(\text{Ѣ}_1, \text{Ѣ}_2, \ldots \text{Ѣ}_n) = \frac{1}{\rho^{\cdot}}\left(\bigoplus_{i=1}^{n} (\rho^{\cdot}\text{Ѣ}_{i})^{\omega^{\cdot}_{i}}\right)$$

$$= \left\{\left\{1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1 - \text{ҕ}_{i})^{\rho^{\cdot}}\right)^{\omega^{\cdot}_{i}}\right)^{\frac{1}{\rho^{\cdot}}} : \text{ҕ}_{i} \in \text{Ѣ}_{i}^{+}\right\}, \left\{-1 - \left(\prod_{i=1}^{n} -\left(-\left(-1 - \left(-(-\text{ҕ}_{i})^{\rho^{\cdot}}\right)\right)\right)^{\omega^{\cdot}_{i}}\right)^{\frac{1}{\rho^{\cdot}}} : \text{ҕ}_{i} \in \text{Ѣ}_{i}^{-}\right\}\right\}$$

$$with\ \rho^{\cdot} > 0$$

C. Definition 3: [25]

Let $\text{Ѣ} = < \text{Ѣ}^{+}, \text{Ѣ}^{-} >$ be a BPVHFE, then the Score function (Accuracy Function) of Ѣ is denoted and defined by:

$$S(\text{Ѣ}) = \frac{1}{\xi_{\text{Ѣ}}}(\xi_{\text{Ѣ}}^{+} + \xi_{\text{Ѣ}}^{-})$$

where $\xi_{\text{Ѣ}}^{+}$ is the sum of elements of Ѣ^{+} and $\xi_{\text{Ѣ}}^{-}$ is the sum of elements of Ѣ^{-}, $\xi_{\text{Ѣ}}$ is the length of Ѣ and $S(\text{Ѣ}) \in [-1, 1]$.

D. Remark 1: [25]

Length of Ѣ^{+} and Ѣ^{-} are not necessarily equal.

For two BPVHFEs Ѣ_1 and Ѣ_2, if

$$S(\text{Ѣ}_1) < S(\text{Ѣ}_2),$$

then Ѣ_1 is said to be minor than Ѣ_2 i.e. $\text{Ѣ}_1 < \text{Ѣ}_2$.

$$S(\text{Ѣ}_1) > S(\text{Ѣ}_2),$$

then Ѣ_1 is said to bigger than Ѣ_2 i.e. $\text{Ѣ}_1 > \text{Ѣ}_2$.

$$S(\text{Ѣ}_1) = S(\text{Ѣ}_2),$$

then Ѣ_1 is indifferent (similar) to Ѣ_2 denoted by $\text{Ѣ}_1 \sim \text{Ѣ}_2$.

III. Ordered Weighted And Hybrid Operators For BPVHFSs

In this section, we define BPVHFOWA operators, BPVHFOWG operators, BPVHFHA operators and BPVHFHG operators. We also explain these operations with the help of examples.

Definition 4:

Let $\mathcal{H}_i (i = 1,2,3,4 \dots n)$ be a set of BPVHFEs and $\mathcal{H}_{\sigma(i)}$ the i^{th} largest among them. Let $\omega' = (\omega'_1, \omega'_2, \omega'_3, \omega'_4, \dots \omega'_n)^T$ be the aggregation associated weight vector of $\mathcal{H}_i (i = 1,2,3,4 \dots n)$ with $\omega'_i \in [0,1]$ and $\sum_{i=1}^{n} \omega'_i = 1$. Then

1. A BPVHFOWA operator is a function $BPVHFOWA: \Psi^n \to \Psi$, such that

$$BPVHFOWA(\mathcal{H}_1, \mathcal{H}_2, \dots \mathcal{H}_n) = \bigoplus_{i=1}^{n} (\omega'_i \mathcal{H}_{\sigma(i)})$$

2. A BPVHFOWG operator is a function $BPVHFOWG: \Psi^n \to \Psi$, such that

$$BPVHFWG(\mathcal{H}_1, \mathcal{H}_2, \dots \mathcal{H}_n) = \bigotimes_{i=1}^{n} (\mathcal{H}_{\sigma(i)})^{\omega'_i}$$

