Some Arithmetic Properties of the q-Euler Numbers and q-Salié Numbers*

Victor J. W. Guo[1] and Jiang Zeng[2]

Institut Camille Jordan, Université Claude Bernard (Lyon I)
F-69622, Villeurbanne Cedex, France
[1]jwguo@eyou.com, [2]zeng@igd.univ-lyon1.fr

Abstract. For $m > n \geq 0$ and $1 \leq d \leq m$, it is shown that the q-Euler number $E_{2m}(q)$ is congruent to $q^{m-n}E_{2n}(q) \bmod (1+q^d)$ if and only if $m \equiv n \bmod d$. The q-Salié number $S_{2n}(q)$ is shown to be divisible by $(1+q^{2r+1})^{\lfloor \frac{n}{2r+1} \rfloor}$ for any $r \geq 0$. Furthermore, similar congruences for the generalized q-Euler numbers are also obtained, and some conjectures are formulated.

AMS Subject Classifications (2000): Primary 05A30, 05A15; Secondary 11A07.

1 Introduction

The Euler numbers E_{2n} may be defined as the coefficients in the Taylor expansion of $2/(e^x + e^{-x})$:

$$\sum_{n=0}^{\infty} E_{2n} \frac{x^{2n}}{(2n)!} = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right)^{-1}.$$

A classical result due to Stern [13] asserts that

$$E_{2m} \equiv E_{2n} \pmod{2^s} \quad \text{if and only if} \quad 2m \equiv 2n \pmod{2^s}.$$

The so-called Salié numbers S_{2n} [7, p. 242] are defined as

$$\sum_{n=0}^{\infty} S_{2n} \frac{x^{2n}}{(2n)!} = \frac{\cosh x}{\cos x}. \tag{1.1}$$

Carlitz [3] first proved that the Salié numbers S_{2n} are divisible by 2^n.

Motivated by the work of Andrews-Gessel [2], Andrews-Foata [1], Désarménien [4], and Foata [5], we are about to study a q-analogue of Stern's result and a q-analogue of Carlitz's result for Salié numbers. A natural q-analogue of the Euler numbers is given by

$$\sum_{n=0}^{\infty} E_{2n}(q) \frac{x^{2n}}{(q;q)_{2n}} = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(q;q)_{2n}} \right)^{-1}, \tag{1.2}$$

*European J. Combin. 27 (2006), 884–895.

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geq 1$ and $(a;q)_0 = 1$.

A recent arithmetic study of Euler numbers and more general q-Euler numbers can be found in [14] and [11]. Note that, in order to coincide with the Euler numbers in [14, 15], our definition of $E_{2n}(q)$ differs by a factor $(-1)^n$ from that in [1, 2, 4, 5].

Theorem 1.1 *Let $m > n \geq 0$ and $1 \leq d \leq m$. Then*

$$E_{2m}(q) \equiv q^{m-n}E_{2n}(q) \pmod{1+q^d} \quad \text{if and only if} \quad m \equiv n \pmod{d}.$$

Since the polynomials $1+q^{2^a d}$ and $1+q^{2^b d}$ $(a \neq b)$ are relatively prime, we derive immediately from the above theorem the following

Corollary 1.2 *Let $m > n \geq 0$ and $2m-2n = 2^s r$ with r odd. Then*

$$E_{2m}(q) \equiv q^{m-n}E_{2n}(q) \pmod{\prod_{k=0}^{s-1}(1+q^{2^k r})}.$$

Define the q-Salié numbers by

$$\sum_{n=0}^{\infty} S_{2n}(q)\frac{x^{2n}}{(q;q)_{2n}} = \sum_{n=0}^{\infty} \frac{q^n x^{2n}}{(q;q)_{2n}} \Big/ \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(q;q)_{2n}}. \tag{1.3}$$

For each positive integer n, write $n = 2^s(2r+1)$ with $r, s \geq 0$ (so s is the 2-adic valuation of n), and set $p_n(q) = 1+q^{2r+1}$. Define

$$P_n(q) = \prod_{k=1}^{n} p_k(q) = \prod_{r\geq 0}(1+q^{2r+1})^{a_{n,r}},$$

where $a_{n,r}$ is the number of positive integers of the form $2^s(2r+1)$ less than or equal to n. The first values of $P_n(q)$ are given in Table 1.

