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SOME ARITHMETIC IDENTITIES INVOLVING DIVISOR FUNCTIONS

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Abstract: For a positive integer n, let

$$\sigma(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d.$$

The explicit evaluation of such arithmetic sums as

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+2b+4c=n}} \sigma(a)\sigma(b)\sigma(c) \qquad \text{and} \qquad \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a+2b=n}} a\sigma(a)\sigma(b)$$

is carried out for all positive integers n.

Keywords: sum of divisors function, Eisenstein series.

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{Z}$ the set of all integers. For $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, we define as usual

$$\sigma_k(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d^k. \tag{1.1}$$

If $n \notin \mathbb{N}$, we set $\sigma_k(n) = 0$. Further, we set $\sigma(n) := \sigma_1(n)$.

For a complex variable q satisfying $|q| < 1$, Ramanujan's Eisenstein series

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$L(q), M(q)$ and $N(q)$ are defined by

$$L(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n, \tag{1.2}$$

$$M(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{1.3}$$

$$N(q) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n. \tag{1.4}$$

Further, for $r \in \mathbb{N}_0$ and $s, k \in \mathbb{N}$, we define

$$[r, s]_k := \sum_{n=1}^{\infty} n^r \sigma_s(n) q^{kn}. \tag{1.5}$$

Lahiri [4, p. 149] has derived the following thirteen identities from the work of Ramanujan [5], [6, Theorem 2.4, pp. 141 - 142]:

$$L^2(q) = 1 - 288[1,1]_1 + 240[0,3]_1, \tag{1.6}$$

$$L^3(q) = 1 - 1728[2,1]_1 + 2160[1,3]_1 - 504[0,5]_1, \tag{1.7}$$

$$L^4(q) = 1 - 6912[3,1]_1 + 10368[2,3]_1 - 4032[1,5]_1 + 480[0,7]_1, \tag{1.8}$$

$$L^5(q) = 1 - 20736[4,1]_1 + 34560[3,3]_1 - 17280[2,5]_1 + 3600[1,7]_1 - 264[0,9]_1, \tag{1.9}$$

$$M^2(q) = 1 + 480[0,7]_1, \tag{1.10}$$

$$L(q)M(q) = 1 + 720[1,3]_1 - 504[0,5]_1, \tag{1.11}$$

$$L^2(q)M(q) = 1 + 1728[2,3]_1 - 2016[1,5]_1 + 480[0,7]_1, \tag{1.12}$$

$$L^3(q)M(q) = 1 + 3456[3,3]_1 - 5184[2,5]_1 + 2160[1,7]_1 - 264[0,9]_1, \tag{1.13}$$

$$L(q)M^2(q) = 1 + 720[1,7]_1 - 264[0,9]_1, \tag{1.14}$$

$$L(q)N(q) = 1 - 1008[1,5]_1 + 480[0,7]_1, \tag{1.15}$$

$$L^2(q)N(q) = 1 - 1728[2,5]_1 + 1440[1,7]_1 - 264[0,9]_1, \tag{1.16}$$

$$M(q)N(q) = 1 - 264[0,9]_1, \tag{1.17}$$

$$M^2(q)N(q) = 1 - 24[0,13]_1. \tag{1.18}$$

In this paper, we are interested in products of the form

$$L^{a_1}(q)L^{a_2}(q^2)L^{a_4}(q^4)M^{b_1}(q)M^{b_2}(q^2)M^{b_4}(q^4)N^{c_1}(q)N^{c_2}(q^2)N^{c_4}(q^4) \tag{1.19}$$

with

$$\begin{cases} a_1, a_2, a_4, b_1, b_2, b_4, c_1, c_2, c_4 \in \mathbb{N}_0, \\ \max(a_1, b_1, c_1) \in \mathbb{N}, \\ \max(a_2, b_2, c_2, a_4, b_4, c_4) \in \mathbb{N}. \end{cases} \tag{1.20}$$

We seek those products which can be expressed as a linear combination of the series $[r,s]_1, [r,s]_2$ and $[r,s]_4$ $(r \in \mathbb{N}_0, s \in \mathbb{N})$. Guided by (1.6)-(1.18), a computer search for such products was carried out in the range

$$\begin{cases} \max(a_1, a_2, a_4) = 5, \\ \max(b_1, b_2, b_4) = 2, \\ \max(c_1, c_2, c_4) = 1, \\ \operatorname{sgn}(a_1) + \operatorname{sgn}(a_2) + \operatorname{sgn}(a_4) + \operatorname{sgn}(b_1) \\ \ +\operatorname{sgn}(b_2) + \operatorname{sgn}(b_4) + \operatorname{sgn}(c_1) + \operatorname{sgn}(c_2) + \operatorname{sgn}(c_4) \leqslant 3. \end{cases} \tag{1.21}$$

The search produced seven possible identities. All seven candidates were subsequently proved and are given in Theorem 1.1. The proof of Theorem 1.1 is given in Section 2.

