Kishore et al. *Fixed Point Theory and Applications* (2017) 2017:10
DOI 10.1186/s13663-017-0603-2

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Some applications via fixed point results in partially ordered S_b-metric spaces

GNV Kishore[1*], KPR Rao[2], D Panthi[3], B Srinuvasa Rao[4] and S Satyanaraya[5]

*Correspondence:
kishore.apr2@gmail.com
[1]Department of Mathematics, K L University, Vaddeswaram, Guntur-522 502, Andhra Pradesh, India
Full list of author information is available at the end of the article

Abstract

In this paper we give some applications to integral equations as well as homotopy theory via fixed point theorems in partially ordered complete S_b-metric spaces by using generalized contractive conditions. We also furnish an example which supports our main result.

Keywords: S_b-metric space; w-compatible pairs; S_b-completeness

1 Introduction

Banach contraction principle in metric spaces is one of the most important results in fixed theory and nonlinear analysis in general. Since 1922, when Stefan Banach [1] formulated the concept of contraction and posted a famous theorem, scientists around the world have published new results related to the generalization of a metric space or with contractive mappings (see [1–24]). Banach contraction principle is considered to be the initial result of the study of fixed point theory in metric spaces.

In the year 1989, Bakhtin introduced the concept of b-metric spaces as a generalization of metric spaces [6]. Later several authors proved so many results on b-metric spaces (see [13–16]). Mustafa and Sims defined the concept of a generalized metric space which is called a G-metric space [12]. Sedghi, Shobe and Aliouche gave the notion of an S-metric space and proved some fixed point theorems for a self-mapping on a complete S-metric space [22]. Aghajani, Abbas and Roshan presented a new type of metric which is called G_b-metric and studied some properties of this metric [2].

Recently Sedghi et al. [20] defined S_b-metric spaces using the concept of S-metric spaces [22].

The aim of this paper is to prove some unique fixed point theorems for generalized contractive conditions in complete S_b-metric spaces. Also, we give applications to integral equations as well as homotopy theory. Throughout this paper R, R^+ and N denote the sets of all real numbers, non-negative real numbers and positive integers, respectively.

First we recall some definitions, lemmas and examples.

2 Preliminaries

Definition 2.1 ([22]) Let X be a non-empty set. An S-metric on X is a function $S : X^3 \to [0, +\infty)$ that satisfies the following conditions for each $x, y, z, a \in X$:

$(S1)$: $0 < S(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z \neq x$,

$(S2)$: $S(x,y,z) = 0$ if and only if $x = y = z$,
$(S3)$: $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$ for all $x,y,z,a \in X$.

Then the pair (X,S) is called an S-metric space.

Definition 2.2 ([20]) Let X be a non-empty set and $b \geq 1$ be a given real number. Suppose that a mapping $S_b : X^3 \to [0,\infty)$ is a function satisfying the following properties:

(S_b1) $0 < S_b(x,y,z)$ for all $x,y,z \in X$ with $x \neq y \neq z \neq x$,
(S_b2) $S_b(x,y,z) = 0$ if and only if $x = y = z$,
(S_b3) $S_b(x,y,z) \leq b(S_b(x,x,a) + S_b(y,y,a) + S_b(z,z,a))$ for all $x,y,z,a \in X$.

Then the function S_b is called an S_b-metric on X and the pair (X,S_b) is called an S_b-metric space.

Remark 2.3 ([20]) It should be noted that the class of S_b-metric spaces is effectively larger than that of S-metric spaces. Indeed each S-metric space is an S_b-metric space with $b = 1$.

The following example shows that an S_b-metric on X need not be an S-metric on X.

Example 2.4 ([20]) Let (X,S) be an S-metric space and $S_*(x,y,z) = S(x,y,z)^p$, where $p > 1$ is a real number. Note that S_* is an S_b-metric with $b = 2^{2(p-1)}$. Also, (X,S_*) is not necessarily an S-metric space.

Definition 2.5 ([20]) Let (X,S_b) be an S_b-metric space. Then, for $x \in X$, $r > 0$, we define the open ball $B_{S_b}(x,r)$ and the closed ball $B_{S_b}[x,r]$ with center x and radius r as follows, respectively:

$$B_{S_b}(x,r) = \{y \in X : S_b(y,y,x) < r\},$$
$$B_{S_b}[x,r] = \{y \in X : S_b(y,y,x) \leq r\}.$$

Lemma 2.6 ([20]) *In an S_b-metric space, we have*

$$S_b(x,x,y) \leq bS_b(y,y,x)$$

and

$$S_b(y,y,x) \leq bS_b(x,x,y).$$

Lemma 2.7 ([20]) *In an S_b-metric space, we have*

$$S_b(x,x,z) \leq 2bS_b(x,x,y) + b^2S_b(y,y,z).$$

Definition 2.8 ([20]) If (X,S_b) is an S_b-metric space, a sequence $\{x_n\}$ in X is said to be:
(1) S_b-Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $S_b(x_n,x_n,x_m) < \epsilon$ for each $m,n \geq n_0$.
(2) S_b-convergent to a point $x \in X$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that $S_b(x_n,x_n,x) < \epsilon$ or $S_b(x,x,x_n) < \epsilon$ for all $n \geq n_0$, and we denote $\lim_{n\to\infty} x_n = x$.

Definition 2.9 ([20]) An S_b-metric space (X, S_b) is called complete if every S_b-Cauchy sequence is S_b-convergent in X.

Lemma 2.10 ([20]) *If (X, S_b) is an S_b-metric space with $b \geq 1$, and suppose that $\{x_n\}$ is S_b-convergent to x, then we have*

$$\text{(i)} \quad \frac{1}{2b} S_b(y,x,x) \leq \lim_{n\to\infty} \inf S_b(y,y,x_n) \leq \lim_{n\to\infty} \sup S_b(y,y,x_n) \leq 2bS_b(y,y,x)$$

and

$$\text{(ii)} \quad \frac{1}{b^2} S_b(x,x,y) \leq \lim_{n\to\infty} \inf S_b(x_n,x_n,y) \leq \lim_{n\to\infty} \sup S_b(x_n,x_n,y) \leq b^2 S_b(x,x,y)$$

for all $y \in X$.

