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Some Applications of Dilatation Invariance to Structural Questions in the Theory of Local Observables

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Abstract. In a dilatation-invariant theory it is shown that there is a unique locally normal dilatation-invariant state. Furthermore a gauge transformation of a local algebra cannot be implemented by a unitary operator from the local algebra. If the local field algebras are factors then so are the local observable algebras. The superselection structure of the theory can be determined locally.

1. Introduction

Various structural features of the algebra of local observables depend on the short-distance behaviour of the theory. Our aim is to study certain of these features by looking at what has become known as the Gell-Mann Low limit of the theory in hommage to the work of Gell-Mann and Low on the high-energy behaviour of quantum electrodynamics [1]. Roughly speaking the Gell-Mann Low limit of a theory is a dilatationally invariant theory with the same short-distance behaviour as the original theory.

In this paper we shall simplify matters by discussing dilatationally invariant theories; such theories are, so to speak, their own Gell-Mann Low limits. This enables the reader to focus his attention on the way the singular short-distance behaviour shapes the theory without being distracted by the technical assumptions on how the theory attains its Gell-Mann Low limits.

As an example of how structural features of the algebra of local observables can depend on the short-distance behaviour of the theory, consider the problem of whether one can measure the total electric charge contained within a sphere of radius R by means of an observation inside the sphere. Formally the operator

$$Q_R = \int_{|\boldsymbol{x}| \leq R} j_0(0, \boldsymbol{x})\, d^3\boldsymbol{x}\,, \qquad (1.1)$$

where j_0 denotes the time-component of the electric current, should do the trick for us. Experience shows that only expressions of the form $\int j_0(t, \boldsymbol{x}) f(t, \boldsymbol{x}) dt\, d^3\boldsymbol{x}$ where f is a suitably smooth function can be expected to exist as an operator. In (1.1) we are trying to take $f(t, \boldsymbol{x}) = \delta(t)\chi_R(\boldsymbol{x})$ where $\chi_R(\boldsymbol{x})$ denotes the characteristic function of the sphere $|\boldsymbol{x}| \leqq R$. The problem here is not the δ-function in the time because this is taken care of by the current conservation law but rather the discontinuity of χ_R at the boundary of the sphere. In this sense the question of whether Q_R is well defined depends on the short-distance behaviour of the theory. We expect that all reasonable theories are so singular at short distances that operators like Q_R are not well defined.

In the algebraic framework where physics is described in terms of the net[1] $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ of local observables it is natural to reformulate the above question in terms of two related questions which are of interest in their own right. The first question is whether a local observable algebra $\mathfrak{A}(\mathcal{O})$ is a factor, i.e. whether the centre of $\mathfrak{A}(\mathcal{O})$ consists of multiples of the identity. The only physically motivated candidates for elements of the centre of $\mathfrak{A}(\mathcal{O}_R)$ known to the author are bounded functions of the Q_R or related objects associated with other charge quantum numbers. The second question arises when we consider j_0 as the generator of gauge transformations of the first kind. Here it is natural to look at the net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ as the gauge-invariant part of the net $\mathcal{O} \to \mathfrak{F}(\mathcal{O})$ of local fields and to ask whether the gauge transformations $\mathfrak{F}(\mathcal{O})$ can be implemented by a unitary operator from $\mathfrak{F}(\mathcal{O})$ itself. Here again there is a natural condidate, namely an operator of the form $\exp i\lambda Q_R$.

The first question is not answered here completely, but it is shown that $\mathfrak{A}(\mathcal{O})$ is a factor whenever $\mathfrak{F}(\mathcal{O})$ is (Theorem 3.6). The second question is answered negatively: local gauge transformations cannot be locally implemented (Theorem 3.2). For the case of free fields this second result has been obtained by Dell' Antonio [2]. To illustrate the influence of short distance behaviour at its simplest, we show that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated double cones with a boundary point in common then there is no locally normal state ω of $\mathfrak{A}$ which factorizes over $\mathcal{O}_1$ and $\mathcal{O}_2$, i.e. such that

$$\omega(A_1 A_2) = \omega(A_1)\,\omega(A_2), \qquad A_1 \in \mathfrak{A}(\mathcal{O}_1), \qquad A_2 \in \mathfrak{A}(\mathcal{O}_2). \tag{1.2}$$

It follows in particular that $\mathfrak{A}(\mathcal{O})$ cannot be Type I.

[1] This is an inclusion-preserving mapping $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ between finite regions in space-time and C^*-algebras. For the purposes of this paper it will suffice to take $\mathcal{O}$ to be a double cone (the intersection of a forward light cone with a backward light cone) and to suppose that $\mathfrak{A}(\mathcal{O})$ is actually a von Neumann algebra. $\mathcal{O}_R$ denotes the double cone based on a sphere of radius R centred at the origin. $\mathfrak{A}$ denotes the global C^*-algebra which is the inductive limit of the local algebras $\mathfrak{A}(\mathcal{O})$.

We also examine the consequences of dilatation invariance for the superselection structure of the theory and show that this structure is determined by the local algebras $\mathfrak{A}(\mathcal{O})$ and $\mathfrak{F}(\mathcal{O})$ for some fixed $\mathcal{O}$ without reference to the corresponding nets.

The results which are deduced here for dilatationally invariant theories can also be proved for theories which possess a Gell-Mann Low limit in a sense which will be made precise in a sequel.

As a by-product of the study of dilatationally invariant theories we show that there is a unique locally normal dilatationally invariant vacuum. Consequently a symmetry commuting with dilatations can only be spontaneously broken if the dilatation invariance is itself spontaneously broken [2].

In a sense the short-distance behaviour of every local relativistic quantum field theory is singular because it is impossible to define the value of a field at a point as an operator in Hilbert space. In fact the whole of our analysis hinges on a result of Wightman [3] which is the algebraic expression of this fact:

$$\bigcap_{\mathcal{O} \ni 0} \mathfrak{A}(\mathcal{O}) = \mathbb{C} I . \tag{1.3}$$

The intersection in (1.3) is taken over all double cones containing the origin as an (interior) point, so (1.3) expresses the fact that there are no bounded local operators based on the origin.

