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Some Aspects of Groups Acting on Finite Posets

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Let P be a finite poset and G a group of automorphisms of P. The action of G on P can be used to define various linear representations of G, and we investigate how these representations are related to one another and to the structure of P. Several examples are analyzed in detail, viz., the symmetric group $\mathfrak{S}_n$ acting on a boolean algebra, $GL_n(q)$ acting on subspaces of an n-dimensional vector space over $GF(q)$, the hyperoctahedral group B_n acting on the lattice of faces of a cross-polytope, and $\mathfrak{S}_n$ acting on the lattice Π_n of partitions of an n-set. Several results of a general nature are also proved. These include a duality theorem related to Alexander duality, a special property of geometric lattices, the behavior of barycentric subdivision, and a method for showing that certain sequences are unimodal. In particular, we give what seems to be the simplest proof to date that the q-binomial coefficient $\left[{k+l \atop k}\right]$ has unimodal coefficients.

Contents.

1. GENERALITIES

Let P be a finite poset (= partially ordered set) and G a (finite) group of automorphisms of P. Thus G is a subgroup of the full group Aut P of automorphisms of P. By definition each element of G permutes the elements of P, so that the action of G on P defines a certain permutation representation of the abstract group G. There are many other actions or representations of G which can be defined in terms of its original action on P, and our main purpose here is to see how these representations interact with each

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other and with the combinatorial structure of P. There are several published instances of this kind of interaction in the literature (which we will say more about later). In most cases the purpose has been to study the representation theory of G. We, on the other hand, will be primarily concerned with the interplay between combinatorics and group theory, with no particular concern for obtaining purely group-theoretic results. In a few cases, however, we will be able to say something new about the representations of G independent of P.

The following notation will be used throughout. The set $\{1, 2,..., n\}$ is denoted $[n]$, so in particular $[0] = \varnothing$. Also let $\mathbb{N} = \{0, 1, 2,...,\}$. The notation $S = \{s_1,..., s_k\}_<$ means that S is the set $\{s_1,..., s_k\}$ of real numbers and that $s_1 < \cdots < s_k$. If S is a set and $T \subset S$, then $S \backslash T$ denotes the complement $\{x \in S: x \notin T\}$.

From the outset we assume that the poset P has a unique minimal element $\hat{0}$, a unique maximal element $\hat{1}$, and that every maximal chain of P has the same length n. Define the *rank function* $r: P \to \{0, 1,..., n\}$ of P by setting $r(x)$ equal to the length of any saturated chain in the interval $[\hat{0}, x] = \{y: \hat{0} \leqslant y \leqslant x\}$. If $x \leqslant y$ in P, then set $r(x, y) = r(y) - r(x)$. If $S \subset [n-1]$, then define the *rank-selected subposet* P_S of P by

$$P_S = \{x \in P: x = \hat{0},\ x = \hat{1}, \text{ or } r(x) \in S\}.$$

If Q is any poset with $\hat{0}$ and $\hat{1}$, then define the *order complex* $\Delta(Q)$ to be the abstract simplicial complex whose vertices are the elements of $Q \backslash \{\hat{0}, \hat{1}\}$, and whose faces (or simplices) are the chains $x_0 < x_1 < \cdots < x_k$ in $Q \backslash \{\hat{0}, \hat{1}\}$. Denote by $\tilde{H}_i(Q)$ the reduced simplicial homology group $\tilde{H}_i(\Delta(Q), \mathbb{C})$. (Recall that for any simplicial complex Δ, $\tilde{H}_{-1}(\Delta, \mathbb{C}) = 0$ unless $\Delta = \varnothing$, while $\tilde{H}_{-1}(\varnothing, \mathbb{C}) \cong \mathbb{C}$ and $\tilde{H}_i(\varnothing, \mathbb{C}) = 0$ for $i \geqslant 0$.)

Now suppose G is a group of automorphisms of P. For any $S \subset [n-1]$, G permutes the maximal chains of P_S. Let α_S^P (or just α_S if no confusion will result) denote this permutation representation of G. The character of a (complex, linear) representation α of G evaluated at $g \in G$ is denoted $\langle \alpha, g \rangle$. Thus $\langle \alpha_S, g \rangle$ is the number of maximal chains of P_S fixed by g. In particular, $\langle \alpha_S, 1 \rangle$ is just the number of maximal chains of P_S. Since every element g of G is order-preserving, G also acts on each reduced homology group $\tilde{H}_i(P_S)$, $-1 \leqslant i \leqslant |S|-1$. Let $\gamma_{S,i}: G \to \mathrm{Hom}(\tilde{H}_i(P_S), \tilde{H}_i(P_S))$ denote this representation of G. Now define a virtual representation $\beta_S = \beta_S^P$ of G by

$$\beta_S = \sum_i (-1)^{|S|-1-i} \gamma_{S,i}. \tag{1}$$

In particular, when $S = \varnothing$ then $\beta_\varnothing$ is the trivial representation, i.e., $\beta_\varnothing(g) = 1$ for all $g \in G$.