In particular, if $x = y$, then we have $\lim_{n\to\infty} S_b(x_n,x_n,y) = 0$.

Now we prove our main results.

3 Results and discussions

Definition 3.1 Let $(X, S_b, \preceq)$ be a partially ordered complete S_b-metric space which is said to be regular if every two elements of X are comparable,

$$\text{i.e., if } x, y \in X \Rightarrow \text{ either } x \preceq y \text{ or } y \preceq x.$$

Definition 3.2 Let $(X, S_b, \preceq)$ be a partially ordered complete S_b-metric space which is also regular; let $f : X \to X$ be a mapping. We say that f satisfies (ψ, ϕ)-contraction if there exist $\psi, \phi : [0,\infty) \to [0,\infty)$ such that

(3.2.1) f is non-decreasing,
(3.2.2) ψ is continuous, monotonically non-decreasing and ϕ is lower semi-continuous,
(3.2.3) $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$,
(3.2.4) $\psi(4b^4 S_b(fx,fx,fy)) \leq \psi(M_f^i(x,y)) - \phi(M_f^i(x,y))$, $\forall x, y \in X$, $x \preceq y$, $i = 3,4,5$ and

$$M_f^5(x,y) = \max\{S_b(x,x,y), S_b(x,x,fx), S_b(y,y,fy), S_b(x,x,fy), S_b(y,y,fx)\},$$

$$M_f^4(x,y) = \max\left\{S_b(x,x,y), S_b(x,x,fx), S_b(y,y,fy), \frac{1}{4b^4}\left[S_b(x,x,fy) + S_b(y,y,fx)\right]\right\},$$

$$M_f^3(x,y) = \max\left\{S_b(x,x,y), \frac{1}{4b^4}\left[S_b(x,x,fx) + S_b(y,y,fy)\right], \frac{1}{4b^4}\left[S_b(x,x,fy) + S_b(y,y,fx)\right]\right\}.$$

Definition 3.3 Suppose that $(X, \preceq)$ is a partially ordered set and f is a mapping of X into itself. We say that f is non-decreasing if for every $x, y \in X$,

$$x \preceq y \text{ implies that } fx \preceq fy. \tag{1}$$

Theorem 3.4 *Let $(X, S_b, \preceq)$ be an ordered complete S_b metric space, which is also regular, and let $f : X \to X$ satisfy (ψ, ϕ)-contraction with $i = 5$. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a unique fixed point in X.*

Proof Since f is a mapping from X into X, there exists a sequence $\{x_n\}$ in X such that

$$x_{n+1} = fx_n, \quad n = 0, 1, 2, 3, \ldots.$$

Case (i): If $x_n = x_{n+1}$, then x_n is a fixed point of f.

Case (ii): Suppose $x_n \neq x_{n+1}$ $\forall n$.

Since $x_0 \preceq fx_0 = x_1$ and f is non-decreasing, it follows that

$$x_0 \preceq fx_0 \preceq f^2x_0 \preceq f^3x_0 \preceq \cdots \preceq f^nx_0 \preceq f^{n+1}x_0 \preceq \cdots.$$

Now

$$\begin{aligned}\psi\left(4b^4 S_b\left(fx_0, fx_0, f^2x_0\right)\right) &= \psi\left(4b^4 S_b(fx_0, fx_0, fx_1)\right)\\ &\leq \psi\left(M_f^5(x_0, x_1)\right) - \phi\left(M_f^5(x_0, x_1)\right),\end{aligned}$$

where

$$\begin{aligned}M_f^5(x_0, x_1) &= \max\left\{\begin{matrix} S_b(x_0, x_0, x_1), S_b(x_0, x_0, fx_0), S_b(x_1, x_1, fx_1) \\ S_b(x_0, x_0, f^2x_0), S_b(fx_0, fx_0, fx_0)\end{matrix}\right\}\\ &= \max\left\{S_b(x_0, x_0, fx_0), S_b\left(fx_0, fx_0, f^2x_0\right), S_b\left(x_0, x_0, f^2x_0\right)\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\psi\left(4b^4 S_b\left(fx_0, fx_0, f^2x_0\right)\right)\\ &\quad\leq \psi\left(\max\left\{S_b(x_0, x_0, fx_0), S_b\left(fx_0, fx_0, f^2x_0\right), S_b\left(x_0, x_0, f^2x_0\right)\right\}\right)\\ &\qquad - \phi\left(\max\left\{S_b(x_0, x_0, fx_0), S_b\left(fx_0, fx_0, f^2x_0\right), S_b\left(x_0, x_0, f^2x_0\right)\right\}\right)\\ &\quad\leq \psi\left(\max\left\{S_b(x_0, x_0, fx_0), S_b\left(fx_0, fx_0, f^2x_0\right), S_b\left(x_0, x_0, f^2x_0\right)\right\}\right).\end{aligned}$$

By the definition of ψ, we have that

$$S_b\left(fx_0, fx_0, f^2x_0\right) \leq \max\left\{\begin{matrix}\frac{1}{4b^4} S_b(x_0, x_0, fx_0) \\ \frac{1}{4b^4} S_b(fx_0, fx_0, f^2x_0) \\ \frac{1}{4b^4} S_b(x_0, x_0, f^2x_0)\end{matrix}\right\}. \tag{2}$$

But

$$\begin{aligned}\frac{1}{4b^4} S_b\left(x_0, x_0, f^2x_0\right) &\leq \frac{1}{4b^4}\left[2b S_b(x_0, x_0, fx_0) + b^2 S_b\left(fx_0, fx_0, f^2x_0\right)\right]\\ &\leq \max\left\{\frac{1}{b^3} S_b(x_0, x_0, fx_0), \frac{1}{2b^2} S_b\left(fx_0, fx_0, f^2x_0\right)\right\}.\end{aligned}$$

