PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 54, January 1976

SOME APPLICATIONS OF LANDWEBER-NOVIKOV OPERATIONS[1]

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ABSTRACT. Previous results on the characteristic numbers of Sp-manifolds are extended in three different ways. I. It is shown that the primitive symplectic Pontrjagin class evaluated on a $4(2^j - 1)$ dimensional Sp-manifold always gives a number divisible by 8. This forms an analogue to a well-known result of Milnor concerning U-manifolds. II. It is shown that some of the results of Floyd as well as an analogue of the previous result can be obtained for 'pseudo-symplectic' manifolds. III. Results are generalised to (Sp,fr) manifolds.

1. $4(2^j - 1)$ **dimensional** Sp**-manifolds.** Let $s_\pi(p)[M]$, π a partition of $n = n(\pi)$, M a $4n(\pi)$ dimensional stably symplectic manifold, denote the normal symplectic Pontrjagin number of M corresponding to the π-symmetrised polynomial in a system of indeterminates for which the symplectic Pontrjagin classes are the elementary symmetric polynomials. Throughout this section we will set $k = 2^j - 1$ and M will denote a $4k$ dimensional stably symplectic manifold.

THEOREM 1.1. $8 \mid s_{(k)}(p)[M]$.

REMARKS. 1. The unitary analogue, $2 \mid s_{(k)}(c)[N]$, N stably unitary is well known; it could be proven by the techniques used below.

2. The techniques of [3] are not adequate by themselves to prove Theorem 1.1.

PROOF. Actually we will prove slightly more: Let π be any partition of k all of whose parts are themselves integers of the form $2^s - 1$. Then $8 \mid s_\pi(p)[M]$.

If $\pi = (a_1, \ldots, a_r)$, let $D(\pi) = \prod_i[(2a_i + 2)!/2]$. Well known fact. $2 \mid \sum_{n(\pi)=k}(s_\pi(p)[M]/D(\pi))$. This is the 'Todd genus' relation of Stong [4] who put things in an 'abnormal' form; using normal rather than tangential numbers makes computation manageable. In particular, we can see that for a fixed k the denominators $D(\pi)$ with maximal number of factors of 2 will be just those for which all parts of π are of the form $2^s - 1$.

By Proposition 4 of [3] it is automatic that $4 \mid s_\pi(p)[M]$ for all π, $n(\pi) = k$. If we can show that $8 \mid s_\pi(p)[M]$ whenever $\pi = (a_1, \ldots, a_r)$, $n(\pi) = k$, $r > 1$

Presented to the Society, January 26, 1975; received by the editors November 1, 1974 and, in revised form, January 24, 1975.

AMS (MOS) subject classifications (1970). Primary 57A70; Secondary 55B20.

Key words and phrases. Landweber-Novikov operation, symplectic manifold, pseudo-symplectic manifold, symplectic-framed manifold.

[1] This research was supported by the Research Foundation of the City University of New York.

and all a_i of the form $2^s - 1$, then it will follow from the above that $8 \mid s_{(k)}(p)[M]$.

Now assume inductively that Theorem 1.1 holds in dimensions less than $4k$; by [4] it certainly holds in dimensions 4 and 12. Let $\bar{\pi} = (a_1, \ldots, a_r)$ be a partition of k with $r > 1$ and all a_i of the form $2^s - 1$. Since $\bar{\pi}$ is a partition of an odd number into odd parts there is a number k' which occurs exactly f times as a part of $\bar{\pi}$, f odd. Let $\bar{\pi}'$ denote the partition obtained from $\bar{\pi}$ by deleting one occurence of k'. Let $S(\bar{\pi}')$ denote the symplectic Landweber-Novikov operation corresponding to $\bar{\pi}'$. Then from the results of [1] on the action of such operations,

$$s_{(k')}(p)\left[S(\bar{\pi}')M\right] = f s_{\bar{\pi}}(p)\left[M\right] + \sum_{\pi} a(\pi,\bar{\pi},\bar{\pi}') s_{\pi}(p)\left[M\right], \tag{1.2}$$

where the summation on the right runs through all partitions π obtained by adding k' to one of the parts of $\bar{\pi}'$, and the coefficients $a(\pi,\bar{\pi},\bar{\pi}')$ are integers which are in fact even as a consequence of the fact that the parts of $\bar{\pi}'$ are all of the form $2^s - 1$. Then by Proposition 4 of [3] (and since $n(\pi)$ is odd), 8 divides the summation term. But by the inductive hypothesis, 8 divides the left side of (1.2). Our assertion and the theorem then follow from the oddness of f.