1.1. THEOREM. *The representations α_S and virtual representations β_S are related by*

$$\alpha_S = \sum_{T \subset S} \beta_T, \tag{2}$$

$$\beta_S = \sum_{T \subset S} (-1)^{|S-T|} \alpha_T. \tag{3}$$

Proof. Let $\tilde{\Lambda}_S(g)$ be the reduced Lefschetz number of the map $g: \Delta(P_S) \to \Delta(P_S)$, i.e.,

$$\tilde{\Lambda}_S(g) = \sum_{n \geq -1} (-1)^n \operatorname{tr} g_*^{S,n},$$

where $g_*^{S,n}$ denotes the induced action of g on $\tilde{H}_n(P_S)$. Thus $\tilde{\Lambda}_S(g) = (-1)^{|S|-1} \langle \beta_S, g \rangle$. Let $\tilde{\chi}_S(g) = \tilde{\chi}(P_S^g)$, the reduced Euler characteristic of the subcomplex of $\Delta(P_S)$ fixed by g. A version of the Hopf–Lefschetz fixed point formula [4, Theorem 1.1] states that

$$\tilde{\Lambda}_S(g) = \tilde{\chi}(P_S^g). \tag{4}$$

(The result [4, Theorem 1.1] is stated for ordinary simplicial homology, but the proof works just as well for reduced simplicial homology. The basic principle behind (4) is well known [28, p. 389; 21; 12], but the first general statement seems to be [4].) By definition of the Euler characteristic,

$$\tilde{\chi}(P_S^g) = \sum_{T \subset S} (-1)^{|T|-1} \langle \alpha_T, g \rangle.$$

Hence

$$(-1)^{|S|-1} \langle \beta_S, g \rangle = \sum_{T \subset S} (-1)^{|T|-1} \langle \alpha_T, g \rangle$$

for all $g \in G$, and (3) follows. The equivalence of (2) and (3) is just the Principle of Inclusion–Exclusion. ∎

Theorem 1.1 may be viewed combinatorially as follows. $\langle \alpha_S, 1 \rangle$ is just the number of maximal chains in P_S, and $\langle \beta_S, 1 \rangle$ is an invariant of P which has been subjected to considerable scrutiny (e.g., [6, 30, 31] and the references therein). Frequently $\langle \beta_S, 1 \rangle$ counts some natural subset of the maximal chains of P_S, and we have a combinatorial interpretation of the formula $\langle \alpha_S, 1 \rangle = \sum_{T \subset S} \langle \beta_T, 1 \rangle$. Now in general if X is the set of inequivalent irreducible representations of G, then we have the decompositions

$$\alpha_S = \sum_{\rho \in X} a_S(\rho) \cdot \rho,$$

$$\beta_S = \sum_{\rho \in X} b_S(\rho) \cdot \rho,$$

where $a_S(\rho)$ and $b_S(\rho)$ are the multiplicities of ρ in α_S and β_S, respectively. In particular,

$$\langle \alpha_S, 1\rangle = \sum_{\rho \in X} a_S(\rho)\langle \rho, 1\rangle, \tag{5}$$

$$\langle \beta_S, 1\rangle = \sum_{\rho \in X} b_S(\rho)\langle \rho, 1\rangle. \tag{6}$$

Thus we can ask whether there are natural, pairwise disjoint subsets of maximal chains of P_S of cardinalities $a_S(\rho)\langle \rho, 1\rangle$ yielding a combinatorial interpretation of (5), and furthermore decompositions of these subsets yielding a combinatorial interpretation of the formula

$$a_S(\rho) = \sum_{T \subset S} b_T(\rho). \tag{7}$$

Thus the group G gives us the opportunity (provided it has more than one element) to try to "refine" the formula $\langle \alpha_S, 1\rangle = \sum_{T \subset S} \langle \beta_T, 1\rangle$ in a combinatorial fashion. We will give several examples of this phenomenon in subsequent sections.