From (2) we have that

$$S_b\left(fx_0, fx_0, f^2x_0\right) \leq \max\left\{\frac{1}{b^3} S_b(x_0, x_0, fx_0), \frac{1}{2b^2} S_b\left(fx_0, fx_0, f^2x_0\right)\right\}.$$

If $\frac{1}{2b^2}S_b(fx_0,fx_0,f^2x_0)$ is maximum, we get a contradiction. Hence

$$S_b\big(fx_0,fx_0,f^2x_0\big) \le \frac{1}{b^3}S_b(x_0,x_0,fx_0). \tag{3}$$

Also

$$\begin{aligned}\psi\big(4b^4S_b\big(f^2x_0,f^2x_0,f^3x_0\big)\big) &= \psi\big(4b^4S_b(fx_1,fx_1,fx_2)\big)\\ &\le \psi\big(M_f^5(x_1,x_2)\big) - \phi\big(M_f^4(x_1,x_2)\big),\end{aligned}$$

where

$$\begin{aligned}M_f^5(x_1,x_2) &= \max\left\{\begin{matrix} S_b(fx_0,fx_0,f^2x_0), S_b(fx_0,fx_0,f^2x_0), S_b(f^2x_0,f^2x_0,f^3x_0) \\ S_b(fx_0,fx_0,f^3x_0), S_b(f^2x_0,f^2x_0,f^2x_0)\end{matrix}\right\}\\ &= \max\left\{S_b\big(fx_0,fx_0,f^2x_0\big), S_b\big(f^2x_0,f^2x_0,f^3x_0\big), S_b\big(fx_0,fx_0,f^3x_0\big)\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\psi\big(4b^4S_b\big(f^2x_0,f^2x_0,f^3x_0\big)\big)\\ &\quad\le \psi\left(\max\left\{\begin{matrix} S_b(fx_0,fx_0,f^2x_0), S_b(f^2x_0,f^2x_0,f^3x_0) \\ S_b(fx_0,fx_0,f^3x_0)\end{matrix}\right\}\right)\\ &\qquad - \phi\left(\max\left\{\begin{matrix} S_b(fx_0,fx_0,f^2x_0), S_b(f^2x_0,f^2x_0,f^3x_0) \\ S_b(fx_0,fx_0,f^3x_0)\end{matrix}\right\}\right)\\ &\quad\le \psi\left(\max\left\{\begin{matrix} S_b(fx_0,fx_0,f^2x_0), S_b(f^2x_0,f^2x_0,f^3x_0) \\ S_b(fx_0,fx_0,f^3x_0)\end{matrix}\right\}\right).\end{aligned}$$

By the definition of ψ, we have that

$$S_b\big(f^2x_0,f^2x_0,f^3x_0\big) \le \max\left\{\begin{matrix} \frac{1}{4b^4}S_b(fx_0,fx_0,f^2x_0) \\ \frac{1}{4b^4}S_b(f^2x_0,f^2x_0,f^3x_0) \\ \frac{1}{4b^4}S_b(fx_0,fx_0,f^3x_0)\end{matrix}\right\}. \tag{4}$$

But

$$\begin{aligned}&\frac{1}{4b^4}S_b\big(fx_0,fx_0,f^3x_0\big)\\ &\quad\le \frac{1}{4b^4}\big[2bS_b\big(fx_0,fx_0,f^2x_0\big) + b^2S_b\big(f^2x_0,f^2x_0,f^3x_0\big)\big]\\ &\quad\le \max\left\{\frac{1}{b^3}S_b\big(fx_0,fx_0,f^2x_0\big), \frac{1}{2b^2}S_b\big(f^2x_0,f^2x_0,f^3x_0\big)\right\}.\end{aligned}$$

From (4) we have that

$$S_b\big(f^2x_0,f^2x_0,f^3x_0\big) \le \max\left\{\frac{1}{b^3}S_b\big(fx_0,fx_0,f^2x_0\big), \frac{1}{2b^2}S_b\big(f^2x_0,f^2x_0,f^3x_0\big)\right\}.$$

If $\frac{1}{2b^2}S_b(f^2x_0,f^2x_0,f^3x_0)$ is maximum, we get a contradiction. Hence

$$\begin{aligned}S_b\big(f^2x_0,f^2x_0,f^3x_0\big) &\le \frac{1}{b^3}S_b\big(fx_0,fx_0,f^2x_0\big)\\ &\le \frac{1}{(b^3)^2}S_b(x_0,x_0,fx_0).\end{aligned}$$

Continuing this process, we can conclude that

$$\begin{aligned}S_b\big(f^nx_0,f^nx_0,f^{n+1}x_0\big) &\le \frac{1}{(b^3)^n}S_b(x_0,x_0,fx_0)\\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

That is,

$$\lim_{n\to\infty} S_b\big(f^nx_0,f^nx_0,f^{n+1}x_0\big) = 0. \tag{5}$$

Now we prove that $\{f^nx_0\}$ is an S_b-Cauchy sequence in (X,S_b). On the contrary, we suppose that $\{f^nx_0\}$ is not S_b-Cauchy. Then there exist $\epsilon > 0$ and monotonically increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$.

$$S_b\big(f^{m_k}x_0,f^{m_k}x_0,f^{n_k}x_0\big) \ge \epsilon \tag{6}$$

and

$$S_b\big(f^{m_k}x_0,f^{m_k}x_0,f^{n_k-1}x_0\big) < \epsilon. \tag{7}$$

From (6) and (7), we have

$$\begin{aligned}\epsilon &\le S_b\big(f^{m_k}x_0,f^{m_k}x_0,f^{n_k}x_0\big)\\ &\le 2bS_b\big(f^{m_k}x_0,f^{m_k}x_0,f^{m_k+1}x_0\big)\\ &\quad + b^2S_b\big(f^{m_k+1}x_0,f^{m_k+1}x_0,f^{n_k}x_0\big).\end{aligned}$$