Table 1: Table of $P_n(q)$.

n	1	3	5	7
$P_n(q)$	$(1+q)$	$(1+q)^2(1+q^3)$	$(1+q)^3(1+q^3)(1+q^5)$	$(1+q)^3(1+q^3)^2(1+q^5)(1+q^7)$
n	2	4	6	8
$P_n(q)$	$(1+q)^2$	$(1+q)^3(1+q^3)$	$(1+q)^3(1+q^3)^2(1+q^5)$	$(1+q)^4(1+q^3)^2(1+q^5)(1+q^7)$

Note that $P_n(1) = 2^n$. The following is a q-analogue of Carlitz's result for Salié numbers:

Theorem 1.3 *For every $n \geq 1$, the polynomial $S_{2n}(q)$ is divisible by $P_n(q)$. In particular, $S_{2n}(q)$ is divisible by $(1+q^{2r+1})^{\lfloor \frac{n}{2r+1} \rfloor}$ for any $r \geq 0$.*

We shall collect some arithmetic properties of Gaussian polynomials or q-binomial coefficients in the next section. The proofs of Theorems 1.1 and 1.3 are given in Sections 3 and 4, respectively. We will give some similar arithmetic properties of the generalized q-Euler numbers in Section 5. Some combinatorial remarks and open problems are given in Section 6.

2 Two properties of Gaussian polynomials

The Gaussian polynomial $\begin{bmatrix} M \\ N \end{bmatrix}_q$ may be defined by

$$\begin{bmatrix} M \\ N \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_M}{(q;q)_N (q;q)_{M-N}}, & \text{if } 0 \le N \le M, \\ 0, & \text{otherwise.} \end{cases}$$

The following result is equivalent to the so-called q-Lucas theorem (see Olive [10] and Désarménien [4, Proposition 2.2]).

Proposition 2.1 *Let m, k, d be positive integers, and write $m = ad + b$ and $k = rd + s$, where $0 \le b, s \le d - 1$. Let ω be a primitive d-th root of unity. Then*

$$\begin{bmatrix} m \\ k \end{bmatrix}_\omega = \binom{a}{r} \begin{bmatrix} b \\ s \end{bmatrix}_\omega.$$

Indeed, we have

$$\begin{aligned} \begin{bmatrix} m \\ k \end{bmatrix}_q &= \prod_{j=1}^{rd+s} \frac{1 - q^{(a-r)d+b-s+j}}{1 - q^j} \\ &= \left(\prod_{j=1}^{s} \frac{1 - q^{(a-r)d+b-s+j}}{1 - q^j} \right) \left(\prod_{j=1}^{rd} \frac{1 - q^{(a-r)d+b+j}}{1 - q^{s+j}} \right). \end{aligned}$$

By definition, we have $\omega^d = 1$ and $\omega^j \ne 1$ for $0 < j < d$. Hence,

$$\lim_{q \to \omega} \prod_{j=1}^{s} \frac{1 - q^{(a-r)d+b-s+j}}{1 - q^j} = \prod_{j=1}^{s} \frac{1 - \omega^{b-s+j}}{1 - \omega^j} = \begin{bmatrix} b \\ s \end{bmatrix}_\omega.$$

Notice that, for any integer k, the set $\{k + j \colon j = 1, \dots, rd\}$ is a *complete system of residues* modulo rd. Therefore,

$$\begin{aligned} \lim_{q \to \omega} \prod_{j=1}^{rd} \frac{1 - q^{(a-r)d+b+j}}{1 - q^{s+j}} &= \lim_{q \to \omega} \frac{(1 - q^{(a-r+1)d})(1 - q^{(a-r+2)d}) \cdots (1 - q^{ad})}{(1 - q^d)(1 - q^{2d}) \cdots (1 - q^{rd})} \\ &= \binom{a}{r}. \end{aligned}$$

Let $\Phi_n(x)$ be the n-th *cyclotomic polynomial*. The following easily proved result can be found in [8, Equation (10)].

Proposition 2.2 *The Gaussian polynomial $\begin{bmatrix} m \\ k \end{bmatrix}_q$ can be factorized into*

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \prod_d \Phi_d(q),$$

where the product is over all positive integers $d \le m$ such that $\lfloor k/d \rfloor + \lfloor (m-k)/d \rfloor < \lfloor m/d \rfloor$.