1.1.1. Theorem

Let $\mathcal{H}_i (i = 1,2,3,4 \dots n)$ be a set of BPVHFEs. Then their aggregated value determined by using BPVHFOWA operator or BPVHFOWG operator is a BPVHFE and

$$BPVHFOWA(\mathcal{H}_1, \mathcal{H}_2, \dots \mathcal{H}_n) = \left\{ \left\{ 1 - \prod_{i=1}^{n} (1 - \zeta_{\sigma(i)})^{\omega'_i} : \zeta_{\sigma(i)} \in \mathcal{H}^+_{\sigma(i)} \right\}, \left\{ - \prod_{i=1}^{n} (-\zeta_{\sigma(i)})^{\omega'_i} : \zeta_{\sigma(i)} \in \mathcal{H}^-_{\sigma(i)} \right\} \right\}$$

$$BPVHFWG(\mathcal{H}_1, \mathcal{H}_2, \dots \mathcal{H}_n) = \left\{ \left\{ \prod_{i=1}^{n} (\zeta_{\sigma(i)})^{\omega'_i} : \zeta_{\sigma(i)} \in \mathcal{H}^+_{\sigma(i)} \right\}, \left\{ -1 - \left(- \prod_{i=1}^{n} \left(-(-1 - \zeta_{\sigma(i)}) \right)^{\omega'_i} \right) : \zeta_{\sigma(i)} \in \mathcal{H}^-_{\sigma(i)} \right\} \right\}$$

1.1.2. Definition

Let $\mathcal{H}_i (i = 1,2,3,4 \dots n)$ be a set of BPVHFEs and $\mathcal{H}_{\sigma(i)}$ the i^{th} largest among them.

Let $\omega' = (\omega'_1, \omega'_2, \omega'_3, \omega'_4, \dots \omega'_n)^T$ be the aggregation associated weight vector of $\mathcal{H}_i (i = 1,2,3,4 \dots n)$ with $\omega'_i \in [0,1]$ and $\sum_{i=1}^{n} \omega'_i = 1$. Then

1. A GBPVHFOWA operator is a function $GBPVHFOWA: \Psi^n \to \Psi$, such that

$$GBPVHFOWA_{\rho'}(\mathcal{H}_1, \mathcal{H}_2, \dots \mathcal{H}_n) = \left(\bigoplus_{i=1}^{n} \left(\omega'_i \mathcal{H}^{\rho'}_{\sigma(i)} \right) \right)^{\frac{1}{\rho'}}$$

$with\ \rho' > 0$

$$= \left\{ \left\{ \left(1 - \prod_{i=1}^{n} \left(1 - \zeta^{\rho'}_{\sigma(i)} \right)^{\omega'_i} \right)^{\frac{1}{\rho'}} : \zeta_{\sigma(i)} \in \mathcal{H}^+_{\sigma(i)} \right\}, \left\{ -1 - \left(- \left(\prod_{i=1}^{n} \left(- \left(-1 - \left(- \left((-\zeta_{\sigma(i)})^{\rho'} \right)^{\omega'_i} \right) \right) \right) \right)^{\frac{1}{\rho'}} \right) : \zeta_{\sigma(i)} \in \mathcal{H}^-_{\sigma(i)} \right\} \right\}$$

2. A GBPVHFOWG operator is a function $GBPVHFOWG: \Psi^n \to \Psi$, such that

$$GBPVHFOWG_{\rho'}(\mathcal{H}_1, \mathcal{H}_2, \dots \mathcal{H}_n) = \frac{1}{\rho'} \left(\bigoplus_{i=1}^{n} (\rho' \mathcal{H}_{\sigma(i)})^{\omega'_i} \right)$$

$with\ \rho' > 0$

$$= \left\{ \left\{ 1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1 - \zeta_{\sigma(i)})^{\rho'} \right)^{\omega'_i} \right)^{\frac{1}{\rho'}} : \zeta_{\sigma(i)} \in \mathcal{H}^+_i \right\}, \left\{ -1 - \left(- \prod_{i=1}^{n} \left(\left(- \left(-1 - \left(-(-\zeta_{\sigma(i)})^{\rho'} \right) \right) \right)^{\omega'_i} \right)^{\frac{1}{\rho'}} \right) : \zeta_{\sigma(i)} \in \mathcal{H}^-_{\sigma(i)} \right\} \right\}$$