2. Dilatation Invariance

The assumptions we make in this paper are, with the exception of dilatation invariance, standard assumptions of algebraic quantum theory for treating the observable algebra as the gauge-invariant part of a field algebra. We list them here briefly and refer the reader for example to the introduction of [4] for a more leisurely treatment of similar assumptions.

The field algebra $\mathfrak{F}$ is assumed to be the global algebra of a net $\mathcal{O} \to \mathfrak{F}(\mathcal{O})$ of von Neumann algebras and to act irreducibly on a Hilbert space $\mathcal{H}$. There is a continuous unitary representation $L \to \mathcal{U}(L)$ of the covering group of the Poincaré group $\mathcal{P}$ on $\mathcal{H}$ which induces automorphisms α_L of the field algebra: $\alpha_L(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(L\mathcal{O})$. There is a unit vector $\Omega \in \mathcal{H}$, the vacuum vector, invariant under the $\mathcal{U}(L)$, $L \in \mathcal{P}$ and inducing the vacuum state ω_0 of $\mathfrak{F}$.

$$\omega_0(F) = (\Omega, F\Omega) . \tag{2.1}$$

[2] In this case the invariant vacuum state is not a pure state and its pure components are not dilatation invariant.

The energy-momentum spectrum is contained in the forward light cone and Ω is supposed to be a cyclic and separating vector for each $\mathfrak{F}(\mathcal{O})$ (the Reeh-Schlieder property).

There is a faithful, strongly continuous unitary representation $g \to \mathscr{V}(g)$ of a compact group $\mathscr{G}$, the gauge group, which commutes with the $\mathscr{U}(L)$, leaves Ω invariant and induces automorphisms β_g of each local field algebra:

$$\beta_g(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(\mathcal{O}), \tag{2.2}$$

$$\mathscr{V}(g)\,\Omega = \Omega\,, \qquad g \in \mathscr{G}\,. \tag{2.3}$$

The net of local observables $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ is just the gauge-invariant part of $\mathcal{O} \to \mathfrak{F}(\mathcal{O})$, i.e.

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{F}(\mathcal{O}) \cap \mathscr{V}(\mathscr{G})'\,. \tag{2.4}$$

To avoid irrelevant complications, we shall assume the simple commutation relations in the field algebra at spacelike separations which would follow if $\mathfrak{F}(\mathcal{O})$ were generated in the usual way by Bose and Fermi fields. The simplest way of expressing this mathematically is to suppose that there is an element $k \in \mathscr{G}$ with $k^2 = e$ so that if we set

$F_+ = \frac{1}{2}(F + \beta_k(F))$ and $F_- = \frac{1}{2}(F - \beta_k(F))$ then

$$\begin{aligned} &F_+ F'_+ - F'_+ F_+ = 0 \\ &F_+ F'_- - F'_- F_+ = 0 \qquad F \in \mathfrak{F}(\mathcal{O}_1),\ F' \in \mathfrak{F}(\mathcal{O}_2),\ \mathcal{O}_1 \subset \mathcal{O}'_2\,, \\ &F_- F'_- + F'_- F_- = 0 \end{aligned} \tag{2.5}$$

Of course F_+ and F_- are just the Bose and Fermi parts of F respectively and β_k is the gauge automorphism leaving Bose fields invariant but changing the sign of Fermi fields. As would be expected from this, we have

2.1 Lemma. *k is in the centre of $\mathscr{G}$.*

Proof. Suppose $\beta_k(F) = F \in \mathfrak{F}(\mathcal{O})$ and let $g \in \mathscr{G}$ then

$$F^* \alpha_x(F) = \alpha_x(F)\, F^*\,, \qquad \mathcal{O} + x \subset \mathcal{O}'\,.$$

Hence if $G = \beta_g(F)$ we have

$$G^* \alpha_x(G) = \alpha_x(G)\, G^*\,, \qquad \mathcal{O} + x \subset \mathcal{O}'\,.$$

This implies that

$$0 = G^*_- \alpha_x(G)_- = G^*_- \alpha_x(G_-)\,, \qquad \mathcal{O} + x \subset \mathcal{O}'\,.$$

But $x \to G^*_- \alpha_x(G_-)\,\Omega$ is the boundary value of a function analytic in the forward tube, so $G^*_- \alpha_x(G_-)\,\Omega = 0$ for all x. Setting $x = 0$ we have by the Reeh-Schlieder property, $G_-^* G_- = 0$ which implies $G_- = 0$. Hence $G = G_+$, i.e. $\beta_k \beta_g(F) = \beta_g(F)$. Thus if E denotes the projection onto the closure of $\{F_+ \Omega : F \in \mathfrak{F}(\mathcal{O})\}$ we have

$$E\mathcal{V}(g)\,E = \mathcal{V}(g)\,E\,.$$

Replacing g by g^{-1} and taking adjoints we get $\mathcal{V}(g)\,E = E\mathcal{V}(g)$. But $\mathcal{V}(k) = 2E - I$ so $\mathcal{V}(gk) = \mathcal{V}(kg)$. As $\mathcal{V}$ is faithful, $gk = kg$ completing the proof.

The starting point of this work is the following:

2.2 Theorem (Wightman [3c) $\bigcap_{\mathcal{O} \ni 0} \mathfrak{F}(\mathcal{O}) = \mathbb{C} I$.

Actually Wightman only considers the Bose case but that is quite sufficient because for the Fermi part of $\bigcap_{\mathcal{O} \ni 0} \mathfrak{F}(\mathcal{O})$ we could for example apply his argument to the square of the two-point function.

2.3 Corollary. *Let ω_0 be the vacuum state and ω any locally normal state of $\mathfrak{F}$ then*

$$\|(\omega_0 - \omega) \upharpoonright \mathfrak{F}(\mathcal{O})\| \to 0$$

as $\mathcal{O}$ shrinks to $\{0\}$.