Theorem 1.1. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(i)
$$L(q)L(q^2) = 1 - 72[1,1]_1 + 48[0,3]_1 - 288[1,1]_2 + 192[0,3]_2,$$

(ii)
$$L(q)L(q^4) = 1 - 36[1,1]_1 + 12[0,3]_1 + 36[0,3]_2 - 576[1,1]_4 + 192[0,3]_4,$$

(iii)
$$L(q)M(q^2) = 1 - 24[0,5]_1 + 1440[1,3]_2 - 480[0,5]_2,$$

(iv)
$$L(q^2)M(q) = 1 + 360[1,3]_1 - 120[0,5]_1 - 384[0,5]_2,$$

(v)
$$\begin{aligned} L(q)L^2(q^2) = 1 &- 144[2,1]_1 + 144[1,3]_1 - 24[0,5]_1 - 2304[2,1]_2 \\ &+ 2592[1,3]_2 - 480[0,5]_2, \end{aligned}$$

(vi)
$$\begin{aligned} L^2(q)L(q^2) = 1 &- 576[2,1]_1 + 648[1,3]_1 - 120[0,5]_1 - 2304[2,1]_2 \\ &+ 2304[1,3]_2 - 384[0,5]_2, \end{aligned}$$

(vii)
$$\begin{aligned} L(q)L(q^2)L(q^4) = 1 &- 72[2,1]_1 + 54[1,3]_1 - 6[0,5]_1 \\ &- 576[2,1]_2 + 684[1,3]_2 - 114[0,5]_2 \\ &- 4608[2,1]_4 + 3456[1,3]_4 - 384[0,5]_4. \end{aligned}$$

Equating the coefficients of q^n $(n \in \mathbb{N})$ in these identities, we obtain seven arithmetic identities. These identities are given in Theorem 1.2. The proof of Theorem 1.2 is given in Section 3.

Theorem 1.2. *Let $n \in \mathbb{N}$. We have*

(i)

$$\sum_{\substack{(a,b)\in\mathbb{N}^2\\ a+2b=n}} \sigma(a)\sigma(b) = \frac{1}{12}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{8}n\right)\sigma(n) + \frac{1}{3}\sigma_3(n/2) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n/2),$$

(ii)

$$\sum_{\substack{(a,b)\in\mathbb{N}^2\\ a+4b=n}} \sigma(a)\sigma(b) = \frac{1}{48}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{16}n\right)\sigma(n) + \frac{1}{16}\sigma_3(n/2) + \frac{1}{3}\sigma_3(n/4) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n/4),$$

(iii)

$$\sum_{\substack{(a,b)\in\mathbb{N}^2\\ a+2b=n}} \sigma(a)\sigma_3(b) = \frac{1}{240}\sigma_5(n) - \frac{1}{240}\sigma(n) + \frac{1}{12}\sigma_5(n/2) + \left(\frac{1}{24} - \frac{1}{8}n\right)\sigma_3(n/2),$$

(iv)

$$\sum_{\substack{(a,b)\in\mathbb{N}^2\\ 2a+b=n}} \sigma(a)\sigma_3(b) = \frac{1}{48}\sigma_5(n) + \left(\frac{1}{24} - \frac{1}{16}n\right)\sigma_3(n) + \frac{1}{15}\sigma_5(n/2) - \frac{1}{240}\sigma(n/2),$$

(v)

$$\begin{aligned}\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+2b+2c=n}} \sigma(a)\sigma(b)\sigma(c) = \frac{1}{576}\sigma_5(n) &+ \left(\frac{1}{144} - \frac{1}{96}n\right)\sigma_3(n)\\ &+ \left(\frac{1}{576} - \frac{1}{96}n + \frac{1}{96}n^2\right)\sigma(n) + \frac{5}{144}\sigma_5(n/2)\\ &+ \left(\frac{13}{288} - \frac{3}{32}n\right)\sigma_3(n/2)\\ &+ \left(\frac{1}{288} - \frac{1}{32}n + \frac{1}{24}n^2\right)\sigma(n/2),\end{aligned}$$