Example 1:

Let $\mathcal{H}_1 = \{\{0.1, 0.2\}, \{-0.3, -0.2\}\}$, $\mathcal{H}_2 = \{\{0.5, 0.6\}, \{-0.2, -0.1\}\}$ and $\mathcal{H}_3 = \{\{0.9, 0.8\}, \{-0.2, -0.1\}\}$ be three BPVHFEs and let $\omega = (0.3, 0.5, 0.2)^{\frac{1}{T}}$ be the aggregation- associated weight vector. Then

$$S(\mathcal{H}_1) = \frac{1}{\ell_{\mathcal{H}_1}} (\xi^+_{\mathcal{H}_1} + \xi^-_{\mathcal{H}_1}) = \frac{1}{2}(0.1 + 0.2 + (-0.3) + (-0.2)) = -0.1$$

$$S(\mathcal{H}_2) = \frac{1}{\ell_{\mathcal{H}_2}} (\xi^+_{\mathcal{H}_2} + \xi^-_{\mathcal{H}_2}) = \frac{1}{2}(0.5 + 0.6 + (-0.2) + (-0.1)) = 0.8$$

$$S(\mathcal{H}_3) = \frac{1}{\ell_{\mathcal{H}_3}} (\xi^+_{\mathcal{H}_3} + \xi^-_{\mathcal{H}_3}) = \frac{1}{2}(0.9 + 0.8 + (-0.2) + (-0.1)) = 0.7$$

Clearly as

$$S(\mathcal{H}_1) < S(\mathcal{H}_3) < S(\mathcal{H}_2)$$

So

$\mathcal{H}_{\sigma(1)} = \mathcal{H}_2 = \{\{0.5, 0.6\}, \{-0.2, -0.1\}\}$,

$\mathcal{H}_{\sigma(2)} = \mathcal{H}_3 = \{\{0.9, 0.8\}, \{-0.2, -0.1\}\}$

and $\mathcal{H}_{\sigma(3)} = \mathcal{H}_1 = \{\{0.1, 0.2\}, \{-0.3, -0.2\}\}$

Now

$$GBPVHFOWA_1(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) = \bigoplus_{i=1}^{3} (\omega'_i \mathcal{H}_{\sigma(i)})$$

$$= \bigcup_{\zeta_1 \in \mathcal{H}_1, \zeta_2 \in \mathcal{H}_2, \zeta_3 \in \mathcal{H}_3} \left\{ \{1 - (1 - \zeta_2)^{0.3} (1 - \zeta_3)^{0.5} (1 - \zeta_1)^{0.2}\}, \left\{ - \left((-(\zeta_2))^{0.3} (-(\zeta_3))^{0.5} (-(\zeta_1))^{0.2} \right) \right\} \right\}$$

$$= \{\{0.748499, 0.754354, 0.644324, 0.652605, 0.764784, 0.77026, 0.667355, 0.675099\} \{-0.21689, -0.2, -0.15337, -0.14142, -0.17617, -0.16245, -0.12457, -0.11487\}\}$$

$$GBPVHFOWA_2(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) = \left(\bigoplus_{i=1}^{3} \left(\omega'_i \mathcal{H}^2_{\sigma(i)} \right) \right)^{\frac{1}{2}}$$

$$=\bigcup_{\hbar_1\in\mathcal{H}_1,\hbar_2\in\mathcal{H}_2,\hbar_3\in\mathcal{H}_3}\left\{\left\{1-\left((1-(\hbar_2)^2)^{0.3}(1-(\hbar_3)^2)^{0.5}(1-(\hbar_1)^2)^{0.2}\right)^{\frac{1}{2}}\right\},\left\{-1-\left(-\left(\left(-\left(-1-\left(-\left(\left((-(\hbar_2))^2\right)^{0.3}\right)\right)\right)\right)^{\frac{1}{2}}\right)\left(\left(-\left(-1-\left(-\left(\left((-(\hbar_3))^2\right)^{0.5}\right)\right)\right)\right)^{\frac{1}{2}}\right)\left(\left(-\left(-1-\left(-\left(\left((-(\hbar_1))^2\right)^{0.2}\right)\right)\right)\right)^{\frac{1}{2}}\right)\right)\right\}\right\}$$