In order for (7) to be interpreted combinatorially in the fashion suggested above, it is certainly necessary that each $b_S(\rho) \geqslant 0$, i.e., that the virtual representation β_S is a "genuine" representation. There is a very natural condition on P which insures that this is the case. Define an arbitrary finite poset P with $\hat{0}$ and $\hat{1}$ to be *Cohen–Macaulay* (over $\mathbb{C}$) if for every interval $I = [x, y]$ of P, we have $\tilde{H}_i(I) = 0$ whenever $i \neq \dim \Delta(I)$. It follows that the question of whether P is Cohen–Macaulay depends only on the geometric realization $|\Delta(P)|$ of $\Delta(P)$ [2, Proposition 3.4; 22]. It is easily seen that if P is Cohen–Macaulay, then every maximal chain of P has the same length n. It is known [2, Theorem 6.4; 22, Sect. 6; 31, Theorem 4.3], moreover, that P_S is also Cohen–Macaulay for any $S \subset [n-1]$. It follows from (1) that $\beta_S = \gamma_{S,s-1}$, where $s = |S|$. In other words:

1.2. THEOREM. *If P is Cohen–Macaulay then β_S is isomorphic to the natural representation of G on the reduced homology group $\tilde{H}_{s-1}(P_S)$.* ∎

In particular, the numbers $b_T(\rho)$, where $T \subset S$ and $\rho \in X$, form a refinement of $a_S(\rho)$ into *non-negative* integers and thus could possibly count something of combinatorial interest.

2. DUALITY

Given $S \subset [n-1]$, let $\bar{S} = [n-1] \backslash S$. Under certain circumstances the representations β_S and $\beta_{\bar{S}}$ will be closely related. We first give conditions under which the degrees of β_S and $\beta_{\bar{S}}$ are related, and then give stronger conditions for the representations themselves to be related. Our first results will be stated in terms of the Möbius function $\mu = \mu_P$ of P [25]. All that needs to be known about μ is that if we have an interval $I = [x, y]$ in P, then

$$\mu(x, y) = \tilde{\chi}(I).$$

If Q is any poset with $\hat{0}$ and $\hat{1}$, then we write $\mu(Q) = \mu_Q(\hat{0}, \hat{1})$. Now note that if $S \subset [n-1]$, then from (1) we have (putting $g = 1$)

$$\langle \beta_S, 1 \rangle = (-1)^{|S|-1} \mu(P_S). \tag{9}$$

The following result appears in [3, Lemma 4.6].

2.1. Lemma. *Let Q be any subposet of P with $\hat{0}$ and $\hat{1}$. Then*

$$\mu(Q) = \Sigma(-1)^k \mu_P(\hat{0}, x_1)\, \mu_P(x_1, x_2) \cdots \mu_P(x_k, \hat{1}),$$

where the sum ranges over all chains $\hat{0} < x_1 < \cdots < x_k < \hat{1}$ in P such that $x_i \notin Q$ for all i. ∎

We now consider a special case of the previous lemma. The poset P (as usual, graded of rank n with $\hat{0}$ and $\hat{1}$, and with rank function r) is said to be *semi-Eulerian* if $\mu(x, y) = (-1)^{r(x,y)}$, whenever $(x, y) \neq (\hat{0}, \hat{1})$. If in addition $\mu(\hat{0}, \hat{1}) = (-1)^{r(\hat{0}, \hat{1})} = (-1)^n$, then P is called *Eulerian*. If the geometric realization of $\Delta(P)$ is a manifold M, then P is semi-Eulerian by elementary topological reasoning. If M is in fact a sphere (or more generally just a homology sphere), then P is Eulerian.

2.2. Proposition (The combinatorial Alexander duality theorem). *Suppose P is semi-Eulerian of rank n, and let Q be any subposet of P containing $\hat{0}$ and $\hat{1}$. Set $\bar{Q} = (P \backslash Q) \cup \{\hat{0}, \hat{1}\}$. Then*

$$\mu(Q) = (-1)^{n-1} \mu(\bar{Q}) + \mu(P) - (-1)^n. \tag{10}$$

In particular, for all $S \subset [n-1]$ we have

$$\langle \beta_{\bar{S}}, 1 \rangle = \langle \beta_S, 1 \rangle + (-1)^{|S|} ((-1)^n - \mu(P)), \tag{11}$$

where $\bar{S} = [n-1] \backslash S$. If in fact P is Eulerian, then

$$\begin{aligned} \mu(Q) &= (-1)^{n-1} \mu(\bar{Q}), \\ \langle \beta_{\bar{S}}, 1 \rangle &= \langle \beta_S, 1 \rangle. \end{aligned} \tag{12}$$