Proof. This is standard: suppose not then there is a sequence $F_n \in \mathfrak{F}(\mathcal{O}_n)$ where $\bigcap_n \mathcal{O}_n = \{0\}$ with $\|F_n\| \leqq 1$ and $\omega_0(F_n) - \omega(F_n) > \delta > 0$. However any weak limit point F of $F_n - \omega_0(F_n)\,I$ must be a multiple of the identity by Theorem 2.2 and trivially $\omega_0(F) = 0$. Hence $F_n - \omega_0(F_n)\,I$ tends weakly to zero. Since ω is locally normal, $\omega(F_n) - \omega_0(F_n) \to 0$ giving a contradiction.

The specific assumption of this paper concerns dilatation invariance. We suppose we have a strongly continuous unitary representation $\lambda \to \mathcal{U}(\lambda)$ of the multiplicative group of the positive real line $\mathbb{R}_+$ satisfying

$$\mathcal{U}(\lambda)\,\Omega = \Omega\,, \tag{2.6}$$

$$\mathcal{U}(\lambda)\,\mathcal{U}(a, \Lambda) = \mathcal{U}(\lambda a, \Lambda)\,\mathcal{U}(\lambda), \qquad \{a, \Lambda\} \in \mathcal{P}\,, \tag{2.7}$$

$$\mathcal{U}(\lambda)\,\mathcal{V}(g) = \mathcal{V}(g)\,\mathcal{U}(\lambda), \qquad g \in \mathcal{G}\,, \tag{2.8}$$

and inducing automorphisms δ_λ of the field algebra with

$$\delta_\lambda(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(\lambda \mathcal{O}) \tag{2.9}$$

2.4 Theorem. a) *ω_0 is the only dilatationally invariant locally normal state.*

b) *If ω is a locally normal state then*

$$\|(\omega \circ \delta_\lambda - \omega_0) \upharpoonright \mathfrak{F}(\mathcal{O})\| \to 0 \quad \text{as} \quad \lambda \to 0\,.$$

Proof. If $\omega \neq \omega_0$ then we can find a double cone $\mathcal{O}$ such that $\|(\omega - \omega_0) \upharpoonright \mathfrak{F}(\mathcal{O})\| \neq 0$ so a) will follow from b). However, by (2.6) and (2.9) we have $\|(\omega \circ \delta_\lambda - \omega_0) \upharpoonright \mathfrak{F}(\mathcal{O})\| = \|(\omega - \omega_0) \upharpoonright \mathfrak{F}(\lambda\mathcal{O})\|$ and the result follows from Corollary 2.3, since it is clearly no loss of generality to suppose that $\mathcal{O}$ contains the origin.

This result has some relevance to questions of spontaneously broken symmetries. If ω_0 were not a pure state we would have a case where dilatation invariance was spontaneously broken. If ω_0 is a pure state and we have an internal symmetry of the system represented by an automorphism α commuting with dilatations, $\alpha \circ \delta_\lambda = \delta_\lambda \circ \alpha$, $\alpha(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(\mathcal{O})$ then $\omega_0 \circ \alpha$ is locally normal and dilatation invariant so $\omega_0 \circ \alpha = \omega_0$ and α is not spontaneously broken.

Another useful corollary of the basic theorem of Wightman in the case of dilatation-invariant theories is

2.5 Corollary. *If $F \in \mathfrak{F}$ then $\delta_\lambda(F)$ tends weakly to $\omega_0(F)I$ as $\lambda \to 0$.*

Proof. It suffices to suppose that F is strictly local then any weak limit point of $\delta_\lambda(F)$ is a multiple of the identity by Theorem 2.2. Since ω_0 is dilatation invariant, this limit point can only be $\omega_0(F)I$.

3. Applications to the Structure of Local Nets

Our first result is elementary and uses nothing more than the dilatation invariance of the observable net.

3.1 Proposition. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two double cones with a boundary point in common. Then there is no locally normal state ω of $\mathfrak{A}$ such that*

$$\omega(AB) = \omega(A)\,\omega(B), \qquad A \in \mathfrak{A}(\mathcal{O}_1), \qquad B \in \mathfrak{A}(\mathcal{O}_2). \tag{3.1}$$

Proof. By translation invariance we may suppose the origin is a common boundary point of $\mathcal{O}_1$ and $\mathcal{O}_2$ then for $\lambda \leqq 1$ we have $\delta_\lambda(\mathfrak{A}(\mathcal{O}_1)) \subset \mathfrak{A}(\mathcal{O}_1)$ and $\delta_\lambda(\mathfrak{A}(\mathcal{O}_2)) \subset \mathfrak{A}(\mathcal{O}_2)$. Hence

$$\omega \circ \delta_\lambda(AB) = \omega \circ \delta_\lambda(A)\,\omega \circ \delta_\lambda(B), \qquad A \in \mathfrak{A}(\mathcal{O}_1), \qquad B \in \mathfrak{A}(\mathcal{O}_2), \qquad \lambda \leqq 1\,.$$

Proceeding to the limit as $\lambda \to 0$ we deduce from Theorem 2.4 that

$$\omega_0(AB) = \omega_0(A)\,\omega_0(B)\,.$$

Pick $B \neq 0$ with $\omega_0(B) = 0$; since Ω is cyclic for $\mathfrak{A}(\mathcal{O}_1)$ in the vacuum sector, we may pick A such that $\omega_0(AB) \neq 0$ giving a contradiction.

This tells us in particular that $\mathfrak{A}(\mathcal{O})$ is not a Type I factor. The negative result of Proposition 3.1 is not surprising: however if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated double cones without a boundary point in common then Buchholz [5] has shown for the free massive neutral scalar field that one can find locally normal states satisfying (3.1).

In the sequel we have occasion to use spacelike separated double cones $\mathcal{O}_1$ and $\mathcal{O}_2$ with a common boundary point and we abbreviate this by saying that $\mathcal{O}_2$ is a spacelike tangent to $\mathcal{O}_1$.