$$=\{\{0.3683, 0.370241, 0.258863, 0.26114, 0.383151, 0.385046, 0.276287, 0.278511\},\{-0.56486, -0.51506, -0.53846, -0.48564, -0.52151, -0.46674, -0.49248, -0.43439\}\}$$

$$GBPVHFOWG_1(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3)=$$

$$BPVHFOWG(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3)=\bigotimes_{i=1}^{3}\left(\mathcal{H}_{\sigma(i)}\right)^{\acute{\omega}_i}$$

$$=\bigcup_{\hbar_1\in\mathcal{H}_1,\hbar_2\in\mathcal{H}_2,\hbar_3\in\mathcal{H}_3}\left\{\{(\hbar_2)^{0.3}(\hbar_3)^{0.5}(\hbar_1)^{0.2}\},\left\{-1-\left(-\left(-(-1-(\hbar_2))\right)^{0.3}\left(-(-1-(\hbar_3))\right)^{0.5}\left(-(-1-(\hbar_1))\right)^{0.2}\right)\right\}\right\}$$

$$=\{\{0.486197, 0.558494, 0.458391, 0.526553, 0.513531, 0.589892, 0.484162, 0.556156\},\{-0.22108, -0.2, -0.17383, -0.15147, -0.19307, -0.17123, -0.14412, -0.12095\}\}$$

$$GBPVHFOWG_2(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3)=\frac{1}{\rho}\left(\bigoplus_{i=1}^{n}\left(\rho\mathcal{H}_{\sigma(i)}\right)^{\acute{\omega}_i}\right)$$

$$=\bigcup_{\hbar_1\in\mathcal{H}_1,\hbar_2\in\mathcal{H}_2,\hbar_3\in\mathcal{H}_3}\left\{\left\{1-\left(1-\left(1-(1-\hbar_2)^2\right)^{0.3}\left(1-(1-\hbar_3)^2\right)^{0.5}\left(1-(1-\hbar_1)^2\right)^{0.2}\right)^{\frac{1}{2}}\right\},\left\{-1-\left(-\left(\left(\left(-\left(-1-\left(-\left((-(\hbar_2))^2\right)\right)\right)\right)^{0.3}\right)^{\frac{1}{2}}\right)\left(\left(\left(-\left(-1-\left(-\left((-(\hbar_3))^2\right)\right)\right)\right)^{0.5}\right)^{\frac{1}{2}}\right)\left(\left(\left(-\left(-1-\left(-\left((-(\hbar_1))^2\right)\right)\right)\right)^{0.2}\right)^{\frac{1}{2}}\right)\right)\right\}\right\}$$

$$=\{\{0.412435, 0.494075, 0.403988, 0.48297, 0.432031, 0.520179, 0.422997, 0.508084\},\{-0.02543, -0.0202, -0.0179, -0.01264, -0.02092, -0.01567, -0.01336, -0.00807\}\}$$

It seems that BPVHFWA, BPVHFWG, GBPVHFWA and GBPVHFWG operators concern with the weight of BPVHF argument and have no concern with their order. On the other hand BPVHFOWA, BPVHFOWG, GBPVHFOWA and GBPVHFOWG operators deal with the weight of the ordered position of each given argument and give no importance to argument itself. Therefore we need to introduce the hybrid aggregation operators for bipolar-valued hesitant fuzzy arguments. These newly defined hybrid aggregation operators give weight to any given argument and their ordered positions too.

Definition 5:

Let $\mathcal{H}_i\,(i=1,2,3,4\ldots n)$ be a collection of BPVHFEs, $\omega=(\omega_1,\omega_2,\omega_3,\omega_4,\ldots\omega_n)^T$ is their weight vector with $\omega_i\in[0,1]$ and $\sum_{i=1}^{n}\omega_i=1$, $'\mathbf{n}'$ is the balancing coefficient and $\acute{\omega}=(\acute{\omega}_1,\acute{\omega}_2,\acute{\omega}_3,\acute{\omega}_4,\ldots\acute{\omega}_n)^T$ be the aggregation associated weight vector of $\mathcal{H}_i\,(i=1,2,3,4\ldots n)$ with $\acute{\omega}_i\in[0,1]$ and $\sum_{i=1}^{n}\acute{\omega}_i=1$. Then