(vi)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+2c=n}} \sigma(a)\sigma(b)\sigma(c) = \frac{5}{576}\sigma_5(n) + \left(\frac{7}{288} - \frac{3}{64}n\right)\sigma_3(n)$$

$$+ \left(\frac{1}{288} - \frac{1}{32}n + \frac{1}{24}n^2\right)\sigma(n)$$

$$+ \frac{1}{36}\sigma_5(n/2) + \left(\frac{1}{36} - \frac{1}{12}n\right)\sigma_3(n/2)$$

$$+ \left(\frac{1}{576} - \frac{1}{48}n + \frac{1}{24}n^2\right)\sigma(n/2),$$

(vii)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+2b+4c=n}} \sigma(a)\sigma(b)\sigma(c) = \frac{1}{2304}\sigma_5(n) + \left(\frac{5}{1152} - \frac{1}{256}n\right)\sigma_3(n)$$

$$+ \left(\frac{1}{576} - \frac{1}{128}n + \frac{1}{192}n^2\right)\sigma(n)$$

$$+ \frac{19}{2304}\sigma_5(n/2) + \left(\frac{23}{1152} - \frac{19}{768}n\right)\sigma_3(n/2)$$

$$+ \left(\frac{1}{576} - \frac{5}{384}n + \frac{1}{96}n^2\right)\sigma(n/2)$$

$$+ \frac{1}{36}\sigma_5(n/4) + \left(\frac{1}{36} - \frac{1}{16}n\right)\sigma_3(n/4)$$

$$+ \left(\frac{1}{576} - \frac{1}{64}n + \frac{1}{48}n^2\right)\sigma(n/4).$$

Parts (i), (ii), (iii) and (iv) of Theorem 1.2 are due to Huard, Ou, Spearman and Williams [3, Theorems 2, 4, 6]. The rest are new. Two further arithmetic identities are deduced from Theorem 1.1 in Section 4 (see Theorem 4.1).

2. Proof of Theorem 1.1

Let q be a complex variable satisfying $|q| < 1$. Define the theta function $\varphi(q)$ (Ramanujan's notation [1, p. 6]) by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{2.1}$$

Further, we define the parameters $x = x(q)$ and $z = z(q)$ by

$$x := 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \qquad z := \varphi^2(q), \tag{2.2}$$

(see [1, pp. 119 - 120]). Then it is known that

$$L(q) = (1-5x)z^2 + 12x(1-x)zz', \tag{2.3}$$
$$L(q^2) = (1-2x)z^2 + 6x(1-x)zz', \tag{2.4}$$
$$L(q^4) = \left(1-\frac{5}{4}x\right)z^2 + 3x(1-x)zz', \tag{2.5}$$
$$M(q) = (1+14x+x^2)z^4, \tag{2.6}$$
$$M(q^2) = (1-x+x^2)z^4, \tag{2.7}$$
$$M(q^4) = \left(1-x+\frac{1}{16}x^2\right)z^4, \tag{2.8}$$
$$N(q) = (1-33x-33x^2+x^3)z^6, \tag{2.9}$$
$$N(q^2) = \left(1-\frac{3}{2}x-\frac{3}{2}x^2+x^3\right)z^6, \tag{2.10}$$
$$N(q^4) = \left(1-\frac{3}{2}x+\frac{15}{32}x^2+\frac{1}{64}x^3\right)z^6, \tag{2.11}$$

(see [1, pp. 125 - 128]), where $z' = \dfrac{dz}{dx}$. From (1.3) and (1.5), we have

$$[0,3]_1 = \sum_{n=1}^{\infty}\sigma_3(n)q^n = -\frac{1}{240} + \frac{1}{240}M(q). \tag{2.12}$$

From (1.4) and (1.5), we have

$$[0,5]_1 = \sum_{n=1}^{\infty}\sigma_5(n)q^n = \frac{1}{504} - \frac{1}{504}N(q). \tag{2.13}$$

Then, from (1.6) and (2.12), we deduce

$$[1,1]_1 = -\frac{1}{288}L^2(q) + \frac{1}{288}M(q). \tag{2.14}$$

From (1.11) and (2.13), we have

$$[1,3]_1 = \frac{1}{720}L(q)M(q) - \frac{1}{720}N(q). \tag{2.15}$$

From (1.7), (2.13) and (2.15), we obtain

$$[2,1]_1 = -\frac{1}{1728}L^3(q) + \frac{1}{576}L(q)M(q) - \frac{1}{864}N(q). \tag{2.16}$$

We now carry out the details of the proof of part (v) of Theorem 1.1. The remaining parts can be proved in a similar manner.