3.2 Theorem. *β_g is an outer automorphism of $\mathfrak{F}(\mathcal{O})$ if $g \neq e$.*

Proof. In fact we prove that if $V \in \mathfrak{F}(\mathcal{O})$ and if

$$VF = \beta_g(F)\, V, \qquad F \in \mathfrak{F}(\mathcal{O}) \tag{3.2}$$

then either $V = 0$ or $g = \mathrm{e}$. If $\mathfrak{F}(\mathcal{O})$ is not a factor this is a more general result, although we choose this form for technical convenience. Now (3.2) and Lemma 2.1 imply

$$\beta_k(V)\, F = \beta_g(F)\, \beta_k(V), \qquad F \in \mathfrak{F}(\mathcal{O}).$$

Hence it suffices to consider the two special cases, $V = \beta_k(V)$ and $V = -\beta_k(V)$. Suppose $V = \beta_k(V)$ and let $\mathcal{O}_1$ be a spacelike tangent double cone to $\mathcal{O}$ then

$$\omega_0(VF'FV^*) = \omega_0(F'\beta_g(F)\, VV^*), \qquad F' \in \mathfrak{F}(\mathcal{O}_1), \qquad F \in \mathfrak{F}(\mathcal{O}). \tag{3.3}$$

Without loss of generality we may take the origin as the point of contact of $\mathcal{O}$ and $\mathcal{O}_1$. Then (3.3) remains valid if F and F' are replaced by $\delta_\lambda(F)$ and $\delta_\lambda(F')$, $\lambda \leqq 1$. Taking the limit as $\lambda \to 0$ we get by Corollary 2.5 and (2.8)

$$\omega_0(VV^*)\, \omega_0(F'F) = \omega_0(VV^*)\, \omega_0(F'\beta_g(F)).$$

So either $\omega_0(VV^*) = 0$ which implies $V = 0$ by the Reeh-Schlieder property or $\omega_0(F'F) = \omega_0(F'\beta_g(F))$ which again by the Reeh-Schlieder property implies $\mathcal{V}(g) = I$. However $\mathcal{V}$ is faithful so $g = e$, or $V = 0$. If $V = -\beta_k(V)$ then we would get in place of (3.3)

$$\omega_0(VF'FV^*) = \omega_0(\beta_k(F')\, \beta_g(F)\, VV^*).$$

If $V \neq 0$ the same argument as above would allow us to conclude that

$$\omega_0(F'F) = \omega_0(\beta_k(F')\, \beta_g(F)) = \omega_0(F'\beta_{kg}(F))$$

and hence that $g = k$. Thus $VV^* = \beta_g(V^*)\, V = \beta_k(V^*)\, V = -V^*V$ so $V = 0$, and this contradiction completes the proof.

We come now to the result on the factoriality of the local observable algebra. We may decompose the representation $\mathcal{V}$ of $\mathcal{G}$ into its irreducible components and let Σ denote the set of equivalence classes of continuous irreducible representations of $\mathcal{G}$ occurring in $\mathcal{V}$. In fact, as shown in [4], every such equivalence class occurs in the decomposition but this is irrelevant to our present discussion.

3.3 Lemma. *If E is a non-zero projection of $\mathfrak{A}(\mathcal{O})$ and if $\sigma \in \Sigma$ then there is $F \in \mathfrak{F}(\mathcal{O})$ which transforms under gauge transformations like an irreducible tensor of type σ and satisfies $EFE \neq 0$.*

Proof. Since $\sigma \in \Sigma$, Ω is cyclic for $\mathfrak{F}(\mathcal{O})$ and $\mathfrak{F}(\mathcal{O})$ is weakly closed we can pick a non-zero tensor F of type σ from $\mathfrak{F}(\mathcal{O})$ (see the discussion preceding Lemma 3.4 below). Let $\mathcal{O}_1$ be a spacelike tangent double cone to $\mathcal{O}$ and pick $F' \in \mathfrak{F}(\mathcal{O}_1)$ such that $\omega_0(F'F) \neq 0$. By Corollary 2.5 if λ is sufficiently small $\omega_0(E\,\delta_\lambda(F'F)\,E) \neq 0$. Taking the origin to be the point of contact of $\mathcal{O}$ and $\mathcal{O}_1$,

$$\delta_\lambda(F') \in \mathfrak{F}(\mathcal{O}_1), \quad \text{so} \quad \omega_0(\delta_\lambda(F')\,E\,\delta_\lambda(F)\,E) = \omega_0(E\,\delta_\lambda(F'F)\,E) \neq 0\,.$$

Hence $E\,\delta_\lambda(F)\,E \neq 0$. Also $\delta_\lambda(F) \in \mathfrak{F}(\mathcal{O})$ and by (2.8) $\delta_\lambda(F)$ is an irreducible tensor of type σ under $\mathcal{G}$.

It now follows from a result of Connes [6; Théorème 2.4.1] that if $\mathfrak{F}(\mathcal{O})$ is a factor and $\mathcal{G}$ is Abelian then $\mathfrak{A}(\mathcal{O})$ is a factor. What follows is an extension of this result to the case where $\mathcal{G}$ is not Abelian; here Lemma 3.3 is not adequate and the appropriate generalization requires a little preliminary work.