1. The BPVHFHA operator is a mapping $BPVHFHA:\Psi^n\to\Psi$ such that

$$\mathrm{BPVHFHA}(\mathcal{H}_1,\mathcal{H}_2,\ldots\mathcal{H}_n)=\bigoplus_{i=1}^{n}\left(\acute{\omega}_i\mathcal{H}_{\sigma(i)}\right)$$

$$=\left\{\left\{1-\prod_{i=1}^{n}\left(1-\hbar_{\sigma(i)}\right)^{\acute{\omega}_i}:\hbar_{\sigma(i)}\in\mathcal{H}^{+}_{\sigma(i)}\right\},\left\{-\prod_{i=1}^{n}\left(-\hbar_{\sigma(i)}\right)^{\acute{\omega}_i}:\hbar_{\sigma(i)}\in\mathcal{H}^{-}_{\sigma(i)}\right\}\right\}$$

where $\ddot{\mathbb{H}}_{\sigma(\mathrm{i})}$ is the r^{th} largest of $\dot{\mathbb{H}} = nw_k\mathbb{H}_k (k = 1, 2, 3, \dots, n)$

2. The BPVHFHG operator is a mapping $BPVHFHA: \Psi^n \to \Psi$ such that

$$\mathrm{BPVHFHG}(\ddot{\mathbb{H}}_1, \ddot{\mathbb{H}}_2, \dots \ddot{\mathbb{H}}_n) = \bigotimes_{\mathrm{i}=1}^{n} (\ddot{\mathbb{H}}_{\sigma(\mathrm{i})})^{\omega'_{\mathrm{i}}}$$

$$= \left\{\left\{\prod_{\mathrm{i}=1}^{n} (\xi_{\sigma(\mathrm{i})})^{\omega'_{\mathrm{i}}} : \xi_{\sigma(\mathrm{i})} \in \ddot{\mathbb{H}}^{+}_{\sigma(\mathrm{i})}\right\}, \left\{-1 - \left(-\prod_{\mathrm{i}=1}^{n} \left(-(-1 - \xi_{\sigma(\mathrm{i})})\right)^{\omega'_{\mathrm{i}}}\right) : \xi_{\sigma(\mathrm{i})} \in \ddot{\mathbb{H}}^{-}_{\sigma(\mathrm{i})}\right\}\right\}$$

here $\ddot{\mathbb{H}}_{\sigma(\mathrm{i})}$ is the r^{th} largest of $\dot{\mathbb{H}} = \mathbb{H}_k^{nw_k} (k = 1, 2, 3, \dots, n)$

3. A GBPVHFA operator is a function $GBPVHFHA: \Psi^n \to \Psi$, such that

$$GBPVHFHA_{\rho'}(\mathbb{H}_1, \mathbb{H}_2, \dots \mathbb{H}_n) = \left(\bigoplus_{\mathrm{i}=1}^{n} \left(\omega'_{\mathrm{i}} \mathbb{H}^{\rho'}_{\sigma(\mathrm{i})}\right)\right)^{\frac{1}{\rho'}}$$

$$with\ \rho' > 0$$

$$= \left\{\left\{\left(1 - \prod_{\mathrm{i}=1}^{n} \left(1 - \xi^{\rho'}_{\sigma(\mathrm{i})}\right)^{\omega'_{\mathrm{i}}}\right)^{\frac{1}{\rho'}} : \xi_{\sigma(\mathrm{i})} \in \mathbb{H}^{+}_{\sigma(\mathrm{i})}\right\},\right.$$

$$\left.\left\{-1 - \left(-\left(\prod_{\mathrm{i}=1}^{n} \left(-\left(-1 - \left(-\left((-\xi_{\sigma(\mathrm{i})})^{\rho'}\right)^{\omega'_{\mathrm{i}}}\right)\right)\right)\right)^{\frac{1}{\rho'}}\right) : \xi_{\sigma(\mathrm{i})} \in \mathbb{H}^{-}_{\sigma(\mathrm{i})}\right\}\right\}$$

here $\mathbb{H}_{\sigma(\mathrm{i})}$ is the m^{th} largest of $\dot{\mathbb{H}} = nw_k\mathbb{H}_k (k = 1, 2, 3, \dots, n)$