So that

$$\begin{aligned}4b^2\epsilon &\le 8b^3S_b\big(f^{m_k}x_0,f^{m_k}x_0,f^{m_k+1}x_0\big)\\ &\quad + 4b^4S_b\big(f^{m_k+1}x_0,f^{m_k+1}x_0,f^{n_k}x_0\big).\end{aligned}$$

Letting $k\to\infty$ and applying ψ on both sides, we have that

$$\begin{aligned}\psi\big(4b^2\epsilon\big) &\le \lim_{k\to\infty}\psi\big(4b^4S_b\big(f^{m_k+1}x_0,f^{m_k+1}x_0,f^{n_k}x_0\big)\big)\\ &= \lim_{k\to\infty}\psi\big(4b^4S_b(fx_{m_k},fx_{m_k},fx_{n_k-1})\big)\\ &\le \lim_{k\to\infty}\psi\big(M_f^5(x_{m_k},x_{n_k-1})\big) - \lim_{k\to\infty}\phi\big(M_f^5(x_{m_k},x_{n_k-1})\big),\end{aligned} \tag{8}$$

where

$$\begin{aligned}
&\lim_{k\to\infty} M_f^5(x_{m_k},x_{n_k-1})\\
&= \lim_{k\to\infty} \max \left\{ \begin{matrix} S_b(f^{m_k}x_0,f^{m_k}x_0,f^{n_k-1}x_0), S_b(f^{m_k}x_0,f^{m_k}x_0,f^{m_k+1}x_0) \\ S_b(f^{n_k-1}x_0,f^{n_k-1}x_0,f^{n_k}x_0), S_b(f^{m_k}x_0,f^{m_k}x_0,f^{n_k}x_0) \\ S_b(f^{n_k-1}x_0,f^{n_k-1}x_0,f^{m_k+1}x_0) \end{matrix} \right\}\\
&< \lim_{k\to\infty} \max \left\{\epsilon, 0, 0, S_b\big(f^{m_k}x_0,f^{m_k}x_0,f^{n_k}x_0\big), S_b\big(f^{n_k-1}x_0,f^{n_k-1}x_0,f^{m_k+1}x_0\big)\right\}.
\end{aligned}$$

But

$$\lim_{k\to\infty} S_b\big(f^{m_k}x_0,f^{m_k}x_0,f^{n_k}x_0\big) \le \lim_{k\to\infty} \left[\begin{matrix} 2bS_b(f^{m_k}x_0,f^{m_k}x_0,f^{n_k-1}x_0) \\ + b^2 S_b(f^{n_k-1}x_0,f^{n_k-1}x_0,f^{n_k}x_0) \end{matrix} \right] < 2b\epsilon.$$

Also

$$\lim_{k\to\infty} S_b\big(f^{n_k-1}x_0,f^{n_k-1}x_0,f^{m_k+1}x_0\big) \le \lim_{k\to\infty} \left[\begin{matrix} 2bS_b(f^{n_k-1}x_0,f^{n_k-1}x_0,f^{m_k}x_0) \\ + b^2 S_b(f^{m_k}x_0,f^{m_k}x_0,f^{m_k+1}x_0) \end{matrix} \right] < 2b^2\epsilon.$$

Therefore

$$\begin{aligned}
\lim_{k\to\infty} M_f^5(x_{m_k},x_{n_k-1}) &\le \max\left\{\epsilon, 2b\epsilon, 2b^2\epsilon\right\}\\
&= 2b^2\epsilon.
\end{aligned}$$

From (8), by the definition of ψ, we have that

$$4b^2\epsilon \le 2b^2\epsilon,$$

which is a contradiction. Hence $\{f^n x_0\}$ is an S_b-Cauchy sequence in complete regular S_b-metric spaces $(X, S_b, \preceq)$. By the completeness of (X, S_b), it follows that the sequence $\{f^n x_0\}$ converges to α in (X, S_b). Thus

$$\lim_{k\to\infty} f^n x_0 = \alpha = \lim_{k\to\infty} f^{n+1}x_0.$$

Since $x_n, \alpha \in X$ and X is regular, it follows that either $x_n \preceq \alpha$ or $\alpha \preceq x_n$.

Now we have to prove that α is a fixed point of f.

Suppose $f\alpha \neq \alpha$, by Lemma (2.10), we have that

$$\frac{1}{2b} S_b(f\alpha,f\alpha,\alpha) \le \lim_{n\to\infty} \inf S_b(f\alpha,f\alpha,f^{n+1}x_0).$$

Now from (3.2.4) and applying ψ on both sides, we have that

$$\begin{aligned}
\psi\big(2b^3 S_b(f\alpha,f\alpha,\alpha)\big) &\le \lim_{n\to\infty} \inf \psi\big(4b^4 S_b\big(f\alpha,f\alpha,f^{n+1}x_0\big)\big)\\
&\le \lim_{n\to\infty} \inf \psi\big(M_f^5(\alpha,x_n)\big) - \lim_{n\to\infty} \inf \phi\big(M_f^5(\alpha,x_n)\big). \qquad (9)
\end{aligned}$$

Here

$$\begin{aligned}\lim_{n\to\infty}\inf M_f^5(\alpha,x_n) &= \lim_{n\to\infty}\inf\max\begin{Bmatrix}S_b(\alpha,\alpha,x_n),S_b(\alpha,\alpha,f\alpha),S_b(x_n,x_n,fx_n)\\ S_b(\alpha,\alpha,fx_n),S_b(x_n,x_n,f\alpha)\end{Bmatrix}\\ &\le \lim_{n\to\infty}\sup\max\left\{0,S_b(\alpha,\alpha,f\alpha),0,0,S_b(x_n,x_n,f\alpha)\right\}\\ &\le \max\left\{S_b(\alpha,\alpha,f\alpha),b^2S_b(\alpha,\alpha,f\alpha)\right\}\\ &\le b^3S_b(f\alpha,f\alpha,\alpha).\end{aligned}$$

Hence from (9) we have that

$$\begin{aligned}\psi\left(2b^3S_b(f\alpha,f\alpha,\alpha)\right) &\le \psi\left(b^3S_b(\alpha,\alpha,f\alpha)\right)-\lim_{n\to\infty}\inf\phi\left(M_f^5(\alpha,x_n)\right)\\ &\le \psi\left(b^3S_b(f\alpha,f\alpha,\alpha)\right),\end{aligned}$$

which is a contradiction. So that α is a fixed point of f.