Given $\sigma \in \Sigma$ let H_σ be a Hilbert space carrying an irreducible unitary representation u_σ of $\mathcal{G}$ of class σ. Consider the space $\mathfrak{F}(\mathcal{O}) \otimes H_\sigma$ as a space of row matrices with entries in $\mathfrak{F}(\mathcal{O})$ by picking a basis $b_1, b_2, \dots b_d$, $d = \dim\sigma$, of H_σ and expressing $\boldsymbol{F} = \sum_i F_i \otimes b_i \in \mathfrak{F}(\mathcal{O}) \otimes H_\sigma$ as $(F_1, F_2, \dots, F_d)$. We may define an action β of $\mathcal{G}$ and δ of $\mathbb{R}_+$ on $\mathfrak{F}(\mathcal{O}) \otimes H_\sigma$ by acting with β_g and δ_λ respectively on the components of $\boldsymbol{F}$. Set

$$\mathfrak{F}_\sigma(\mathcal{O}) = \{\boldsymbol{F} \in \mathfrak{F}(\mathcal{O}) \otimes H_\sigma : \beta_g(\boldsymbol{F}) = \boldsymbol{F}\,u_\sigma(g),\ g \in \mathcal{G}\}\,, \tag{3.4}$$

where $\boldsymbol{F}\,u_\sigma(g)$ denotes the matrix multiplication of $\boldsymbol{F}$ on the right by $u_\sigma(g)$. Given $b \in H_\sigma$ define

$$m_{b_i,b}(F) = \int \langle u_\sigma(g)\,b, b_i\rangle\, \beta_g(F)\, d\mu(g)\,. \tag{3.5}$$

An elementary computation shows that if $F \in \mathfrak{F}(\mathcal{O})$ and $\boldsymbol{F} = (F_i)$ where $F_i = m_{b_i,b}(F)$ then $\boldsymbol{F} \in \mathfrak{F}_\sigma(\mathcal{O})$. This construction and the cyclicity of Ω for $\mathfrak{F}(\mathcal{O})$ can be used to produce non-zero elements of $\mathfrak{F}_\sigma(\mathcal{O})$, (compare [4; Lemma 3.4] and [7; Theorem 4]).

3.4 Lemma. *Given $\boldsymbol{F}, \boldsymbol{F}' \in \mathfrak{F}_\sigma(\mathcal{O})$ then*

$$\omega_0(F_i F_j'^*) = \delta_{ij}\,\frac{1}{d} \sum_k \omega_0(F_k F_k'^*)\,.$$

Proof.

$$\begin{aligned}\omega_0(F_i F_j'^*) &= \int \omega_0(\beta_g(F_i)\,\beta_g(F_j'^*))\,d\mu(g)\\ &= \int \sum_{k,l} u_\sigma(g)_{ki}\,\overline{u}_\sigma(g)_{lj}\,\omega_0(F_k F_l'^*)\,d\mu(g)\,.\end{aligned}$$

The orthogonality relations for the irreducible representation u_σ now give the required result.

We also consider the algebra $\mathfrak{F}(\mathcal{O})\otimes\mathscr{B}(H_\sigma)$ as the algebra of $d\times d$-matrices with entries in $\mathfrak{F}(\mathcal{O})$ and define an action β of $\mathscr{G}$ on $\mathfrak{F}(\mathcal{O})\otimes\mathscr{B}(H_\sigma)$ by letting β_g act on each component of the matrix. The basic technical difference between the Abelian and the non-Abelian case is that in the former case the support projections of irreducible tensors are in $\mathfrak{A}(\mathcal{O})$ whereas in the latter case they must be considered as elements of $\mathfrak{F}(\mathcal{O})\otimes\mathscr{B}(H_\sigma)$. We now state the appropriate generalization of Lemma 3.3.

3.5 Lemma. *Let $\sigma\in\Sigma$ and $0\neq \boldsymbol{F}\in\mathfrak{F}_\sigma(\mathcal{O})$. Let $E'\in\mathfrak{F}(\mathcal{O})\otimes\mathscr{B}(H_\sigma)$ be the support projection of $\boldsymbol{F}$ and $E'\leqq E\otimes I_\sigma$ where E is a projection from $\mathfrak{A}(\mathcal{O})$ then there exists a $\boldsymbol{G}\in\mathfrak{F}_\sigma(\mathcal{O})$ such that $E\boldsymbol{G}E'\neq 0$.*

Proof. Since $\boldsymbol{F}$ is an irreducible tensor and k is in the center of $\mathscr{G}$ by Lemma 2.1, we must have $\beta_k(\boldsymbol{F})=\boldsymbol{F}u_\sigma(k)=\pm\boldsymbol{F}$. But $\beta_k(E')$ is the support projection of $\beta_k(\boldsymbol{F})$ so $\beta_k(E')=E'$. Having established this, we may forget about $\boldsymbol{F}$. Let $\mathcal{O}_1$ be a spacelike tangent double cone to $\mathcal{O}$. By the Reeh-Schlieder property and Lemma 3.4 we may pick $\boldsymbol{G}\in\mathfrak{F}_\sigma(\mathcal{O})$ and $\boldsymbol{G}'\in\mathfrak{F}_\sigma(\mathcal{O}_1)$ such that $\omega_0(G_jG_k'^*)=\delta_{jk}$. Taking the origin to be the point of contact of $\mathcal{O}_1$ and $\mathcal{O}$ we have for

$$\lambda\leqq 1,\quad \delta_\lambda(\boldsymbol{G})\in\mathfrak{F}_\sigma(\mathcal{O})\quad\text{and}\quad \delta_\lambda(\boldsymbol{G}')\in\mathfrak{F}_\sigma(\mathcal{O}_1)\,.$$

Now

$$\begin{aligned}\omega_0(E\delta_\lambda(\boldsymbol{G})\,E'\,\delta_\lambda(\boldsymbol{G}')^*)&\equiv\sum_{k,j}\omega_0(E\delta_\lambda(G_k)\,E'_{kj}\delta_\lambda(G_j')^*)\\&=\sum_{k,j}\omega_0(E\delta_\lambda(G_kG_j'^*)\,E'_{kj})\end{aligned}$$

and by Corollary 2.5 as $\lambda\to 0$

$$\omega_0(E\delta_\lambda(\boldsymbol{G})\,E'\,\delta_\lambda(\boldsymbol{G}')^*)\to\sum_{k,l}\omega_0(G_kG_j'^*)\,\omega_0(EE'_{kj})=\sum_k\omega_0(EE'_{kk})\,.$$

Since $E'\leqq E\otimes I_\sigma$, $E'_{kj}E=E'_{kj}$ so $\sum_k\omega_0(EE'_{kk})=\sum_k\omega_0(E'_{kk})$. But E' is a non-zero projection and ω_0 is faithful on $\mathfrak{F}(\mathcal{O})$, so $\sum_k\omega_0(E'_{kk})>0$. Hence for sufficiently small λ, $E\delta_\lambda(\boldsymbol{G})\,E'\neq 0$.