Proof. If P is semi-Eulerian, then

$$\begin{aligned} \mu_P(\hat{0}, x_1)\,\mu_P(x_1, x_2) \cdots \mu_P(x_k, \hat{1}) &= (-1)^n, \qquad k \geqslant 1 \\ &= \mu(P), \qquad k = 0 \end{aligned}$$

for all chains $\hat{0} < x_1 < \cdots < x_k < \hat{1}$ in P. Hence from Lemma 2.1,

$$\mu(Q) = \Sigma(-1)^{k+n} + \mu(P) - (-1)^n,$$

where the sum ranges over all chains $\hat{0} < x_1 < \cdots < x_k < \hat{1}$ in $\bar{Q}$. Thus

$$\begin{aligned} \mu(Q) &= (-1)^{n-1}\,\tilde{\chi}(\Delta(\bar{Q})) + \mu(P) - (-1)^n \\ &= (-1)^{n-1}\mu(\bar{Q}) + \mu(P) - (-1)^n, \end{aligned}$$

which is (10). Picking $Q = P_S$, $\bar{Q} = P_{\bar{S}}$, and using (8) yields (11). Since $\mu(P) = (-1)^n$ when P is Eulerian, (12) follows. ∎

2.3. Corollary. *A semi-Eulerian poset P of odd rank n is Eulerian.*

Proof. Substitute $\bar{Q}$ for Q in (10) and rearrange to obtain

$$\mu(Q) = (-1)^{n-1}\,\mu(\bar{Q}) + (-1)^n\,(\mu(P) - (-1)^n).$$

Comparing with (10) yields $\mu(P) - (-1)^n = (-1)^n\,(\mu(P) - (-1)^n)$, so $\mu(P) = (-1)^n$ if n is odd. (There are many other ways to prove this corollary.) ∎

According to Corollary 2.2, when P is Eulerian the virtual representations β_S and $\beta_{\bar{S}}$ have the same degree, and we can ask what is the precise relationship between them. For this purpose we need a stronger condition on P so that the actual Alexander duality theorem can be invoked. (This explains why we have called Corollary 2.2 "the combinatorial Alexander duality theorem"—a special case of it follows directly from the usual Alexander duality theorem.) Suppose therefore that the geometric realization $|\Delta(P)|$ of $\Delta(P)$ is the sphere $\mathbb{S}^{n-2}$ (or more generally a homology $(n-2)$-sphere). Thus $\tilde{H}_{n-2}(P) \cong \mathbb{C}$. If $g \in \operatorname{Aut} P$, then denote by $\operatorname{sgn} g$ the degree of g, i.e., the induced map $\tilde{H}_{n-2}(P) \to^{g_*} \tilde{H}_{n-2}(P)$ is given by multiplication by $\operatorname{sgn} g$. Since g is an automorphism, $\operatorname{sgn} g = \pm 1$, depending on whether g is orientation-preserving or reversing (with respect to some given orientation of $\Delta(P)$). If g can be identified with some element of the orthogonal group $O(n-1, \mathbb{R})$ (i.e., if one has a triangulation $|\Delta(P)| \to \mathbb{S}^{n-2} = \{x \in \mathbb{R}^{n-1} : \|x\| = 1\}$ such that the automorphism g of $\Delta(P)$ is induced by some element of $O(n-1, \mathbb{R})$), then $\operatorname{sgn} g = \det g$. If G is a group of automorphisms of $\Delta(P)$, then the function $\operatorname{sgn}: G \to \{\pm 1\}$ is a character of G

of degree one. A special case of the next result (which does not require Alexander duality to prove) appears in [29, Lemma 7].

2.4. THEOREM. *Suppose that the geometric realization of $\Delta(P)$ is the sphere $\mathbb{S}^{n-2}$ (or more generally a homology $(n-2)$-sphere), so in particular P is Cohen–Macaulay. Let $G \subset \operatorname{Aut} P$. If $S \subset [n-1]$, then the representations β_S and $\beta_{\bar{S}}$ of G are related by*

$$\beta_S = (\operatorname{sgn})\, \beta_{\bar{S}}.$$

Proof. Let $g \in G$, and set $d = \operatorname{sgn} g$. By the Alexander duality theorem for simplicial complexes contained in a sphere (or homology sphere), we have an isomorphism

$$\tilde{H}^{n-k}(P_{\bar{S}}) \xrightarrow[\cong]{\phi} \tilde{H}_{k-1}(P_S).$$

By the standard naturality properties of ϕ, the following diagram commutes:

$$\begin{array}{ccc} \tilde{H}^{n-k}(P_{\bar{S}}) & \xrightarrow[\cong]{\phi} & \tilde{H}_{k-1}(P_S) \\ \Big\downarrow {\scriptstyle g^*_{\bar{S},n-k}} & & \Big\downarrow {\scriptstyle g^{S,k-1}_*} \\ \tilde{H}^{n-k}(P_{\bar{S}}) & \xrightarrow[\cong]{d\cdot\phi} & \tilde{H}_{k-1}(P_S) \end{array}.$$