Appealing to (2.3), (2.4), (2.6), (2.7), (2.9), (2.10), (2.13), (2.15) and (2.16), we obtain

$$\begin{aligned}
&1-144[2,1]_1+144[1,3]_1-24[0,5]_1-2304[2,1]_2+2592[1,3]_2-480[0,5]_2\\
&\quad=1-144\Big(-\frac{1}{1728}L^3(q)+\frac{1}{576}L(q)M(q)-\frac{1}{864}N(q)\Big)\\
&\qquad+144\Big(\frac{1}{720}L(q)M(q)-\frac{1}{720}N(q)\Big)-24\Big(\frac{1}{504}-\frac{1}{504}N(q)\Big)\\
&\qquad-2304\Big(-\frac{1}{1728}L^3(q^2)+\frac{1}{576}L(q^2)M(q^2)-\frac{1}{864}N(q^2)\Big)\\
&\qquad+2592\Big(\frac{1}{720}L(q^2)M(q^2)-\frac{1}{720}N(q^2)\Big)-480\Big(\frac{1}{504}-\frac{1}{504}N(q^2)\Big)\\
&\quad=\frac{1}{12}L^3(q)-\frac{1}{20}L(q)M(q)+\frac{1}{70}N(q)+\frac{4}{3}L^3(q^2)-\frac{2}{5}L(q^2)M(q^2)\\
&\qquad+\frac{2}{105}N(q^2)\\
&\quad=\frac{1}{12}\Big((1-5x)z^2+12x(1-x)zz'\Big)^3\\
&\qquad-\frac{1}{20}\Big((1-5x)z^2+12x(1-x)zz'\Big)(1+14x+x^2)z^4\\
&\qquad+\frac{1}{70}(1-33x-33x^2+x^3)z^6+\frac{4}{3}\Big((1-2x)z^2+6x(1-x)zz'\Big)^3\\
&\qquad-\frac{2}{5}\Big((1-2x)z^2+6x(1-x)zz'\Big)(1-x+x^2)z^4\\
&\qquad+\frac{2}{105}\Big(1-\frac{3}{2}x-\frac{3}{2}x^2+x^3\Big)z^6\\
&\quad=(1-2x)^2(1-5x)z^6+12x(1-x)(1-2x)(2-7x)z^5z'\\
&\qquad+36x^2(1-x)^2(5-13x)z^4z'^2+432x^3(1-x)^3z^3z'^3\\
&\quad=\Big((1-5x)z^2+12x(1-x)zz'\Big)\Big((1-2x)z^2+6x(1-x)zz'\Big)^2\\
&\quad=L(q)L^2(q^2),
\end{aligned}$$

as asserted.

We remark that in the course of the proof of part (v), we proved the new identity

$$\begin{aligned}
L(q)L^2(q^2)&=\frac{1}{12}L^3(q)-\frac{1}{20}L(q)M(q)+\frac{1}{70}N(q)+\frac{4}{3}L^3(q^2)\\
&\qquad-\frac{2}{5}L(q^2)M(q^2)+\frac{2}{105}N(q^2).
\end{aligned}\tag{2.17}$$

Similarly, from the proof of part (i), we have

$$L(q)L(q^2)=\frac{1}{4}L^2(q)+L^2(q^2)-\frac{1}{20}M(q)-\frac{1}{5}M(q^2),\tag{2.18}$$

from part (ii)

$$L(q)L(q^4)=\frac{1}{8}L^2(q)+2L^2(q^4)-\frac{3}{40}M(q)+\frac{3}{20}M(q^2)-\frac{6}{5}M(q^4), \tag{2.19}$$

from part (iii)

$$L(q)M(q^2)=2L(q^2)M(q^2)+\frac{1}{21}N(q)-\frac{22}{21}N(q^2), \tag{2.20}$$

and from part (iv)

$$L(q^2)M(q)=\frac{1}{2}L(q)M(q)-\frac{11}{42}N(q)+\frac{16}{21}N(q^2). \tag{2.21}$$