Suppose that α^* is another fixed point of f such that $\alpha\neq\alpha^*$.

Consider

$$\begin{aligned}\psi\left(4b^4S_b(\alpha,\alpha,\alpha^*)\right) &\le \psi\left(M_f^5(\alpha,\alpha^*)\right)-\phi\left(M_f^5(\alpha,\alpha^*)\right)\\ &= \psi\left(\max\left\{S_b(\alpha,\alpha,\alpha^*),S_b(\alpha^*,\alpha^*,\alpha)\right\}\right)\\ &\quad-\phi\left(\max\left\{S_b(\alpha,\alpha,\alpha^*),S_b(\alpha^*,\alpha^*,\alpha)\right\}\right)\\ &\le \psi\left(bS_b(\alpha,\alpha,\alpha^*)\right),\end{aligned}$$

which is a contradiction.

Hence α is a unique fixed point of f in (X,S_b). □

Example 3.5 Let $X=[0,1]$ and $S:X\times X\times X\to\mathbb{R}^+$ by $S_b(x,y,z)=(|y+z-2x|+|y-z|)^2$ and $\preceq$ by $a\preceq b \iff a\le b$, then $(X,S_b,\preceq)$ is a complete ordered S_b-metric space with $b=4$. Define $f:X\to X$ by $f(x)=\frac{x}{32\sqrt{2}}$. Also define $\psi,\phi:\mathbb{R}^+\to\mathbb{R}^+$ by $\psi(t)=t$ and $\phi(t)=\frac{t}{2}$.

$$\begin{aligned}\psi\left(4b^4S_b(fx,fx,fy)\right) &= 4b^4\left(|fx+fy-2fx|+|fx-fy|\right)^2\\ &= 4b^4\left(2\left|\frac{x}{32\sqrt{2}}-\frac{y}{32\sqrt{2}}\right|\right)^2\\ &= \frac{4b^4}{8b^4}S_b(x,x,y)\\ &\le \frac{1}{2}M_f^5(x,y)\\ &\le \psi\left(M_f^5(x,y)\right)-\phi\left(M_f^5(x,y)\right),\end{aligned}$$

where

$$M_f^5(x,y)=\max\left\{S_b(x,x,y),S_b(x,x,fx),S_b(y,y,fy),S_b(x,x,fy),S_b(y,y,fx)\right\}.$$

Hence, all the conditions of Theorem 3.4 are satisfied and 0 is a unique fixed point of f.

Theorem 3.6 *Let $(X, S_b, \preceq)$ be an ordered complete S_b metric space, and let $f : X \to X$ satisfy (ψ, ϕ)-contraction with $i = 3$ or 4. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a unique fixed point in X.*

Proof Follows along similar lines of Theorem 3.4 if we take $M_f^3(x,y)$ or $M_f^4(x,y)$ in place of $M_f^5(x,y)$ in Theorem 3.4. □

Theorem 3.7 *Let $(X, S_b, \preceq)$ be an ordered complete S_b metric space, and let $f : X \to X$ satisfy*

$$4b^4 S_b(fx, fx, fy) \le M_f^i(x,y) - \varphi\big(M_f^i(x,y)\big),$$

where $\varphi : [0,\infty) \to [0,\infty)$ and $i = 3$ or 4 or 5. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a unique fixed point in X.

Proof The proof follows from Theorems 3.4 and 3.6 by taking $\psi(t) = t$ and $\phi(t) = \varphi(t)$. □

Theorem 3.8 *Let $(X, S_b, \preceq)$ be an ordered complete S_b metric space, and let $f : X \to X$ satisfy*

$$S_b(fx, fx, fy) \le \lambda M_f^i(x,y),$$

where $\lambda \in [0, \frac{1}{4b^4})$ and $i = 3, 4, 5$. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a unique fixed point in X.

3.1 Application to integral equations

In this section, we study the existence of a unique solution to an initial value problem as an application to Theorem 3.4.

Theorem 3.9 *Consider the initial value problem*

$$x^1(t) = T\big(t, x(t)\big), \quad t \in I = [0,1], x(0) = x_0, \tag{10}$$

where $T : I \times [\frac{x_0}{4}, \infty) \to [\frac{x_0}{4}, \infty)$ and $x_0 \in \mathbb{R}$. Then there exists a unique solution in $C(I, [\frac{x_0}{4}, \infty))$ for initial value problem (10).