By the universal coefficient theorem, we may identify $\tilde{H}^{n-k}(P_{\bar{S}})$ with $\operatorname{Hom}_{\mathbb{C}}(\tilde{H}_{n-k}(P_{\bar{S}}), \mathbb{C})$. Hence we have a pairing

$$\tilde{H}_{k-1}(P_S) \times \tilde{H}_{n-k}(P_{\bar{S}}) \xrightarrow{\alpha} \mathbb{C}$$

satisfying $\alpha(g_* a, g_* b) = d \cdot \alpha(a, b)$. It follows that $\langle \beta_S, g \rangle = (\operatorname{sgn} g)\langle \beta_{\bar{S}}, g^{-1} \rangle$. But $\beta_{\bar{S}}$ is a linear combination of the permutation representations α_T, so $\langle \beta_{\bar{S}}, g^{-1} \rangle = \langle \beta_{\bar{S}}, g \rangle$, and the proof follows. ∎

3. GEOMETRIC LATTICES

Before turning to examples, we wish to mention one additional result of a general nature. Recall that a *lattice* is a poset for which any two elements x and y have a least upper bound (join) $x \vee y$ and greatest lower bound (meet) $x \wedge y$. A finite lattice is *semimodular* if whenever x and y cover $x \wedge y$ (i.e., no z satisfies $x \wedge y < z < x$ or $x \wedge y < z < y$), then $x \vee y$ covers both x and y. A finite lattice is *geometric* if it is semimodular and every element is a join of atoms (elements covering $\hat{0}$). It is well known [7, 10] that a finite geometric lattice is Cohen–Macaulay (and therefore graded).

3.1. THEOREM. *Let L be a finite geometric lattice of rank n, and let $G \subset \operatorname{Aut} L$. Then there exists a basis B for $\tilde{H}_{n-2}(L)$ such that for all $g \in G$, the matrix of $g_*: \tilde{H}_{n-2}(L) \to \tilde{H}_{n-2}(L)$ with respect to B has all entries equal to* 0, 1, *or* -1.

Proof. Let A be a *base* of L, i.e., a set of n atoms whose join is $\hat{1}$. Björner [7, Sect. 4] shows how to associate with A a cycle $\rho_A \in \tilde{H}_{n-2}(L)$. He constructs a family $\mathbb{B}$ of bases of L such that $B = \{\rho_A \mid A \in \mathbb{B}\}$ is a basis for $\tilde{H}_{n-2}(L)$. He shows (Eq. (4.3)) that for any base A, if ρ_A is written as a linear combination of elements of B, then every coefficient is 0 or ± 1. Since any $g \in G$ permutes the bases of L, the proof follows. ∎

4. BOOLEAN ALGEBRAS

In this section and the next we present known examples which incorporate the theory of the previous sections. They will prove useful as an aid to understanding more complicated examples in subsequent sections.

Define $\mathscr{B}_n$ to be the set of all subsets of $[n]$, ordered by inclusion. $\mathscr{B}_n$ is a graded poset with $\hat{0}$ and $\hat{1}$ (actually, a distributive lattice) of rank n, which we call a *boolean algebra* of rank n. The symmetric group $\mathfrak{S}_n$ of all permutations of $[n]$ acts on $\mathscr{B}_n$ in an obvious way. In order to describe the representations α_S and β_S, we first must review some facts concerning the representations of $\mathfrak{S}_n$, such as may be found in [36, 37]. The (complex) irreducible representations $\{\lambda\}$ of $\mathfrak{S}_n$ are in one-to-one correspondence with the *partitions* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of n. Here $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$ and $\Sigma \lambda_i = n$, and we write $\lambda \vdash n$ or $|\lambda| = n$. If $\mu = (\mu_1, \ldots, \mu_k)$ is any sequence of positive integers summing to n, then set $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots \times \mathfrak{S}_{\mu_k} \subset \mathfrak{S}_n$ (the obvious imbedding). The action of $\mathfrak{S}_n$ on the cosets of $\mathfrak{S}_\mu$ (equivalently, the induction of the trivial representation of $\mathfrak{S}_\mu$ to $\mathfrak{S}_n$) affords a certain permutation representation $\Delta(\mu)$ of $\mathfrak{S}_n$, called a *Frobenius representation.* For each $\lambda \vdash n$, the multiplicity $K_{\lambda\mu}$ of $\{\lambda\}$ in $\Delta(\mu)$ depends only on the *multiset* (= set with repeated elements allowed) of entries of μ, not on their order. The number $K_{\lambda\mu}$ (especially when $\mu_1 \geqslant \cdots \geqslant \mu_k$, i.e., when $\mu \vdash n$) is called a *Kostka number*, and is equal to the number of reverse column-strict plane partitions (RCSPP) π of shape λ and type μ. This means that $\pi = (\pi_{ij})$ is a planar array of positive integers π_{ij}, with λ_i left-justified entries in the ith row, subject to the conditions that the numbers weakly increase along each row, strictly increase down each column, and exactly μ_i of them are equal to