The four identities (2.18) - (2.21) are due to Cheng and Williams [2, Theorems 3.1 and 4.1(i),(ii)]. The corresponding identities for parts (vi) and (vii), namely,

$$\begin{aligned}L^2(q)L(q^2)=&\frac{1}{3}L^3(q)-\frac{1}{10}L(q)M(q)+\frac{1}{210}N(q)+\frac{4}{3}L^3(q^2)\\&-\frac{4}{5}L(q^2)M(q^2)+\frac{8}{35}N(q^2)\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}L(q)L(q^2)L(q^4)=&\frac{1}{24}L^3(q)-\frac{1}{20}L(q)M(q)+\frac{17}{840}N(q)+\frac{1}{3}L^3(q^2)\\&-\frac{1}{20}L(q^2)M(q^2)-\frac{2}{35}N(q^2)+\frac{8}{3}L^3(q^4)\\&-\frac{16}{5}L(q^4)M(q^4)+\frac{136}{105}N(q^4)\end{aligned} \tag{2.23}$$

are new.

3. Proof of Theorem 1.2

We only prove part (vi) of Theorem 1.2 as the remaining parts can be proved in a similar manner.

Using (1.2) we have

$$\begin{aligned}13824\sum_{n=1}^{\infty}\left(\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\a+b+2c=n}}\sigma(a)\sigma(b)\sigma(c)\right)q^n&=24^3\sum_{(a,b,c)\in\mathbb{N}^3}\sigma(a)\sigma(b)\sigma(c)q^{a+b+2c}\\&=\left(24\sum_{a=1}^{\infty}\sigma(a)q^a\right)^2\left(24\sum_{c=1}^{\infty}\sigma(c)q^{2c}\right)\\&=(1-L(q))^2(1-L(q^2))\\&=1-2L(q)+L^2(q)-L(q^2)\\&\quad+2L(q)L(q^2)-L^2(q)L(q^2).\end{aligned}$$

Appealing to (1.2), (1.6), (1.2) (with q replaced by q^2), Theorem 1.1(i) and Theorem 1.1(vi) for the series expansions in powers of q of $L(q)$, $L^2(q)$, $L(q^2)$, $L(q)L(q^2)$ and $L^2(q)L(q^2)$ respectively, and then equating the coefficients of q^n $(n \in \mathbb{N})$, we obtain the asserted formula on dividing by 13824. ■

4. Two Arithmetic Identities

Theorem 1.1 enables us to prove two more new identities involving the divisor function σ.

Theorem 4.1. *Let $n \in \mathbb{N}$. We have*

$$\sum_{\substack{(a,b)\in\mathbb{N}^2\\ a+2b=n}} a\sigma(a)\sigma(b) = \frac{1}{24}n\sigma_3(n) + \left(\frac{1}{24}n - \frac{1}{12}n^2\right)\sigma(n) + \frac{1}{6}n\sigma_3(n/2) - \frac{1}{12}n^2\sigma(n/2)$$

and

$$\sum_{\substack{(a,b)\in\mathbb{N}^2\\ 2a+b=n}} a\sigma(a)\sigma(b) = \frac{1}{48}n\sigma_3(n) - \frac{1}{48}n^2\sigma(n) + \frac{1}{12}n\sigma_3(n/2) + \left(\frac{1}{48}n - \frac{1}{12}n^2\right)\sigma(n/2).$$

Proof. Differentiating (1.2) with respect to q, we obtain

$$qL'(q) = -24\sum_{n=1}^{\infty} n\sigma(n)q^n = -24[1,1]_1. \tag{4.1}$$

Hence, by (1.6) and (2.12), we deduce

$$\begin{aligned} L^2(q) &= 1 - 288[1,1]_1 + 240[0,3]_1 \\ &= 1 + 12qL'(q) + 240\left(-\frac{1}{240} + \frac{1}{240}M(q)\right) \\ &= 12qL'(q) + M(q), \end{aligned}$$

so

$$qL'(q) = \frac{1}{12}L^2(q) - \frac{1}{12}M(q), \tag{4.2}$$

which is a well-known identity.