Indeed, using the factorization $q^n - 1 = \prod_{d|n} \Phi_d(q)$, we have

$$(q;q)_m = (-1)^m \prod_{k=1}^{m} \prod_{d|k} \Phi_d(q) = (-1)^m \prod_{d=1}^{m} \Phi_d(q)^{\lfloor m/d \rfloor},$$

and so

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{(q;q)_m}{(q;q)_k (q;q)_{m-k}} = \prod_{d=1}^{m} \Phi_d(q)^{\lfloor m/d \rfloor - \lfloor k/d \rfloor - \lfloor (m-k)/d \rfloor}.$$

Proposition 2.2 now follows from the obvious fact that

$$\lfloor \alpha + \beta \rfloor - \lfloor \alpha \rfloor - \lfloor \beta \rfloor = 0 \quad \text{or} \quad 1, \quad \text{for } \alpha, \beta \in \mathbb{R}.$$

3 Proof of Theorem 1.1

Multiplying both sides of (1.2) by $\sum_{n=0}^{\infty} x^{2n}/(q;q)_{2n}$ and equating coefficients of x^{2m}, we see that $E_{2m}(q)$ satisfies the following recurrence relation:

$$E_{2m}(q) = -\sum_{k=0}^{m-1} \begin{bmatrix} 2m \\ 2k \end{bmatrix}_q E_{2k}(q). \tag{3.1}$$

This enables us to obtain the first values of the q-Euler numbers:

$$\begin{aligned}
E_0(q) &= -E_2(q) = 1, \\
E_4(q) &= q(1+q)(1+q^2) + q^2, \\
E_6(q) &= -q^2(1+q^3)(1+4q+5q^2+7q^3+6q^4+5q^5+2q^6+q^7) + q^3.
\end{aligned}$$

We first establish the following result.

Lemma 3.1 *Let $m > n \geq 0$ and $1 \leq d \leq m$. Then*

$$E_{2m}(q) \equiv q^{m-n} E_{2n}(q) \pmod{\Phi_{2d}(q)} \quad \textit{if and only if} \quad m \equiv n \pmod{d}. \tag{3.2}$$

Proof. It is easy to see that Lemma 3.1 is equivalent to

$$E_{2m}(\zeta) = \zeta^{m-n} E_{2n}(\zeta) \quad \text{if and only if} \quad m \equiv n \pmod{d}, \tag{3.3}$$

where $\zeta \in \mathbb{C}$ is a $2d$-th primitive root of unity.

We proceed by induction on m. Statement (3.3) is trivial for $m = 1$. Suppose it holds for every number less than m. Let $n < m$ be fixed. Write $m = ad + b$ with $0 \leq b \leq d-1$, then $2m = a(2d) + 2b$. By Proposition 2.1, we see that

$$\begin{bmatrix} 2m \\ 2k \end{bmatrix}_\zeta = \binom{a}{r} \begin{bmatrix} 2b \\ 2s \end{bmatrix}_\zeta, \quad \text{where} \quad k = rd + s,\ 0 \leq s \leq d-1. \tag{3.4}$$

Hence, by (3.1) and (3.4), we have

$$
\begin{aligned}
E_{2m}(\zeta) &= -\sum_{k=0}^{m-1} \begin{bmatrix} 2m \\ 2k \end{bmatrix}_\zeta E_{2k}(\zeta) \\
&= -\sum_{r=0}^{a} \sum_{s=0}^{b-\delta_{a\,r}} \binom{a}{r} \begin{bmatrix} 2b \\ 2s \end{bmatrix}_\zeta E_{2rd+2s}(\zeta), \\
&= -\sum_{s=0}^{b} \sum_{r=0}^{a-\delta_{b\,s}} \binom{a}{r} \begin{bmatrix} 2b \\ 2s \end{bmatrix}_\zeta E_{2rd+2s}(\zeta), \qquad (3.5)
\end{aligned}
$$

where $\delta_{i\,j}$ equals 1 if $i = j$ and 0 otherwise.

By the induction hypothesis, we have

$$E_{2rd+2s}(\zeta) = \zeta^{rd} E_{2s}(\zeta) = (-1)^r E_{2s}(\zeta). \qquad (3.6)$$

Thus,

$$\sum_{r=0}^{a} \binom{a}{r} \begin{bmatrix} 2b \\ 2s \end{bmatrix}_\zeta E_{2rd+2s}(\zeta) = \begin{bmatrix} 2b \\ 2s \end{bmatrix}_\zeta E_{2s}(\zeta) \sum_{r=0}^{a} \binom{a}{r} (-1)^r = 0.$$

Therefore, Equation (3.5) implies that

$$E_{2m}(\zeta) = (-1)^a E_{2b}(\zeta) = \zeta^{m-b} E_{2b}(\zeta). \qquad (3.7)$$