3.6 Theorem. *If $\mathfrak{F}(\mathcal{O})$ is a factor then $\mathfrak{A}(\mathcal{O})$ is a factor.*

Proof. It suffices to show that if E_1 and E_2 are non-zero projections in $\mathfrak{A}(\mathcal{O})$ then we can find $A\in\mathfrak{A}(\mathcal{O})$ with $E_1AE_2\neq 0$. Since $\mathfrak{F}(\mathcal{O})$ is a factor there is an $F'\in\mathfrak{F}(\mathcal{O})$ with $E_1F'E_2\neq 0$. Without loss of generality we may suppose that F' is an irreducible tensor of type σ say. Hence there is an $\boldsymbol{F}'\in\mathfrak{F}_\sigma(\mathcal{O})$ such that $\boldsymbol{F}=E_1\boldsymbol{F}'E_2\otimes I_\sigma\neq 0$. Let E' denote the support of $\boldsymbol{F}$. By Lemma 3.5 there is a $\boldsymbol{G}\in\mathfrak{F}_\sigma(\mathcal{O})$ such that $E_2\boldsymbol{G}E'\neq 0$. Since E'

is the support of F this implies that $FG^*E_2 \neq 0$. Hence $E_1 FG^*E_2 = FG^*E_2 \neq 0$. However $FG^* \in \mathfrak{A}(\mathcal{O})$, since $F, G \in \mathfrak{F}_\sigma(\mathcal{O})$ so we have proved our result.

We may in fact generalize the above result somewhat: let E be a central projection of $\mathfrak{A}(\mathcal{O})$ so that $EA(I-E)=0$ for all $A \in \mathfrak{A}(\mathcal{O})$. The argument in the proof of Theorem 3.6 shows that $EF(I-E)=0$ for all $F \in \mathfrak{F}(\mathcal{O})$. Hence E is in the centre of $\mathfrak{F}(\mathcal{O})$. Conversely if $E \in \mathfrak{A}(\mathcal{O})$ and E is in the centre of $\mathfrak{F}(\mathcal{O})$ then E is a fortiori in the centre of $\mathfrak{A}(\mathcal{O})$. Thus

$$\text{centre}\,(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O}) \cap \text{centre}\,(\mathfrak{F}(\mathcal{O})) \tag{3.6}$$

which is the appropriate generalization of Theorem 3.6 if $\mathfrak{F}(\mathcal{O})$ is not a factor.

In view of the classification of factors it is natural to ask whether $\mathfrak{F}(\mathcal{O})$ and $\mathfrak{A}(\mathcal{O})$ must necessarily be factors of the same type. While there are at least partial answers one can give to this question they fall outside the scope of this paper because dilatation invariance plays no further rôle in the reasoning.

4. Superselection Structure

Dilatation invariance also has certain consequences for the superselection structure of the theory. The analysis in this section makes use of results obtained in [8] on the relationship between the observable net and the field net. These results depend, at least in the case of non-Abelian gauge groups, on a certain "duality" relation

$$\mathfrak{F}(\mathcal{O})' \cap \mathcal{V}(\mathcal{G})' = \mathfrak{A}(\mathcal{O}')^- \tag{4.1}$$

which is known to be valid in the usual free field models. In particular we shall use the following notation and results taken from [8].

Consider finite-dimensional subspaces $H \subset \bigcup_{\mathcal{O}} \mathfrak{F}(\mathcal{O})$ satisfying

a) if $\psi_1, \psi_2 \in H$, $\psi_1^* \psi_2$ is a multiple of the identity,

b) if $g \in \mathcal{G}$ then $\beta_g(H) = H$

c) if E is a projection in $\mathcal{B}(\mathcal{H})$ and $E\psi = \psi$ for all $\psi \in H$ then $E = I$.

In virtue of a) and b) H is a Hilbert space with respect to the operator norm and $\psi \to \beta_g(\psi)$ is a continuous unitary representation of $\mathcal{G}$ on H. Further if $\{\psi_i\}$ is an orthonormal basis of H, $\sum_i \psi_i \psi_i^* = I$ by c) and if we set

$$\varrho(F) = \sum_i \psi_i F \psi_i^*, \qquad F \in \mathfrak{F} \tag{4.2}$$

then $\varrho(\mathfrak{A}) \subset \mathfrak{A}$ and $\varrho : \mathfrak{A} \to \mathfrak{A}$ is a localized morphism in the terminology of [9]. H is entirely determined by ϱ and we write $H = H(\varrho)$; in fact

$H(\varrho)=\left\{\psi\in\bigcup_{\mathcal{O}}\mathfrak{F}(\mathcal{O}):\psi A=\varrho(A)\psi,\ A\in\mathfrak{A}\right\}$. Set

$$\Delta'(\mathcal{O})=\{\varrho:H(\varrho)\subset\mathfrak{F}(\mathcal{O})\}. \tag{4.3}$$

In the first result we also need the concept of a twisted field algebra introduced in [4; Section 3]. We set

$$\mathfrak{F}^t(\mathcal{O})=\{F_++\mathscr{V}(k)F_-:F\in\mathfrak{F}(\mathcal{O})\}. \tag{4.4}$$

$\mathfrak{F}^t(\mathcal{O})$ is just the transform of $\mathfrak{F}(\mathcal{O})$ under the unitary operator W where $2W=(1+i)I+(1-i)\mathscr{V}(k)$.