We now prove the first identity of the theorem. Multiplying (4.2) by $L(q^2)$, we obtain

$$qL'(q)L(q^2) = \frac{1}{12}L^2(q)L(q^2) - \frac{1}{12}L(q^2)M(q).$$

Appealing to Theorem 1.1 for the values of $L^2(q)L(q^2)$ and $L(q^2)M(q)$, we obtain

$$qL'(q)L(q^2) = -48[2,1]_1 + 24[1,3]_1 - 192[2,1]_2 + 192[1,3]_2.$$

Now, by (4.2) and (1.2), we have

$$\begin{aligned} qL'(q)L(q^2) &= \Big(-24\sum_{e=1}^{\infty} a\sigma(a)q^a\Big)\Big(1-24\sum_{b=1}^{\infty}\sigma(b)q^{2b}\Big) \\ &= -24[1,1]_1 + 576\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{(a,b)\in\mathbb{N}^2 \\ a+2b=n}} a\sigma(a)\sigma(b)\Bigg)q^n, \end{aligned}$$

so

$$\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{(a,b)\in\mathbb{N}^2 \\ a+2b=n}} a\sigma(a)\sigma(b)\Bigg)q^n$$
$$= \frac{1}{24}[1,1]_1 - \frac{1}{12}[2,1]_1 + \frac{1}{24}[1,3]_1 - \frac{1}{3}[2,1]_2 + \frac{1}{3}[1,3]_2.$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain the first identity.

We now prove the second identity. Replacing q by q^2 in (4.2) and then multiplying by $L(q)$, we obtain

$$q^2L'(q^2)L(q) = \frac{1}{12}L(q)L^2(q^2) - \frac{1}{12}L(q)M(q^2).$$

Appealing to Theorem 1.1 for the values of $L(q)L^2(q^2)$ and $L(q)M(q^2)$, we have

$$q^2L'(q^2)L(q) = 12[1,3]_1 - 12[2,1]_1 + 96[1,3]_2 - 192[2,1]_2.$$

Now

$$\begin{aligned} q^2L'(q^2)L(q) &= -24\sum_{a=1}^{\infty} a\sigma(a)q^{2a}\Big(1-24\sum_{b=1}^{\infty}\sigma(b)q^b\Big) \\ &= -24[1,1]_2 + 576\sum_{n=1}^{\infty}\Big(\sum_{\substack{(a,b)\in\mathbb{N}^2 \\ 2a+b=n}} a\sigma(a)\sigma(b)\Big)q^n, \end{aligned}$$

so

$$\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{(a,b)\in\mathbb{N}^2 \\ 2a+b=n}} a\sigma(a)\sigma(b)\Bigg)q^n$$
$$= \frac{1}{24}[1,1]_2 + \frac{1}{48}[1,3]_1 - \frac{1}{48}[2,1]_1 + \frac{1}{6}[1,3]_2 - \frac{1}{3}[2,1]_2.$$

Equating the coefficients of q^n $(n \in \mathbb{N})$, we obtain the second identity. ∎

The second identity of Theorem 4.1 can be deduced from the first identity as follows:

$$\begin{aligned}
\sum_{\substack{(a,b)\in\mathbb{N}^2\\ 2a+b=n}} a\sigma(a)\sigma(b) &= \sum_{\substack{(a,b)\in\mathbb{N}^2\\ a+2b=n}} b\sigma(a)\sigma(b) = \sum_{\substack{(a,b)\in\mathbb{N}^2\\ a+2b=n}} \left(\frac{n-a}{2}\right)\sigma(a)\sigma(b)\\
&= \frac{n}{2}\sum_{\substack{(a,b)\in\mathbb{N}^2\\ a+2b=n}} \sigma(a)\sigma(b) - \frac{1}{2}\sum_{\substack{(a,b)\in\mathbb{N}^2\\ a+2b=n}} a\sigma(a)\sigma(b)\\
&= \frac{n}{2}\left(\frac{1}{12}\sigma_3(n) + \left(\frac{1}{24}-\frac{1}{8}n\right)\sigma(n) + \frac{1}{3}\sigma_3(n/2) + \left(\frac{1}{24}-\frac{1}{4}n\right)\sigma(n/2)\right)\\
&\quad - \frac{1}{2}\left(\frac{1}{24}n\sigma_3(n) + \left(\frac{1}{24}n - \frac{1}{12}n^2\right)\sigma(n) + \frac{1}{6}n\sigma_3(n/2) - \frac{1}{12}n^2\sigma(n/2)\right)\\
&= \frac{1}{48}n\sigma_3(n) - \frac{1}{48}n^2\sigma(n) + \frac{1}{12}n\sigma_3(n/2) + \left(\frac{1}{48}n - \frac{1}{12}n^2\right)\sigma(n/2)
\end{aligned}$$

as claimed.

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