From (3.7) we see that

$$E_{2m}(\zeta) = \zeta^{m-n} E_{2n}(\zeta) \quad \Longleftrightarrow \quad E_{2n}(\zeta) = \zeta^{n-b} E_{2b}(\zeta).$$

By the induction hypothesis, the latter equality is also equivalent to

$$n \equiv b \pmod{d} \quad \Longleftrightarrow \quad m \equiv n \pmod{d}.$$

This completes the proof. ∎

Since

$$1 + q^d = \frac{q^{2d} - 1}{q^d - 1} = \frac{\prod_{k|2d} \Phi_k(q)}{\prod_{k|d} \Phi_k(q)} = \prod_{\substack{k|d \\ 2k\nmid d}} \Phi_{2k}(q),$$

and any two different cyclotomic polynomials are relatively prime, Theorem 1.1 follows from Lemma 3.1.

Remark. The sufficiency part of (3.2) is equivalent to Désarménien's result [4]:

$$E_{2km+2n}(q) \equiv (-1)^m E_{2n}(q) \pmod{\Phi_{2k}(q)}.$$

4 Proof of Theorem 1.3

Recall that the q-tangent numbers $T_{2n+1}(q)$ are defined by

$$\sum_{n=0}^{\infty} T_{2n+1}(q)\frac{x^{2n+1}}{(q;q)_{2n+1}} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(q;q)_{2n+1}} \Bigg/ \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(q;q)_{2n}} .$$

Foata [5] proved that $T_{2n+1}(q)$ is divisible by $D_n(q)$, where

$$D_n(q) = \begin{cases} \prod_{k=1}^{n} Ev_k(q), & \text{if } n \text{ is odd,} \\ (1+q^2)\prod_{k=1}^{n} Ev_k(q), & \text{if } n \text{ is even,} \end{cases}$$

and

$$Ev_n(q) = \prod_{j=0}^{s}(1+q^{2^j r}), \quad \text{where } n = 2^s r \text{ with } r \text{ odd.}$$

Notice that this implies that $T_{2n+1}(q)$ is divisible by both $(1+q)^n$ and $(-q;q)_n$, a result due to Andrews and Gessel [2].

To prove our theorem we need the following relation relating $S_{2n}(q)$ to $T_{2n+1}(q)$.

Lemma 4.1 *For every $n \geq 1$, we have*

$$\sum_{k=0}^{n}(-1)^k q^k \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q S_{2k}(q)S_{2n-2k}(q) = T_{2n-1}(q)(1-q^{2n}). \tag{4.1}$$

Proof. Replacing x by $q^{1/2}ix$ ($i=\sqrt{-1}$) in (1.3), we obtain

$$\sum_{n=0}^{\infty} S_{2n}(q)\frac{(-1)^n q^n x^{2n}}{(q;q)_{2n}} = \sum_{n=0}^{\infty}\frac{(-1)^n q^{2n} x^{2n}}{(q;q)_{2n}} \Bigg/ \sum_{n=0}^{\infty}\frac{q^n x^{2n}}{(q;q)_{2n}}. \tag{4.2}$$

Multiplying (1.3) with (4.2), we get

$$\left(\sum_{n=0}^{\infty} S_{2n}(q)\frac{x^{2n}}{(q;q)_{2n}}\right)\left(\sum_{n=0}^{\infty} S_{2n}(q)\frac{(-1)^n q^n x^{2n}}{(q;q)_{2n}}\right) \tag{4.3}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{q^{2n}x^{2n}}{(q;q)_{2n}} \Bigg/ \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(q;q)_{2n}}$$

$$= 1 + x\sum_{n=1}^{\infty}(-1)^{n-1}\frac{q^{2n-1}x^{2n-1}}{(q;q)_{2n-1}} \Bigg/ \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(q;q)_{2n}}$$

$$= 1 + x\sum_{n=0}^{\infty} T_{2n+1}(q)\frac{x^{2n+1}}{(q;q)_{2n+1}}. \tag{4.4}$$

Equating the coefficients of x^{2n} in (4.3) and (4.4), we are led to (4.1). ∎

It is easily seen that $P_n(q)$ is the *least common multiple* of the polynomials $(1+q^{2r+1})^{\lfloor \frac{n}{2r+1} \rfloor}$ $(r \geq 0)$. For any $r \geq 0$, there holds

$$1+q^{2r+1} = \frac{q^{4r+2}-1}{q^{2r+1}-1} = \frac{\prod_{d|(4r+2)} \Phi_d(q)}{\prod_{d|(2r+1)} \Phi_d(q)} = \prod_{d|(2r+1)} \Phi_{2d}(q).$$