i. For instance, if $\lambda = (4, 2, 1)$ and $\mu = (3, 2, 1, 1)$, then $K_{\lambda\mu} = 4$, corresponding to

$$\begin{array}{llll} 1\ 1\ 1\ 2 & 1\ 1\ 1\ 2 & 1\ 1\ 1\ 3 & 1\ 1\ 1\ 4 \\ 2\ 3 & 2\ 4 & 2\ 2 & 2\ 2 \\ 4 & 3 & 4 & 3 \end{array} \tag{13}$$

There is alternative description of $K_{\lambda\mu}$ which is more useful for our purposes. A *standard Young tableau* (SYT) of *shape* $\lambda \vdash n$ is an RCSPP of type $(1, 1,..., 1)$, i.e., with the distinct entries $1, 2,..., n$. The number f^λ of SYT of shape λ is the degree of the irreducible representation $\{\lambda\}$ of $\mathfrak{S}_n$. Given any RCSPP π of shape λ and type μ, replace the 1's in π with $1, 2,..., \mu_1$ from left-to-right. Then replace the 2's with $\mu_1 + 1, \mu_1 + 2,..., \mu_1 + \mu_2$ from left-to-right, etc. We get a SYT τ of shape λ such that if $i + 1$ appears in a row below i, then $i = \mu_1 + \mu_2 + \cdots + \mu_j$ for some j. We call an i for which $i + 1$ appears in a lower row a *descent* of τ, and we call the set of all descents of τ the *descent set* $D(\tau)$. For instance, the four RCSPP of (13) are converted to the SYTs

$$\begin{array}{llll} 1\ 2\ 3\ 5 & 1\ 2\ 3\ 5 & 1\ 2\ 3\ 6 & 1\ 2\ 3\ 7 \\ 4\ 6 & 4\ 7 & 4\ 5 & 4\ 5 \\ 7 & 6 & 7 & 6 \end{array}$$

with descent sets $\{3, 5, 6\}$, $\{3, 5\}$, $\{3, 6\}$, $\{3, 5\}$, respectively. Conversely, suppose we are given an SYT τ of shape $\lambda \vdash n$ and a sequence $\mu = (\mu_1, \mu_2,..., \mu_k)$ of non-negative integers with $\Sigma\mu_i = n$, such that $D(\tau) \subset \{\mu_1, \mu_1 + \mu_2,..., \mu_1 + \cdots + \mu_{k-1}\}$. If we replace $1, 2,..., \mu_1$ in τ with 1's, then $\mu_1 + 1, \mu_1 + 2,..., \mu_1 + \mu_2$ with 2's, etc., we get a RCSPP π of shape λ and type μ. Thus we have the following result, which is certainly implicit in [29, Sect. 6] and seems fairly well known, but does not seem to have been explicitly stated before.

4.1. PROPOSITION. *Let $S = \{n_1,..., n_k\}_< \subset [n-1]$, and set $\mu = (n_1, n_2 - n_1,..., n_k - n_{k-1}, n - n_k)$. Then the number of SYT τ of shape λ satisfying $D(\tau) \subset S$ is the Kostka number $K_{\lambda\mu}$.* ▮

Proposition 4.1 is closely related to *Schensted's correspondence*. This is a well-known one-to-one correspondence between the group $\mathfrak{S}_n$ of permutations π of $[n]$ and the set of all pairs (σ, τ) of SYT of the same shape and with n entries each. We are only interested here in τ and write $\Sigma_2(\pi) = \tau$. The precise description of this correspondence appears, e.g., in [26; 19, Sect. 5.1.4]. If $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ (where $a_i = \pi(i)$), then the *descent set* $D(\pi)$ of π is defined by:

$$D(\pi) = \{i \mid a_i > a_{i+1}\} \subset [n-1].$$

Schützenberger [27] has shown, in our notation, that if $\Sigma_2(\pi) = \tau$, then $D(\pi) = D(\tau)$. There follows from Proposition 4.1:

4.2. Proposition. *Let $S \subset [n-1]$ be as in Proposition 4.1, and let $\lambda \vdash n$. The number of permutations $\pi \in \mathfrak{S}_n$ such that $D(\pi) \subset S$ and such that $\Sigma_2(\pi)$ has shape λ is $K_{\lambda\mu} f^\lambda$, where $K_{\lambda\mu}$ is the Kostka number and f^λ is the number of SYTs of shape λ.* ∎

We now return to the action of $\mathfrak{S}_n$ of $\mathscr{B}_n$. $\mathscr{B}_n$ is well known to be Cohen–Macaulay; in fact, $\Delta(\mathscr{B}_n)$ is just the first barycentric subdivision of the boundary of an $(n-2)$-simplex. Let $S = \{n_1, n_2, ..., n_k\}_< \subset [n-1]$, and let $\mu = (n_1, n_2 - n_1, ..., n_k - n_{k-1}, n - n_k)$. The subgroup of $\mathfrak{S}_n$ fixing a given maximal chain of $(\mathscr{B}_n)_S$ is just $\mathfrak{S}_\mu$, so α_S is just the Frobenius representation $\Delta(\mu)$. The following result is due to Solomon [29, Sect. 6], though he does not state it quite so explicitly.

4.3. Theorem. *Let $\lambda \vdash n$ and let S be as above. Then the multiplicity $b_S(\lambda)$ of $\{\lambda\}$ in the representation β_S of $\mathfrak{S}_n$ is equal to the number of SYT of shape λ and descent set S. Equivalently, $f^\lambda b_S(\lambda)$ is equal to the number of permutations $\pi \in \mathfrak{S}_n$ such that $D(\pi) = S$ and such that $\Sigma_2(\pi)$ has shape λ.*

Proof. Let $a'_S(\lambda)$ (resp., $b'_S(\lambda)$) be the number of SYT τ of shape λ with $D(\tau) \subset S$ (resp. $D(\tau) = S$). Thus by Proposition 4.1, $a'_S(\lambda) = K_{\lambda\mu}$, with $\mu = (n_1, n_2 - n_1, ..., n - n_k)$ as above. Let $a_S(\lambda)$ be the multiplicity of $\{\lambda\}$ in α_S. Thus from Theorem 1.1,

$$a_S(\lambda) = \sum_{T \subset S} b_T(\lambda), \qquad \text{for all} \quad S \subset [n-1]. \tag{14}$$

Since $\alpha_S = \Delta(\lambda)$, we have $a_S(\lambda) = a'_S(\lambda)$. But clearly

$$a'_S(\lambda) = \sum_{T \subset S} b'_T(\lambda), \qquad \text{for all} \quad S \subset [n-1].$$

Since (14) uniquely determines $b_T(\lambda)$, it follows that $b_T(\lambda) = b'_T(\lambda)$. ∎

Theorem 4.3 in effect accomplishes the following. The representation $\alpha_{[n-1]}$ is just the regular representation of $\mathfrak{S}_n$, of degree $n!$, and the formula $\alpha_{[n-1]} = \sum_{S \subset [n-1]} \beta_S$ decomposes the regular representation into a sum of 2^{n-1} smaller representations. The multiplicity of $\{\lambda\}$ in $\alpha_{[n-1]}$ is just f^λ, the number of SYT of shape λ. Hence we would like to assign to each SYT τ of shape λ a set $S \subset [n-1]$ such that the number of τ's which are assigned S is the multiplicity $b_S(\lambda)$ of $\{\lambda\}$ in β_S. Theorem 4.3 shows that we may take $S = D(\tau)$. Alternatively, we would like to assign to every $\pi \in \mathfrak{S}_n$ a partition $\lambda \vdash n$ and a set $S \subset [n-1]$ such that the number of π's which are assigned (λ, S) is $b_S(\lambda) f^\lambda$. Again Theorem 4.3 shows that we may take $S = D(\pi)$ and λ to be the shape of $\Sigma_2(\pi)$. Moreover, these two ways to determine $b_S(\lambda)$ are essentially equivalent, i.e., $D(\pi) = D(\tau)$, where $\tau = \Sigma_2(\pi)$. These propitious circumstances will be exactly duplicated in Section 6 for the hyperoctahedral group (Weyl group of type B_n).