4. A GBPVHFHG operator is a function $GBPVHFHG: \Psi^n \to \Psi$, such that

$$GBPVHFHG_{\rho'}(\ddot{\mathbb{H}}_1, \ddot{\mathbb{H}}_2, \dots \ddot{\mathbb{H}}_n) = \frac{1}{\rho'}\left(\bigoplus_{\mathrm{i}=1}^{n} (\rho' \ddot{\mathbb{H}}_{\sigma(\mathrm{i})})^{\omega'_{\mathrm{i}}}\right)$$

$$with\ \rho' > 0$$

$$= \left\{\left\{1 - \left(1 - \prod_{\mathrm{i}=1}^{n} \left(1 - (1 - \xi_{\sigma(\mathrm{i})})^{\rho'}\right)^{\omega'_{\mathrm{i}}}\right)^{\frac{1}{\rho'}} : \xi_{\sigma(\mathrm{i})} \in \ddot{\mathbb{H}}^{+}_{\mathrm{i}}\right\},\right.$$

$$\left.\left\{-1 - \left(-\prod_{\mathrm{i}=1}^{n} \left(\left(-\left(-1 - \left(-(-\xi_{\sigma(\mathrm{i})})^{\rho'}\right)\right)\right)^{\acute{\omega}_{\mathrm{i}}}\right)^{\frac{1}{\rho'}}\right) : \xi_{\sigma(\mathrm{i})} \in \ddot{\mathbb{H}}^{-}_{\sigma(\mathrm{i})}\right\}\right\}$$

here $\ddot{\mathbb{H}}_{\sigma(\mathrm{i})}$ is the m^{th} largest of $\dot{\mathbb{H}} = \mathbb{H}_k^{nw_k} (k = 1, 2, 3, \dots, n)$

IV. Multi-Attribute Decision Making Based On Bipolar Valued Hesitant Fuzzy Sets

In this section, we describe a brief technique to deal with MADM problems using BPVHF aggregation operator in which a DM provide information in bipolar valued hesitant fuzzy decision matrix and every element is characterized by BPVHFE.

Consider that we have n substitutes $\mathrm{A}_{\mathrm{i}} (\mathrm{i} = 1,2,3,4, \dots n)$ with m attributes $\kappa_j (j = 1,2,3, \dots m)$ and assume that $w = (w_1, w_2, \dots, w_m)$ be the weight vector such that $w_j \in [-1, 1],\ j = 1,2,3 \dots m$ and $\sum_{j=1}^{m} \omega'_j = 1$. The decision makers assigned values in the form of BPVHFEs $(\mathbb{H}_{\mathrm{i}j})$ for the substitutes A_{i} under the attributes κ_j in the state of being anonymous.

The DM method is based on the following steps:

Step 1:

This is the step of decision matrix formation as each alternative A_{i} has assigned some values in the form of BPVHFEs $(\mathbb{H}_{\mathrm{i}j})$ under some attributes κ_j.

Step 2:

In this step, BPVHFE $\alpha_{\mathrm{i}} (\mathrm{i} = 1,2,3, \dots, n)$ can be obtained for the substitutes $\mathrm{A}_{\mathrm{i}} (\mathrm{i} = 1,2,3, \dots, n)$ by using bipolar-valued hesitant fuzzy hybrid operators.

Step 3:

By applying score function (accuracy function) we get the accuracy value $\mathcal{S}(\alpha_{\mathrm{i}}), (\mathrm{i} = 1,2,3, \dots, n)$ of $\alpha_{\mathrm{i}}\ (\mathrm{i} = 1,2,3, \dots, 4)$.

Step 4:

To get the most suitable alternative $\mathrm{A}_{\mathrm{i}} (\mathrm{i} = 1,2,3, \dots, n)$, we established an order in the score values $\mathcal{S}(\alpha_{\mathrm{i}}), (\mathrm{i} = 1,2,3, \dots, n)$.

Example 2:

A cricket board needs a head coach for their cricket team. The cricket board advertised the post in a news paper and a number of candidates applied for the post. Based on their history in the cricket field initially 4 candidates are called for an interview. The cricket board is going to appoint a coach who possesses the qualities like hard working, Creative, Committed and skillful. For the better future of cricket in the country, the cricket board needs to appoint the most suitable head coach.