It follows that

$$P_n(q) = \prod_{r \geq 0} \Phi_{4r+2}(q)^{\lfloor \frac{n}{2r+1} \rfloor}.$$

Theorem 1.3 is trivial for $n = 1$. Suppose it holds for all integers less than n. In the summation of the left-hand side of (4.1), combining the first and last terms, we can rewrite Equation (4.1) as follows:

$$(1+(-1)^n q^n) S_{2n}(q) + \sum_{k=1}^{n-1} (-1)^k q^k \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q S_{2k}(q) S_{2n-2k}(q) = T_{2n-1}(q)(1-q^{2n}). \tag{4.5}$$

For every k $(1 \leq k \leq n-1)$, by the induction hypothesis, the polynomial $S_{2k}(q)S_{2n-2k}(q)$ is divisible by

$$P_k(q) P_{n-k}(q) = \prod_{r \geq 0} \Phi_{4r+2}(q)^{\lfloor \frac{k}{2r+1} \rfloor + \lfloor \frac{n-k}{2r+1} \rfloor}.$$

And by Proposition 2.2, we have

$$\begin{bmatrix} 2n \\ 2k \end{bmatrix}_q = \prod_{d=1}^{2n} \Phi_d(q)^{\lfloor 2n/d \rfloor - \lfloor 2k/d \rfloor - \lfloor (2n-2k)/d \rfloor},$$

which is clearly divisible by

$$\prod_{r \geq 0} \Phi_{4r+2}(q)^{\lfloor \frac{n}{2r+1} \rfloor - \lfloor \frac{k}{2r+1} \rfloor - \lfloor \frac{n-k}{2r+1} \rfloor}.$$

Hence, the product $\begin{bmatrix} 2n \\ 2k \end{bmatrix}_q S_{2k}(q) S_{2n-2k}(q)$ is divisible by

$$\prod_{r \geq 0} \Phi_{4r+2}(q)^{\lfloor \frac{n}{2r+1} \rfloor} = P_n(q).$$

Note that $P_{n-1}(q) \mid D_{n-1}(q)$ and $p_n(q) \mid (1-q^{2n})$. Therefore, by (4.5) and the aforementioned result of Foata, we immediately have

$$P_n(q) \mid (1+(-1)^n q^n) S_{2n}(q).$$

Since $P_n(q)$ is relatively prime to $(1+(-1)^n q^n)$, we obtain $P_n(q) \mid S_{2n}(q)$.

Remark. Since $S_0(q) = 1$ and $S_2(q) = 1+q$, using (4.5) and the divisibility of $T_{2n+1}(q)$, we can prove by induction that $S_{2n}(q)$ is divisible by $(1+q)^n$ without using the divisibility property of Gaussian polynomials.

5 The generalized q-Euler numbers

The generalized Euler numbers may be defined by

$$\sum_{n=0}^{\infty} E_{kn}^{(k)} \frac{x^{kn}}{(kn)!} = \left(\sum_{n=0}^{\infty} \frac{x^{kn}}{(kn)!} \right)^{-1}.$$

Some congruences for these numbers are given in [6, 9]. A q-analogue of generalized Euler numbers is given by

$$\sum_{n=0}^{\infty} E_{kn}^{(k)}(q) \frac{x^{kn}}{(q;q)_{kn}} = \left(\sum_{n=0}^{\infty} \frac{x^{kn}}{(q;q)_{kn}} \right)^{-1},$$

or, recurrently,

$$E_0^{(k)}(q) = 1, \quad E_{kn}^{(k)}(q) = -\sum_{j=0}^{n-1} \begin{bmatrix} kn \\ kj \end{bmatrix}_q E_{kj}^{(k)}(q), \quad n \geq 1. \tag{5.1}$$

Note that $E_{kn}^{(k)}(q)$ is equal to $(-1)^n f_{nk,k}(q)$ studied by Stanley [12, p. 148, Equation (57)].

Theorem 5.1 *Let $m > n \geq 0$ and $1 \leq d \leq m$. Let $k \geq 1$, and let $\zeta \in \mathbb{C}$ be a $2kd$-th primitive root of unity. Then*

$$E_{km}^{(k)}(\zeta^2) = \zeta^{k(m-n)} E_{kn}^{(k)}(\zeta^2) \tag{5.2}$$

if and only if

$$m \equiv n \pmod{d}.$$

The proof is by induction on m and using the recurrence definition (5.1). Since it is analogous to the proof of (3.3), we omit it here. Note that ζ^2 in Theorem 5.1 is a kd-th primitive root of unity. Therefore, when k is even or $m \equiv n \bmod 2$, Equation (5.2) is equivalent to

$$E_{km}^{(k)}(q) \equiv q^{\frac{k(m-n)}{2}} E_{kn}^{(k)}(q) \pmod{\Phi_{kd}(q)}.$$

As mentioned before,

$$1 + q^{2^k d} = \prod_{\substack{i \mid 2^k d \\ 2i \nmid 2^k d}} \Phi_{2i}(q),$$

and we obtain the following theorem and its corollaries.