2. **Pseudo-symplectic manifolds.** We call a U-manifold pseudo-symplectic if some nonzero multiple of its class in MU_* is in the image of MSp_*; this will be the case if and only if every Chern number of the manifold involving an odd Chern class vanishes. Let Ps_* be the subring of MU_* consisting of such classes. Let j, p, d be the maps in the cofibration sequence of spectra

$$MSp \xrightarrow{j} MU \xrightarrow{p} MU/MSp \xrightarrow{d} SMSp.$$

There is a well-defined map h_*^{Ps}: $Ps_* \to H_*(MSp)$ obtained by restricting the Hurewicz homomorphism h_*^{MU}: $MU_* \to H_*(MU)$ to Ps_* and then composing with j_*^{-1}. We regard the sympletic Pontrjagin numbers as defined on Ps_*. Note that $\operatorname{Im} h_*^{Sp} \subset \operatorname{Im} h_*^{Ps} \subset H_*(MSp)$ (inclusions strict) and that $\operatorname{Im} h_*^{Ps}/\operatorname{Im} h_*^{Sp}$ gives the torsion elements of $\operatorname{Im} p_*$ in the (MU,MSp) long exact bordism sequence.

LEMMA 2.1. *Let $S(\pi)$: $MSp \to S^{4n(\pi)}MSp$ be a symplectic Landweber-Novikov operation. Then we can find some U-bordism operation T: $MU \to S^{4n(\pi)}MU$ such that $T \circ j = S^{4n(\pi)}j \circ S(\pi)$.*

PROOF. Treat $S^{4n(\pi)}j \circ S(\pi)$ as a class in $MU^{4n(\pi)}(MSp)$. Now $d_*(S^{4n(\pi)}j \circ S(\pi)) = 0$ in $MU^{4n(\pi)+1}(MU/MSp)$ (since that group is trivial), so by exactness there must exist $T \in MU^{4n(\pi)}(MU)$ such that $j_*(T) = S^{4n(\pi)}j \circ S(\pi)$.

This 'compatibility' lemma implies that $\operatorname{Im} h_*^{Ps}$ is closed under the action of the symplectic Landweber-Novikov operations.

THEOREM 2.2. *Let M be a $4k$ dimensional pseudo-symplectic manifold. Then*
(i) $2 \mid s_{\pi}(p)[M]$ *if $n(\pi)$ is odd or if $\pi = (2^j)$;*
(ii) $4 \mid s_{(k)}(p)[M]$ *if $k = 2^j - 1$.*

REMARK. Floyd first studied pseudo-symplectics (they are the 'related manifolds' of the title of [2]) and part (i) was proved by him by rather different methods.

PROOF. Exactly as for symplectics in [3] except that one has weaker low-dimensional divisibility properties to feed into the machinery so that statements involving 4 become statements involving 2 while those involving 2 become vacuous. By the same token, part (ii) is done on the model of Theorem 1.1 above.

3. (Sp, fr) **manifolds.** Let $h_*^{Sp,fr}\colon MSp/fr_* \to H_*(MSp/fr)$ be the Hurewicz map for MSp/fr, the spectrum representing (Sp,fr) bordism. We wish to obtain divisibility conditions on characteristic numbers of (Sp,fr) manifolds. One would expect to use a compatibility lemma which showed that Im $h_*^{Sp,fr}$ is closed under the action of symplectic Landweber-Novikov operations, find some 'starting' conditions and proceed as with the symplectic and pseudo-symplectic cases.

Actually something happens which makes our work easier (and our results stronger). If $n(\pi) > 0$ then $S(\pi)$ can be lowered to a map $S(\pi)'\colon MSp/fr \to S^{4n(\pi)}MSp$ so that $S(\pi)$ actually sends the (Sp,fr) classes into full-fledged Sp-classes. Thus *all* the divisibility conditions of [3] hold equally for (Sp,fr) manifolds except in the starting dimensions:

THEOREM 3.1. *Let M be a $4k(Sp,fr)$ manifold. Then*
(i) $4 \mid s_\pi(p)[M]$ *if* $n(\pi) > 1$ *and odd or if* $\pi = (2^j), j > 1$;
(ii) $2 \mid s_\pi(p)[M]$ *if* $n(\pi) > 2$ *and* $\equiv 2$ (4) *or if* $\pi = (2^j, 2^j), j > 1$.

The proof of Theorem 1.1 does not carry over to the (Sp,fr) case.

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