Proof The integral equation corresponding to initial value problem (10) is

$$x(t) = x_0 + 3b^2 \int_0^t T\big(s, x(s)\big)\, ds.$$

Let $X = C(I, [\frac{x_0}{4}, \infty))$ and $S_b(x,y,z) = (|y+z-2x| + |y-z|)^2 forx, y \in X$. Define $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$, $\phi(t) = \frac{5t}{9}$. Define $f : X \to X$ by

$$f(x)(t) = \frac{x_0}{3b^2} + \int_0^t T\big(s, x(s)\big)\, ds. \tag{11}$$

Now

$$\begin{aligned}
&\psi\left(4b^4 S_b\big(fx(t),fx(t),fy(t)\big)\right)\\
&\quad = 4b^4\left\{\left|fx(t)+fy(t)-2fx(t)\right| + \left|fx(t)-fy(t)\right|\right\}^2\\
&\quad = 16b^4\left|fx(t)-fy(t)\right|^2\\
&\quad = \frac{16b^4}{9b^4}\left|x_0 + 3b^2\int_0^t T\big(s,x(s)\big)\,ds - y_0 - 3b^2\int_0^t T\big(s,y(s)\big)\,ds\right|^2\\
&\quad = \frac{16}{9}\left|x(t)-y(t)\right|^2\\
&\quad = \frac{4}{9}S(x,x,y)\\
&\quad \le \frac{4}{9}M_f^5(x,y)\\
&\quad = \psi\left(M_f^5(x,y)\right) - \phi\left(M_f^5(x,y)\right),
\end{aligned}$$

where

$$M_f^5(x,y) = \max\left\{S_b(x,x,y), S_b(x,x,fx), S_b(y,y,fy), S_b(x,x,fy), S_b(y,y,fx)\right\}.$$

It follows from Theorem 3.4 that f has a unique fixed point in X. □

3.2 Application to homotopy

In this section, we study the existence of a unique solution to homotopy theory.

Theorem 3.10 *Let (X,S_b) be a complete S_b-metric space, U be an open subset of X and $\overline{U}$ be a closed subset of X such that $U \subseteq \overline{U}$. Suppose that $H : \overline{U} \times [0,1] \to X$ is an operator such that the following conditions are satisfied*:

(i) *$x \neq H(x,\lambda)$ for each $x \in \partial U$ and $\lambda \in [0,1]$ (here ∂U denotes the boundary of U in X)*,

(ii) *$\psi(4b^4 S_b(H(x,\lambda),H(x,\lambda),H(y,\lambda))) \le \psi(S_b(x,x,y)) - \phi(S_b(x,x,y))$ $\forall x,y \in \overline{U}$ and $\lambda \in [0,1]$, where $\psi : [0,\infty) \to [0,\infty)$ is continuous, non-decreasing and $\phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous with $\phi(t) > 0$ for $t > 0$*,

(iii) *there exists $M \ge 0$ such that*

$$S_b\big(H(x,\lambda),H(x,\lambda),H(x,\mu)\big) \le M|\lambda-\mu|$$

for every $x \in \overline{U}$ and $\lambda,\mu \in [0,1]$.

Then $H(\cdot,0)$ has a fixed point if and only if $H(\cdot,1)$ has a fixed point.

Proof Consider the set

$$A = \left\{\lambda \in [0,1] : x = H(x,\lambda) \text{ for some } x \in U\right\}.$$

Since $H(\cdot,0)$ has a fixed point in U, we have that $0 \in A$. So that A is a non-empty set.

We will show that A is both open and closed in $[0,1]$, and so, by the connectedness, we have that $A = [0,1]$. As a result, $H(\cdot,1)$ has a fixed point in U. First we show that A is closed in $[0,1]$. To see this, let $\{\lambda_n\}_{n=1}^{\infty} \subseteq A$ with $\lambda_n \to \lambda \in [0,1]$ as $n \to \infty$.

We must show that $\lambda \in A$. Since $\lambda_n \in A$ for $n = 1, 2, 3, \ldots$, there exists $x_n \in U$ with $x_n = H(x_n, \lambda_n)$.

Consider

$$\begin{aligned} S_b(x_n, x_n, x_{n+1}) &= S_b\big(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_{n+1})\big) \\ &\le 2bS_b\big(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_n)\big) \\ &\quad + b^2 S_b\big(H(x_{n+1},\lambda_n), H(x_{n+1},\lambda_n), H(x_{n+1},\lambda_{n+1})\big) \\ &\le S_b\big(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_n)\big) + M|\lambda_n - \lambda_{n+1}|. \end{aligned}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} S_b(x_n, x_n, x_{n+1}) \le \lim_{n\to\infty} S_b\big(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_n)\big) + 0.$$

Since ψ is continuous and non-decreasing, we obtain

$$\begin{aligned} \lim_{n\to\infty} \psi\big(4b^4 S_b(x_n, x_n, x_{n+1})\big) &\le \lim_{n\to\infty} \psi\big(4b^4 S_b\big(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_n)\big)\big) \\ &\le \lim_{n\to\infty} \big[\psi\big(S_b(x_n, x_n, x_{n+1})\big) - \phi\big(S_b(x_n, x_n, x_{n+1})\big)\big]. \end{aligned}$$

By the definition of ψ, it follows that

$$\lim_{n\to\infty} \big(4b^4 - 1\big) S_b(x_n, x_n, x_{n+1}) \le 0.$$

So that

$$\lim_{n\to\infty} S_b(x_n, x_n, x_{n+1}) = 0. \tag{12}$$

Now we prove that $\{x_n\}$ is an S_b-Cauchy sequence in (X, d_p). On the contrary, suppose that $\{x_n\}$ is not S_b-Cauchy.

There exists $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$,

$$S_b(x_{m_k}, x_{m_k}, x_{n_k}) \ge \epsilon \tag{13}$$

and

$$S_b(x_{m_k}, x_{m_k}, x_{n_k - 1}) < \epsilon. \tag{14}$$

From (13) and (14), we obtain

$$\begin{aligned} \epsilon &\le S_b(x_{m_k}, x_{m_k}, x_{n_k}) \\ &\le 2bS_b(x_{m_k}, x_{m_k}, x_{m_k+1}) + b^2 S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}). \end{aligned}$$

Letting $k \to \infty$ and applying ψ on both sides, we have that

$$\psi\big(2b^2\epsilon\big) \le \lim_{n\to\infty} \psi\big(4b^4 S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})\big). \tag{15}$$

But

$$\begin{aligned}
&\lim_{n\to\infty} \psi\bigl(4b^4 S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})\bigr) \\
&\quad = \lim_{n\to\infty} \psi\bigl(S_b\bigl(4b^4 H(x_{m_k+1}, \lambda_{m_k+1}), H(x_{m_k+1}, \lambda_{m_k+1}), H(x_{n_k}, \lambda_{n_k})\bigr)\bigr) \\
&\quad \le \lim_{n\to\infty} \bigl[\psi\bigl(S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})\bigr) - \phi\bigl(S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})\bigr)\bigr].
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \bigl(4b^4 - 1\bigr) S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}) \le 0.$$

Thus

$$\lim_{n\to\infty} S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}) = 0.$$

Hence from (15) and the definition of ψ, we have that

$$\epsilon \le 0,$$

which is a contradiction.