Theorem 5.2 *Let $k \geq 1$. Let $m > n \geq 0$ and $1 \leq d \leq m$. Then*

$$E_{2^k m}^{(2^k)}(q) \equiv q^{2^{k-1}(m-n)} E_{2^k n}^{(2^k)}(q) \pmod{1 + q^{2^{k-1} d}} \quad \textit{if and only if} \quad m \equiv n \pmod{d}.$$

Corollary 5.3 *Let $k \geq 1$. Let $m > n \geq 0$ and $m - n = 2^{s-1}r$ with r odd. Then*

$$E^{(2^k)}_{2^k m}(q) \equiv q^{2^{k-1}(m-n)} E^{(2^k)}_{2^k n}(q) \pmod{\prod_{i=0}^{s-1}(1 + q^{2^{k+i-1}r})}.$$

Corollary 5.4 *Let k, m, n, s be as above. Then*

$$E^{(2^k)}_{2^k m} \equiv E^{(2^k)}_{2^k n} \pmod{2^s}.$$

Furthermore, numerical evidence seems to suggest the following congruence conjecture for generalized Euler numbers.

Conjecture 5.5 *Let $k \geq 1$. Let $m > n \geq 0$ and $m - n = 2^{s-1}r$ with r odd. Then*

$$E^{(2^k)}_{2^k m} \equiv E^{(2^k)}_{2^k n} + 2^s \pmod{2^{s+1}}.$$

This conjecture is clearly a generalization of Stern's result, which corresponds to the $k = 1$ case.

6 Concluding remarks

We can also consider the following variants of the q-Salié numbers:

$$\sum_{n=0}^{\infty} \overline{S}_{2n}(q) \frac{x^{2n}}{(q;q)_{2n}} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(q;q)_{2n}} \Bigg/ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(q;q)_{2n}}, \tag{6.1}$$

$$\sum_{n=0}^{\infty} \widehat{S}_{2n}(q) \frac{x^{2n}}{(q;q)_{2n}} = \sum_{n=0}^{\infty} \frac{q^{2n} x^{2n}}{(q;q)_{2n}} \Bigg/ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(q;q)_{2n}}, \tag{6.2}$$

$$\sum_{n=0}^{\infty} \widetilde{S}_{2n}(q) \frac{x^{2n}}{(q;q)_{2n}} = \sum_{n=0}^{\infty} \frac{q^{n^2} x^{2n}}{(q;q)_{2n}} \Bigg/ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(q;q)_{2n}}. \tag{6.3}$$

Multiplying both sides of (6.1)–(6.3) by $\sum_{n=0}^{\infty}(-1)^n x^{2n}/(q;q)_{2n}$ and equating coefficients of x^{2n}, we obtain

$$\overline{S}_{2n}(q) = 1 - \sum_{k=0}^{n-1} (-1)^{n-k} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \overline{S}_{2k}(q), \tag{6.4}$$

$$\widehat{S}_{2n}(q) = q^{2n} - \sum_{k=0}^{n-1} (-1)^{n-k} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \widehat{S}_{2k}(q), \tag{6.5}$$

$$\widetilde{S}_{2n}(q) = q^{n^2} - \sum_{k=0}^{n-1} (-1)^{n-k} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \widetilde{S}_{2k}(q). \tag{6.6}$$

This gives

$$\overline{S}_0(q)=1,\quad \overline{S}_2(q)=2,\quad \overline{S}_4(q)=2(1+q^2)(1+q+q^2),$$
$$\widehat{S}_0(q)=1,\quad \widehat{S}_2(q)=1+q^2,\quad \widehat{S}_4(q)=q(1+q^2)(1+3q+q^2+q^3),$$
$$\widetilde{S}_0(q)=1,\quad \widetilde{S}_2(q)=1+q,\quad \widetilde{S}_4(q)=q(1+q)(1+q^2)(2+q),$$

and

$$\overline{S}_6(q)=2(1+q^2)(1+q+2q^2+4q^3+6q^4+6q^5+6q^6+5q^7+4q^8+2q^9+q^{10}),$$
$$\widehat{S}_6(q)=q^2(1+q^2)^2(1+4q+7q^2+6q^3+6q^4+6q^5+5q^6+2q^7+q^8),$$
$$\widetilde{S}_6(q)=q^2(1+q)(1+q^2)(1+q^3)(2+4q+5q^2+4q^3+3q^4+q^5).$$