4.1 Proposition. *Let $\mathcal{O}_1$ and $\mathcal{O}$ be two spacelike tangent double cones to $\mathcal{O}_2$ with $\mathcal{O}_1\subset\mathcal{O}$. Then*

$$\mathfrak{A}(\mathcal{O}_1)'\cap\mathfrak{F}(\mathcal{O})=\mathfrak{F}^t(\mathcal{O}_1)'\cap\mathfrak{F}(\mathcal{O}).$$

Proof. Since $\mathfrak{A}(\mathcal{O}_1)\subset\mathfrak{F}^t(\mathcal{O}_1)$ it is enough to show that

$$\mathfrak{A}(\mathcal{O}_1)'\cap\mathfrak{F}(\mathcal{O})\subset\mathfrak{F}^t(\mathcal{O}_1)'.$$

By [8; Theorem 2.3 and Proposition 3.4] it suffices to show that if $U\in\mathfrak{A}(\mathcal{O}_1)'\cap\mathfrak{F}(\mathcal{O})$ and $\varrho\in\Delta'(\mathcal{O}_1)$ then U commutes with $WH(\varrho)W^{-1}$. We may also suppose that ϱ is irreducible or equivalently that $H(\varrho)$ consists of irreducible tensors and that U is unitary. As usual we take the origin to be the point of contact of $\mathcal{O}$, $\mathcal{O}_1$, and $\mathcal{O}_2$. If $\psi_1,\dots,\psi_d$ is an orthonormal basis for $H(\varrho)$ and $\psi\in H(\varrho)$, $\delta_\lambda(\psi)=V_\lambda\psi$ where $V_\lambda=\sum_{i=1}^{d}\delta_\lambda(\psi_i)\psi_i^*$. V_λ is gauge invariant by (2.8) so $V_\lambda\in\mathfrak{A}(\mathcal{O}_1)$, $\lambda\leqq 1$. By Lemma 3.4 and the Reeh-Schlieder property we may pick $F_j\in\mathfrak{F}(\mathcal{O}_2)$ with $\omega_0(F_j\psi_i)=\delta_{ij}$. Since ψ_i is an irreducible tensor, $\beta_k(\psi_i)=\pm\psi_i$. If $\beta_k(\psi_i)=\psi_i$ then we may suppose $\beta_k(F_j)=F_j$. Hence

$$\begin{aligned}\omega_0(U^{-1}\delta_\lambda(F_j\psi_i)U)&=\omega_0(\delta_\lambda(F_j)V_\lambda U^{-1}\psi_iU)\\&=\sum_{r=1}^{d}\omega_0(\delta_\lambda(F_j\psi_r)\psi_r^*U^{-1}\psi_iU).\end{aligned}$$

Proceeding to the limit $\lambda\to 0$ using Corollary 2.5 we get

$$\delta_{ij}=\omega_0(\psi_j^*U^{-1}\psi_iU).$$

If $i=j$ we are in a case where equality holds in the Cauchy-Schwarz inequality. Hence $U\psi_i\Omega=\psi_iU\Omega$ and by the separating property of the vacuum, $U\psi_i=\psi_iU$. If $\beta_k(\psi_i)=-\psi_i$ then we may suppose $\beta_k(F_j)=-F_j$ and the same argument leads to $\beta_k(U)\psi_i=\psi_iU$ or $U\mathscr{V}(k)\psi_i=\mathscr{V}(k)\psi_iU$. Hence in either case U commutes with $WH(\varrho)W^{-1}$ as required.

In the case of free Bose fields it is possible to describe $\mathfrak{F}^t(\mathcal{O}_1)'\cap\mathfrak{F}(\mathcal{O})=\mathfrak{F}(\mathcal{O}_1)'\cap\mathfrak{F}(\mathcal{O})$ as the algebra of some local region [10; (1.13) and

(1.18)]. Hence $\mathfrak{A}(\mathcal{O}_1)' \cap \mathfrak{A}(\mathcal{O})$ which is just the gauge-invariant part of $\mathfrak{F}^t(\mathcal{O}_1)' \cap \mathfrak{F}(\mathcal{O})$ is the observable algebra of the same local region. Presumably the analogous results hold for free Fermi fields.

4.2 Corollary. $\mathfrak{A}(\mathcal{O})' \cap \mathfrak{F}(\mathcal{O}) = \mathfrak{F}(\mathcal{O})' \cap \mathfrak{F}(\mathcal{O})$.

Proof. Setting $\mathcal{O} = \mathcal{O}_1$ in Proposition 4.1 we get $\mathfrak{A}(\mathcal{O})' \cap \mathfrak{F}(\mathcal{O}) = \mathfrak{F}^t(\mathcal{O})' \cap \mathfrak{F}(\mathcal{O})$. Since $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{F}(\mathcal{O})$ we deduce that $\mathfrak{F}^t(\mathcal{O})' \cap \mathfrak{F}(\mathcal{O}) \supset \mathfrak{F}(\mathcal{O})' \cap \mathfrak{F}(\mathcal{O})$. However the Bose parts of $\mathfrak{F}^t(\mathcal{O})'$ and $\mathfrak{F}(\mathcal{O})'$, i.e. the parts which commute with $\mathscr{V}(k)$, are identical. We complete the proof by showing that $\mathfrak{F}^t(\mathcal{O})' \cap \mathfrak{F}(\mathcal{O})$ contains no Fermi part. Suppose $F \in \mathfrak{F}^t(\mathcal{O})' \cap \mathfrak{F}(\mathcal{O})$ and $\beta_k(F) = -F$ then $F = \mathscr{V}(k)\,G$ with $G \in \mathfrak{F}(\mathcal{O})'$. Hence $F^*F = F^*\mathscr{V}(k)\,G = -\mathscr{V}(k)\,GF^* = -FF^*$ so $F = 0$ as required.

Note that in the course of the above proof we have shown that the centre of $\mathfrak{F}(\mathcal{O})$ is invariant under β_k, i.e. is purely Bose. Theorems 3.6 and (3.2) are immediate consequences of this corollary. We will now show that if $\mathfrak{F}(\mathcal{O})$ is a factor, Corollary 4.2 without further use of dilatation invariance implies that the superselection structure may be described in terms of $\mathfrak{F}(\mathcal{O})$ and $\mathfrak{A}(\mathcal{O})$ without reference to the observable and field nets.