Let $\mathrm{A} = \{\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4\}$ be the set of substitutes and $\mathrm{\Sigma} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ be the set of attributes and let $w = (0.25, 0.23, 0.35, 0.17)^T$ be the weight vector of the attributes $\mathrm{\Sigma}_{\mathrm{i}} (\mathrm{i} = 1,2,3,4)$ and $\acute{\omega} = (0.3, 0.4, 0.2, 0.1)^T$ be the aggregation associated weight vector.

Conclusion

In our treatise, we successfully apply aggregation operators of BPVHFSs in a DM problem. The results clearly indicate that either we use BPVHF hybrid averaging operators or BPVHF hybrid geometric operators, we get the same results. Hence these two aggregation operation can be very useful in DM especially in two-sided DM. Future research will involve the generalization to bipolar-valued hesitant neutrosophic information and its aggregation operators.

References

[1] Atanassov K, Intuitionistic Fuzzy Sets, Fuzzy Sets and Systems, 20 (1986) 87-96.

[2] Atanassov K. and Gargov G, interval –valued intuitionistic fuzzy sets, fuzzy sets and systems, 31 (1989) 343-349.

[3] Chauhan A. and Vaish R, a comparative study on decision making methods with Interval Data, Hindawi Publishing Corporation Journal of Computational Engineering (2014).

[4] Chen T, A comparative analysis of score functions for multiple criteria decision-making in intuitionistic fuzzy settings, Information Sciences, 181 (2011) 3652-3676.

[5] Chen N, Xu Z. S, and Xia M, Interval-valued hesitant preference relations and their applications to group decision making, Knowledge-Based Systems, 37 (2013) 528–540.

[6] Chen N, Xu Z. S, properties of interval-valued hesitant fuzzy sets, journal of intelligent and fuzzy systems (2014).

[7] De S. K, Biswas R and Roy A.R, Some operations on intuitionistic fuzzy sets, Fuzzy Sets and Systems (2000) 477–484.

[8] Dubois D, Prade H, Fuzzy sets and systems: theory and applications, Academic Press NewYork 1980.

[9] Dubois D, Prade H, Interval-valued Fuzzy Sets, Possibility Theory and Imprecise Probability, Institute of research in computer science from Toulouse France.

[10] Farhadinia B, information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets, Information Sciences 2013 129-144.

[11] Herrera-Viedma, Alonso E, Chiclana S, Herrera F, A consensus model for group decision making with incomplete fuzzy preference relations, IEEE Transactions on Fuzzy Systems 15 2007 863–877.

[12] Hong, D. H. and Choi, Multi-criteria fuzzy decision making problems based on vague set theory, Fuzzy Sets and Systems, 21, 2000 1-17.

[13] Hwang C.L, and Yoon K, Multiple Attribute Decision Making Methods and Application Springer New York 1981.

[14] Klutho S, Mathematical Decision Making an Overview of the Analytic Hierarchy Process 2013.

[15] Lee K. M, Bipolar-Valued Fuzzy sets and their operations, Proc. Int. Conf. on intelligent Technologies, Bangkok, Thailand, 2000, 307-312.

[16] Li D, Multiattribute decision making models and methods using intuitionistic fuzzy sets, Journal of Computer and System Sciences, 70, 2005 73-85.

[17] Liu, H.W and Wang G.J, multi-criteria decision making methods based on intuitionistic fuzzy sets, European journal of operational research 2007 220-233.

[18] Muhammad A, Saleem A and Kifayat U. Bipolar Fuzzy Soft Sets and its application in decision making problem, Journal of intelligent and fuzzy system, 27 (2014) 729-742.

[19] Vladareanu, V; Dumitrache, I; Vladareanu, L; Sacala, IS; Tont, G; Moisescu, MA, "Versatile Intelligent Portable Robot Control Platform Based on Cyber Physical Systems Principles", Studies In Informatics And Control, Volume: 24 Issue: 4 Pages: 409-418 Published: DEC 2015, WOS:000366543700005, ISSN: 1220-1766.