For $n\geq 1$ define three sequences of polynomials:

$$\overline{Q}_n(q):=\prod_{r\geq 1}\Phi_{4r}(q)^{\lfloor\frac{n}{2r}\rfloor},$$
$$\widehat{Q}_n(q):=\begin{cases}\overline{Q}_n(q), & \text{if } n \text{ is even},\\ (1+q^2)\overline{Q}_n(q), & \text{if } n \text{ is odd},\end{cases}$$
$$\widetilde{Q}_n(q):=(1+q)(1+q^2)\cdots(1+q^n).$$

Note that $\overline{Q}_n(q)$ is the least common multiple of the polynomials $(1+q^{2r})^{\lfloor\frac{n}{2r}\rfloor}$, $r\geq 1$ (see Table 2).

Table 2: Table of $\overline{Q}_n(q)$.

n	1	3	5	7
$\overline{Q}_n(q)$	1	$1+q^2$	$(1+q^2)^2(1+q^4)$	$(1+q^2)^2(1+q^4)(1+q^6)$
n	2	4	6	8
$\overline{Q}_n(q)$	$1+q^2$	$(1+q^2)^2(1+q^4)$	$(1+q^2)^2(1+q^4)^2(1+q^6)$	$(1+q^2)^3(1+q^4)^2(1+q^6)(1+q^8)$

From (6.4)–(6.6), it is easy to derive by induction that for $n\geq 1$,

$$2\mid \overline{S}_{2n}(q),\quad (1+q^2)\mid \widehat{S}_{2n}(q),\quad (1+q)\mid \widetilde{S}_{2n}(q).$$

Moreover, the computation of the first values of these polynomials seems to suggest the following stronger result.

Conjecture 6.1 *For $n\geq 1$, we have the following divisibility properties:*

$$\overline{Q}_n(q)\mid \overline{S}_{2n}(q),\quad \widehat{Q}_n(q)\mid \widehat{S}_{2n}(q),\quad \widetilde{Q}_n(q)\mid \widetilde{S}_{2n}(q).$$

Similarly to the proof of Lemma 4.1, we can obtain

$$\left(\sum_{n=0}^{\infty}\widehat{S}_{2n}(q)\frac{x^{2n}}{(q;q)_{2n}}\right)\left(\sum_{n=0}^{\infty}\widehat{S}_{2n}(q)\frac{(-1)^n q^{2n}x^{2n}}{(q;q)_{2n}}\right)=1-qx^2+(1+q)x\sum_{n=0}^{\infty}T_{2n+1}(q)\frac{x^{2n+1}}{(q;q)_{2n+1}},$$

which yields

$$\sum_{k=0}^{n}(-1)^k q^{2k}\begin{bmatrix}2n\\2k\end{bmatrix}_q\widehat{S}_{2k}(q)\widehat{S}_{2n-2k}(q)=T_{2n-1}(q)(1+q)(1-q^{2n}),\quad n\geq 2. \qquad (6.7)$$

However, it seems difficult to use (6.7) to prove directly the divisibility of $\widehat{S}_{2n}(q)$ by $\widehat{Q}_n(q)$, because when n is even $1+(-1)^n q^{2n}$ is in general not relatively prime to $\widehat{Q}_n(q)$.

Finally it is well-known that $E_{2n}(q)$ has a nice combinatorial interpretation in terms of generating functions of alternating permutations. Recall that a *permutation* $x_1x_2\cdots x_{2n}$ of $[2n]:=\{1,2,\ldots,2n\}$ is called *alternating*, if $x_1<x_2>x_3<\cdots>x_{2n-1}<x_{2n}$. As usual, the number of *inversions* of a permutation $x=x_1x_2\cdots x_n$, denoted $\mathrm{inv}(x)$, is defined to the number of pairs (i,j) such that $i<j$ and $x_i>x_j$. It is known (see [12, p. 148, Proposition 3.16.4]) that

$$(-1)^n E_{2n}(q)=\sum_{\pi}q^{\mathrm{inv}(\pi)},$$

where π ranges over all the alternating permutations of $[2n]$. It would be interesting to find a combinatorial proof of Theorem 1 within the alternating permutations model.