Hence $\{x_n\}$ is an S_b-Cauchy sequence in (X, S_b) and, by the completeness of (X, S_b), there exists $\alpha \in U$ with

$$\lim_{n\to\infty} x_n = \alpha = \lim_{n\to\infty} x_{n+1}, \tag{16}$$

$$\begin{aligned}
\psi\bigl(2b^3 S_b\bigl(H(\alpha, \lambda), H(\alpha, \lambda), \alpha\bigr)\bigr) &\le \lim_{n\to\infty} \inf \psi\bigl(4b^4 S_b\bigl(H(\alpha, \lambda), H(\alpha, \lambda), H(x_n, \lambda)\bigr)\bigr) \\
&\le \lim_{n\to\infty} \inf\bigl[\psi\bigl(S_b(\alpha, \alpha, x_n)\bigr) - \phi\bigl(S_b(\alpha, \alpha, x_n)\bigr)\bigr] \\
&= 0.
\end{aligned}$$

It follows that $\alpha = H(\alpha, \lambda)$.

Thus $\lambda \in A$. Hence A is closed in $[0, 1]$.

Let $\lambda_0 \in A$. Then there exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$.

Since U is open, there exists $r > 0$ such that $B_{S_b}(x_0, r) \subseteq U$.

Choose $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ such that $|\lambda - \lambda_0| \le \frac{1}{M^n} < \epsilon$.

Then, for $x \in \overline{B_p(x_0, r)} = \{x \in X / S_b(x, x, x_0) \le r + b^2 S_b(x_0, x_0, x_0)\}$,

$$\begin{aligned}
&S_b\bigl(H(x, \lambda), H(x, \lambda), x_0\bigr) \\
&\quad = S_b\bigl(H(x, \lambda), H(x, \lambda), H(x_0, \lambda_0)\bigr) \\
&\quad \le 2b S_b\bigl(H(x, \lambda), H(x, \lambda), H(x, \lambda_0)\bigr) + b^2 S_b\bigl(H(x, \lambda_0), H(x, \lambda_0), H(x_0, \lambda_0)\bigr) \\
&\quad \le 2bM|\lambda - \lambda_0| + b^2 S_b\bigl(H(x, \lambda_0), H(x, \lambda_0), H(x_0, \lambda_0)\bigr) \\
&\quad \le \frac{2b}{M^{n-1}} + b^2 S_b\bigl(H(x, \lambda_0), H(x, \lambda_0), H(x_0, \lambda_0)\bigr).
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$S_b\big(H(x,\lambda),H(x,\lambda),x_0\big) \le b^2 S_b\big(H(x,\lambda_0),H(x,\lambda_0),H(x_0,\lambda_0)\big).$$

Since ψ is continuous and non-decreasing, we have

$$\begin{aligned}\psi\big(S_b\big(H(x,\lambda),H(x,\lambda),x_0\big)\big) &\le \psi\big(4b^2 S_b\big(H(x,\lambda),H(x,\lambda),x_0\big)\big)\\ &\le \psi\big(4b^4 S_b\big(H(x,\lambda_0),H(x,\lambda_0),H(x_0,\lambda_0)\big)\big)\\ &\le \psi\big(S_b(x,x,x_0)\big) - \phi\big(S_b(x,x,x_0)\big)\\ &\le \psi\big(S_b(x,x,x_0)\big).\end{aligned}$$

Since ψ is non-decreasing, we have

$$\begin{aligned}S_b\big(H(x,\lambda),H(x,\lambda),x_0\big) &\le S_b(x,x,x_0)\\ &\le r + b^2 S_b(x_0,x_0,x_0).\end{aligned}$$

Thus, for each fixed $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, $H(\cdot,\lambda) : \overline{B_p(x_0,r)} \to \overline{B_p(x_0,r)}$.

Since also (ii) holds and ψ is continuous and non-decreasing and ϕ is continuous with $\phi(t) > 0$ for $t > 0$, then all the conditions of Theorem (3.10) are satisfied.

Thus we deduce that $H(\cdot,\lambda)$ has a fixed point in $\overline{U}$. But this fixed point must be in U since (i) holds.

Thus $\lambda \in A$ for any $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$.

Hence $(\lambda_0 - \epsilon, \lambda_0 + \epsilon) \subseteq A$ and therefore A is open in $[0,1]$.

For the reverse implication, we use the same strategy. □

Corollary 3.11 *Let (X,p) be a complete partial metric space, U be an open subset of X and $H : \overline{U} \times [0,1] \to X$ with the following properties*:

(1) $x \neq H(x,t)$ *for each* $x \in \partial U$ *and each* $\lambda \in [0,1]$ (*here* ∂U *denotes the boundary of* U *in* X),

(2) *there exist* $x,y \in \overline{U}$ *and* $\lambda \in [0,1], L \in [0, \frac{1}{4b^4})$ *such that*

$$S_b\big(H(x,\lambda),H(x,\lambda),H(y,\mu)\big) \le L S_b(x,x,y),$$

(3) *there exists* $M \ge 0$ *such that*

$$S_b\big(H(x,\lambda),H(x,\lambda),H(x,\mu)\big) \le M|\lambda - \mu|$$

for all $x \in \overline{U}$ *and* $\lambda,\mu \in [0,1]$.

If $H(\cdot,0)$ *has a fixed point in* U, *then* $H(\cdot,1)$ *has a fixed point in* U.