[20] Vladareanu, V; Schiopu, P; Vladareanu, L , "Theory and Application of Extension Hybrid Force-Position Control In Robotics", University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics Volume: 76 Issue: 3 Pages: 43-54 Published: 2014, WOS:000340335100004, ISSN: 1223-7027

[21] Shabir M, Khan I. A, Interval-valued fuzzy ideals generated by an interval valued fuzzy subset in ordered semigroups, Mathware and soft computing 15 2008 263-272.

[22] Torra V, Narukawa Y, On hesitant fuzzy sets and decision, In The 18th IEEE International Conference on Fuzzy Systems, Jeju Island, Korea 2009 1378–1382.

[23] Torra V, Hesitant fuzzy sets, International Journal of Intelligent Systems 25 2010 529–539.

[24] Tahir M, Muhammad M, On bipolar-valued fuzzy subgroups, World Applied Sciences Journal 27(12) (2013) 1806-1811.

[25] Tahir M, Kifayat U, Qaiser K, bipolar-valued hesitant fuzzy sets and their applications in multi-attribute decision making, international journal of algebra and statistics, Submitted.

[26] Wang W and Xin X, Distance measure between intuitionistic fuzzy sets, Pattern Recognition Letters, 26 (13) 2005 2063-2069

[27] Smarandache F., Vladareanu L., "Applications of Neutrosophic Logic to Robotics - An Introduction", The 2011 IEEE International Conference on Granular Computing Kaohsiung, Taiwan, Nov. 8-10, 2011, pp. 607-612, ISBN 978-1-4577-0370-6

[28] Victor Vladareanua, Radu I. Munteanub, Ali Mumtazc, Florentin Smarandached and Luige Vladareanua , "The optimization of intelligent control interfaces using Versatile Intelligent Portable Robot Platform", Procedia Computer Science 65 (2015): 225 – 232, ELSEVIER, www.sciencedirect.com, doi:10.1016/j.procs.2015.09.115.

[29] Wu D, Mendel J. M, A comparative study of ranking methods, similarity measures and uncertainty measures for interval type-2 fuzzy sets, Information Sciences 179 2009 1169–1192.

[30] Xia M, Xu Z.S, Hesitant fuzzy information aggregation in decision making, international journal of approximate reasoning 2011 395-407.

[31] Xia M. M, Xu Z. S, Chen, N, Some hesitant fuzzy aggregation operators with their application in group decision making, Group Decision and Negotiation 22 2013 259–279.

[32] Xia M. M, Xu Z. S, Managing hesitant information in GDM problems under fuzzy and multiplicative preference relations, International Journal of Uncertainty, Fuzziness and Knowledge-based Systems 21 2013 865–897.

[33] Xu Z. S, A method based on linguistic aggregation operators for group decision making with linguistic preference relations, Information Sciences 166 2004 19–30.

[34] Xu Z. S, Yager R. R, Some geometric aggregation operators based on intuitionistic fuzzy sets, International Journal of General Systems 35, 2006 417–433.

[35] Xu Z. S, Intuitionistic fuzzy aggregation operators, IEEE Transactions on Fuzzy Systems 15 20071179-1187.

[36] Xu Z and Chen J, an approach to group decision making based on interval-valued intuitionistic fuzzy judgment matrices, System Engineer-Theory and Practice 2007.

[37] Ye J, Improved method of multicriteria fuzzy decision-making based on vague sets, Computer-Aided Design 39 (2) 2007 164-169.

[38] Ye J, Multicriteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment, Expert Systems with Applications vol. 36 no. 3 2009 6899-6902.

[39] Zadeh L, Fuzzy sets, Information and Control 8 (3) 1965 338.353.

[40] Zadeh L, The concept of a linguistic variable and its application to approximate reasoning-I, Information Sciences vol. 8 (3) 1975 199-249.

[41] Zhang X, Liu P. D, Method for multiple attribute decision-making under risk with interval numbers, International Journal of Fuzzy Systems 12 2010 237–242.

[42] Zhang Z, interval-valued intuitionistic hesitant fuzzy aggregation operators and their application in group decision-making, journal of applied mathematics 2013.

[43] Zheng P, A note on operations of hesitant fuzzy sets, International Journal of Computational Intelligence Systems, 8(2) 2015 226-239.

[44] Zhu B, Xu Z. S and Xia M. M, Hesitant fuzzy geometric Bonferroni means, Information Sciences 2012 72–85.