A permutation $x=x_1x_2\cdots x_{2n}$ of $[2n]$ is said to be a *Salié permutation*, if there exists an even index $2k$ such that $x_1x_2\cdots x_{2k}$ is alternating and $x_{2k}<x_{2k+1}<\cdots<x_{2n}$, and x_{2k-1} is called the *last valley* of x. It is known (see [7, p. 242, Exercise 4.2.13]) that $\frac{1}{2}S_{2n}$ is the number of Salié permutations of $[2n]$.

Proposition 6.2 *For every $n\geq 1$ the polynomial $\frac{1}{2}\overline{S}_{2n}(q)$ is the generating function for Salié permutations of $[2n]$ by number of inversions.*

Proof. Substituting (1.2) into (6.1) and comparing coefficients of x^{2n} on both sides, we obtain

$$\overline{S}_{2n}(q)=\sum_{k=0}^{n}\begin{bmatrix}2n\\2k\end{bmatrix}_q(-1)^k E_{2k}(q). \qquad (6.8)$$

As $\begin{bmatrix}2n\\2k\end{bmatrix}_q$ is the generating function for the permutations of $1^{2k}2^{2n-2k}$ by number of inversions (see e.g. [12, p. 26, Proposition 1.3.17]), it is easily seen that $\begin{bmatrix}2n\\2k\end{bmatrix}_q(-1)^k E_{2k}(q)$ is the generating function for permutations $x=x_1x_2\cdots x_{2n}$ of $[2n]$ such that $x_1x_2\cdots x_{2k}$ is alternating and $x_{2k+1}\cdots x_{2n}$ is increasing with respect to number of inversions. Notice that such a permutation x is a Salié permutation with the last valley x_{2k-1} if $x_{2k}<x_{2k+1}$ or x_{2k+1} if $x_{2k}>x_{2k+1}$. Therefore, the right-hand side of (6.8) is twice the generating function for Salié permutations of $[2n]$ by number of inversions. This completes the proof. ∎

It is also possible to find similar combinatorial interpretations for the other q-Salié numbers, which are left to the interested readers.

Acknowledgment. The second author was supported by EC's IHRP Programme, within Research Training Network "Algebraic Combinatorics in Europe," grant HPRN-CT-2001-00272.

References

[1] G. E. Andrews and D. Foata, Congruences for the q-Euler numbers, European J. Combin. **1** (1980), 283–287.

[2] G. E. Andrews and I. Gessel, Divisibility properties of the q-tangent numbers, Proc. Amer. Math. Soc. **68** (1978), 380–384.

[3] L. Carlitz, The coefficients of $\cosh x/\cos x$, Monatsh. Math. **69** (1965), 129–135.

[4] J. Désarménien, Un analogue des congruences de Kummer pour les q-nombres d'Euler, European J. Combin. **3** (1982), 19–28.

[5] D. Foata, Further divisibility properties of the q-tangent numbers, Proc. Amer. Math. Soc. **81** (1981), 143–148.

[6] I. M. Gessel, Some congruences for generalized Euler numbers, Canad. J. Math. **35** (1983), 687–709.

[7] I. P. Goulden and D. M. Jackson, Combinatorial Enumeration, reprint of the 1983 original, Dover Publications, Inc., Mineola, NY, 2004.

[8] D. Knuth and H. Wilf, The power of a prime that divides a generalized binomial coefficient, J. Reine Angew. Math. **396** (1989), 212–219.

[9] D. J. Leeming and R. A. MacLeod, Some properties of generalized Euler numbers, Canad. J. Math. **33** (1981), 606–617.

[10] G. Olive, Generalized powers, Amer. Math. Monthly **72** (1965) 619–627.

[11] B. E. Sagan and P. Zhang, Arithmetic properties of generalized Euler numbers, Southeast Asian Bull. Math. **21** (1997), 73–78.

[12] R. P. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge Studies in Advanced Mathematics, **49**, Cambridge University Press, Cambridge, 1997.

[13] M. A. Stern, Zur Theorie der Eulerschen Zahlen, J. Reine Angew. Math. **79** (1875), 67–98.

[14] Z.-W. Sun, On Euler numbers modulo powers of two, J. Number Theory, to appear.

[15] S. S. Wagstaff, Jr., Prime divisors of the Bernoulli and Euler numbers, In: Number Theory for the Millennium, III (Urbana, IL, 2000), A K Peters, Natick, MA, 2002, pp. 357–374.