Proof Proof follows by taking $\psi(x) = x, \phi(x) = x - Lx$ with $L \in [0, \frac{1}{4b^4})$ in Theorem (3.10). □

4 Conclusions

In this paper we conclude some applications to homotopy theory and integral equations by using fixed point theorems in partially ordered S_b-metric spaces.

Acknowledgements
Authors are thankful to referees for their valuable suggestions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, K L University, Vaddeswaram, Guntur-522 502, Andhra Pradesh, India. [2]Department of Mathematics, Acharya Nagarjuna University, Guntur-522 502, Andhra Pradesh, India. [3]Department of Mathematics, Nepal Sanskrit University, Valmeeki Campus, Exhibition Road, Kathmandu, Nepal. [4]Department of Mathematics, Dr. B. R. Ambedkar University, Srikakulam-532 410, Andhra Pradesh, India. [5]Department of Computing, Adama Science and Technology University, Adama, Ethiopia.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 24 April 2017 Accepted: 22 May 2017 Published online: 05 June 2017

References

1. Banach, S: Theorie des Operations Lineaires. Manograic Mathematic Zne, Warsaw (1932)
2. Aghajani, A, Abbas, M, Roshan, JR: Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces. Filomat **28**(6), 1087-1101 (2014)
3. Ansari, AH, Ege, O, Randenović, S: Some fixed point results on complex valued G_b-metric spaces. RACSAM (2017). doi:10.1007/s13398-017-0391-x
4. Anasari, AH, Vetro, P, Randenović, S: Integration of type pair (H, F) upclass in fixed point result for $GP_{(\Lambda,\Theta)}$ contractive mappings. Filomat **31**(8), 2211-2218 (2017)
5. Arshad, M, Kadelburg, Z, Randenović, S, Shoaib, A, Shukla, S: α-dominated mappings on dislocated metric spaces and fixed point results. Filomat **31**(11), 3041-3056 (2017)
6. Bakhtin, IA: The contraction mapping principle in quasimetric spaces. Funct. Anal. Unianowsk Gos. Ped. Inst. **30**, 26-37 (1989)
7. Czerwik, S: Contraction mapping in *b*-metric spaces. Acta Math. Inform. Univ. Ostrav. **1**, 5-11 (1993)
8. Došenović, T, Randenović, S: Some critical remarks on the paper: "An essential remark on fixed point results on multiplicative metric spaces". J. Adv. Math. Stud. **10**(1), 20-24 (2017)
9. Jaradat, MMM, Mustafa, Z, Ansari, AH, Kumari, PS, Djekić, DD, Jaradat, HM: Some fixed point results for $F_{\alpha-\omega,\varphi}$-generalized cyclic contractions on metric-like space with applications to graphs and integral equations. J. Math. Anal. **8**(1), 28-45 (2017)
10. Kishore, GNV, Rao, KPR, Hima Bindu, VML: Suzuki type unique common fixed point theorem in partial metric spaces by using (C): condition with rational expressions. Afr. Math. (2016). doi:10.1007/s13370-017-0484-x
11. Lakshmikantham, V, Lj, C: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70**(12), 4341-4349 (2009)
12. Mustafa, Z, Sims, B: A new approach to generalized metric spaces. J. Nonlinear Convex Anal. **7**(2), 289-297 (2006)
13. Mustafa, Z, Roshan, JR, Parvaneh, V, Kadelburg, Z: Some common fixed point results in ordered partial b-metric space. J. Inequal. Appl. **2013**, 562 (2013)
14. Mustafa, Z, Roshan, JR, Parvaneh, V, Kadelburg, Z: Fixed point theorems for weakly T-Chatterjea and weakly T-Kannan contractions in b-metric spaces. J. Inequal. Appl. **2014**, 46 (2014)
15. Mustafa, Z, Jaradat, M, Ansari, AH, Popovi, BZ, Jaradat, H: *C*-class functions with new approach on coincidence point results for generalized (ψ, ϕ)-weakly contractions in ordered *b*-metric spaces. SpringerPlus **5**, 802 (2016)
16. Mustafa, Z, Jaradata, MMM, Jaradat, HM: Some common fixed point results of graphs on *b*-metric space. J. Nonlinear Sci. Appl. **9**(6), 4838-4851 (2016)
17. Mustafa, Z, Huang, H, Randenović, S: Some remarks on the paper "Some fixed point generalizations are not real generalizations". J. Adv. Math. Stud. **9**(1), 110-116 (2016)
18. Sedghi, S, Altun, I, Shobe, N, Salahshour, M: Some properties of *S*-metric space and fixed point results. Kyungpook Math. J. **54**, 113-122 (2014)
19. Rohen, Y, Došenović, T, Randenović, S: A note on the paper "A fixed point theorems in S_b-metric spaces". Filomat **31**(11), 3335-3346 (2017)
20. Sedghi, S, Gholidahneh, A, Došenović, T, Esfahani, J, Randenović, S: Common fixed point of four maps in S_b-metric spaces. J. Linear Topol. Algebra **5**(2), 93-104 (2016)
21. Sedghi, S, Rezaee, MM, Došenović, T, Sü, R: Common fixed point theorems for contractive mappings satisfying ϕ-maps in *S*-metric spaces. Acta Univ. Sapientiae Math. **8**(2), 298-311 (2016)
22. Sedghi, S, Shobe, N, Aliouche, A: A generalization of fixed point theorem in *S*-metric spaces. Mat. Vesn. **64**, 258-266 (2012)
23. Sedghi, S, Shobe, N, Došenović, T: Fixed point results in *S*-metric spaces. Nonlinear Funct. Anal. Appl. **20**(1), 55-67 (2015)
24. Zhou, M, Liu, XL, Randenović, S: S-γ-φ-contractive type mappings in *S*-metric spaces. J. Nonlinear Sci. Appl. **10**, 1613-1639 (2017)