4.3 Theorem. *Let $\mathfrak{F}(\mathcal{O})$ be a factor. Suppose $\varrho, \varrho' \in \Delta'(\mathcal{O})$, $S \in \mathfrak{A}(\mathcal{O})$ and*

$$S\varrho(A) = \varrho'(A)\,S\,, \qquad A \in \mathfrak{A}(\mathcal{O})\,.$$

Then S intertwines ϱ and ϱ' globally, i.e.

$$S\varrho(A) = \varrho'(A)\,S\,, \qquad A \in \mathfrak{A}\,.$$

Proof. Let $\psi \in H(\varrho)$, $\psi' \in H(\varrho')$ then

$$\psi'^* S\psi A = \psi'^* S\varrho(A)\,\psi = \psi'^* \varrho'(A)\,S\psi = A\psi'^* S\psi\,, \qquad A \in \mathfrak{A}(\mathcal{O})\,.$$

Hence $\psi'^* S\psi \in \mathfrak{A}(\mathcal{O})' \cap \mathfrak{F}(\mathcal{O}) = \mathbb{C}I$ by Corollary 4.2 since $\mathfrak{F}(\mathcal{O})$ is a factor. Consequently $S = \sum_{i,j} c_{ij}\psi'_j\psi_i^*$ where $\{\psi_i\}$ and $\{\psi'_j\}$ are orthonormal bases for $H(\varrho)$ and $H(\varrho')$ respectively and $c_{ij}I = \psi_j'^* S\psi_i$. Thus S may be identified with a gauge-invariant linear mapping from $H(\varrho)$ to $H(\varrho')$ and [8; Proposition 3.5] shows that S intertwines from ϱ to ϱ' as required.

In particular this Theorem shows that if ϱ and ϱ' are inequivalent irreducible morphisms they remain inequivalent and irreducible when restricted to $\mathfrak{A}(\mathcal{O})$; i.e.

$$\varrho(\mathfrak{A}(\mathcal{O}))' \cap \mathfrak{A}(\mathcal{O}) = \mathbb{C}I \quad \text{and} \quad S\varrho(A) = \varrho'(A)\,S\,, \qquad A \in \mathfrak{A}(\mathcal{O})$$

implies $S = 0$. In other words, the charge quantum numbers in $\mathscr{H}$ can be identified with the *local* equivalence classes of irreducible localized morphisms. Bearing in mind the results of [9], Theorem 4.3 not only shows that we may describe charge quantum numbers but also charge conjugation, the "addition law" for charges and the statistics parameter by looking at the localized morphisms over a single local algebra $\mathfrak{A}(\mathcal{O})$.

One point remains to be checked to substantiate our claim that the superselection structure can be determined in terms of a single local algebra rather than the whole net: we must show that the concept of localized morphism itself can be expressed in local terms. However in the framework of this paper, we have no input which could tell us that $\mathcal{H}$ contains all charge quantum numbers or, in other words, that $\mathfrak{F}$ is a maximal field net. Hence we shall have to be content with a purely local criterion for identifying morphisms of $\Delta'(\mathcal{O})$.

4.4 Theorem. *Let $\mathfrak{F}(\mathcal{O})$ be a factor and let ϱ map $\mathfrak{A}(\mathcal{O})$ into $\mathfrak{A}(\mathcal{O})$ then ϱ is the restriction to $\mathfrak{A}(\mathcal{O})$ of an element of $\Delta'(\mathcal{O})$ if and only if there is a finite-dimensional subspace $H \subset \mathfrak{F}(\mathcal{O})$ satisfying a) and c) above such that*

$$\psi A = \varrho(A)\psi, \quad A \in \mathfrak{A}(\mathcal{O}), \quad \psi \in H. \tag{4.5}$$

Proof. Suppose we can find H with the stated properties and let $\psi_1, \ldots, \psi_d$ be an orthonormal basis for H then $\varrho(A) = \sum_{i=1}^{d} \psi_i A \psi_i^*$, $A \in \mathfrak{A}(\mathcal{O})$. We use this equation to define $\varrho(A)$ for $A \in \mathfrak{A}$ and ϱ will then be a localized morphism provided $\varrho(\mathfrak{A}) \subset \mathfrak{A}$. Now if ψ, $\psi' \in H$ and $A \in \mathfrak{A}(\mathcal{O})$

$$\psi'^* \beta_g(\psi) A = \psi'^* \beta_g(\psi A) = \psi'^* \beta_g(\varrho(A)\psi) = A\psi'^* \beta_g(\psi).$$

Hence $\psi' \beta_g(\psi) \in \mathfrak{A}(\mathcal{O})' \cap \mathfrak{F}(\mathcal{O})$ and must be a multiple of the identity by Corollary 4.2. Thus $\beta_g(\psi) = \sum_{i=1}^{d} \psi_i^* \beta_g(\psi)\psi_i \in H$. Consequently β_g induces a unitary representation of $\mathcal{G}$ on H. Hence

$$\beta_g(\varrho(A)) = \sum_{i=1}^{d} \beta_g(\psi_i) A \beta_g(\psi_i)^* = \varrho(A) \quad \text{for} \quad A \in \mathfrak{A},$$

so $\varrho(\mathfrak{A}) \subset \mathfrak{A}$ and $\varrho \in \Delta'(\mathcal{O})$. The converse is just the definition of $\Delta'(\mathcal{O})$.

Note that the essential step in the proof is to show that $\beta_g(H) = H$. If we had postulated this we would not have been able to claim a purely local criterion for identifying the elements of $\Delta'(\mathcal{O})$ without first giving a local characterization of gauge transformations.

In conclusion, it should be stressed that we have never made essential use of the fact that δ_λ is an automorphism. The effect of δ_λ on the net structure [Eq. (2.9)], the commutativity of δ_λ and gauge automorphisms and the fact that certain limits as $\lambda \to 0$ give non-vanishing multiples of the identity are the important ingredients of the proofs. For this reason the validity of the results presented here is not limited to dilatation-invariant theories but apply to all theories having a well-behaved Gell-Mann Low limit.

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