Note that if the SYT τ has descent set S, then the transpose τ' has descent set $\bar{S} = [n-1] \backslash S$. If $\{\lambda\}$ is an irreducible representation of $\mathfrak{S}_n$ and λ' is the conjugate partition to λ, then it is well known that $\{\lambda'\} = (\mathrm{sgn})\{\lambda\}$, where sgn denotes the sign representation of $\mathfrak{S}_n$ (the irreducible representation $\{1^n\}$). Hence we see that $\beta_{\bar{S}} = (\mathrm{sgn})\,\beta_S$. On the other hand, we have already mentioned that $|\Delta(\mathscr{B}_n)|$ is an $(n-2)$-sphere $\mathbb{S}^{n-2}$. The degree of $\pi \in \mathfrak{S}_n$ acting on $\Delta(\mathscr{B}_n)$ is just $\mathrm{sgn}\,\pi$, so that Theorem 2.4 gives an independent proof (avoiding any special knowledge of $\mathfrak{S}_n$) that $\beta_{\bar{S}} = (\mathrm{sgn})\,\beta_S$.

5. The Subspace Lattice $\mathscr{B}_n(q)$

Let $\mathscr{B}_n(q)$ denote the lattice of subspaces, ordered by inclusion, of an n-dimensional vector space $V_n(q)$ over the finite field $GF(q)$. $\mathscr{B}_n(q)$ may be regarded as a "q-analogue" of the boolean algebra $\mathscr{B}_n$, since in general a counting formula involving $\mathscr{B}_n(q)$ reduces to a corresponding formula for $\mathscr{B}_n$ when we put $q = 1$. The group $GL_n(q)$ of all linear automorphisms of $V_n(q)$ acts on $\mathscr{B}_n(q)$ in an obvious way. If $S \subset [n-1]$ as in Proposition 4.1, then define the representation $\Delta(\mu)_q$ of $GL_n(q)$ to be the permutation representation of $GL_n(q)$ on the maximal chains of $\mathscr{B}_n(q)_S$, or equivalently, the induction of the trivial representation of $GL_n(q)_\mu$ to $GL_n(q)$, where $GL_n(q)_\mu$ is the subgroup of $\mathrm{GL}_n(q)$ fixing a given maximal chain of $\mathscr{B}_n(q)_S$. Thus $\Delta(\mu)_q$ in the q-analogue of the representation $\Delta(\mu)$ of $\mathfrak{S}_n$, and $\Delta(\mu)_q = \alpha_S^{\mathscr{B}_n(q)}$. Steinberg [34] has shown that the representation $\Delta(\mu)_q$ decomposes into irreducibles parallel to the decomposition of $\Delta(\mu)$ into irreducibles. More precisely, for each $\lambda \vdash n$ there exists an irreducible representation $\{\lambda\}_q$ of $GL_n(q)$ (but unlike the situation for $\mathfrak{S}_n$, there are many more irreducible representations of $GL_n(q)$) such that the multiplicity of $\{\lambda\}_q$ in $\Delta(\mu)_q$ is the Kostka number $K_{\lambda\mu}$, and no other irreducibles occur in $\Delta(\mu)_q$. Thus the proof of Theorem 4.3 can be duplicated to yield:

5.1. Theorem. *Let $\lambda \vdash n$ and let S be as above. Then the multiplicity $b_S(\lambda, q)$ of $\{\lambda\}_q$ in the representation β_S of $GL_n(q)$ is equal to the number of SYT of shape λ and descent set S.* ∎

When $S = [n-1]$ (so $\mu = (1, 1,..., 1)$) then the representation $\{\mu\}_q$ of $GL_n(q)$ is the *Steinberg representation*, of degree $q^{\binom{n}{2}}$. Since $\mathscr{B}_n(q)$ is well known and easily seen to be a geometric lattice, we recover from Theorem 3.1 the known result that the Steinberg representation of $GL_n(q)$ can be realized by matrices with entries $0, \pm 1$.

6. The Lattice of Faces of a Cross-Polytope

Let $\mathscr{C}_n$ denote the set of all faces (including $\varnothing$ and K_n) of the n-dimensional cross-polytope K_n, ordered by inclusion. (K_n may be defined as the convex hull of the points $\pm e_i$, $1 \leqslant i \leqslant n$, where $e_1,..., e_n$ are the unit coordinate vectors in $\mathbb{R}^n$. It is the natural n-dimensional generalization of the octahedron K_3. The dual $\mathscr{C}_n^*$ of $\mathscr{C}_n$ is just the lattice of faces of an n-cube.) $\mathscr{C}_n$ is an Eulerian lattice of rank $n+1$. The group of symmetries of K_n is the hyperoctahedral group B_n (or Weyl group of type B_n—we are denoting the group by its "type") of order $2^n n!$. Thus B_n acts on $\mathscr{C}_n$, and we wish to describe the representations α_S and β_S for all $S \subset [n]$. First we will review some facts concerning the representations of B_n. Our basic reference is [13].

The irreducible representations of B_n can be put in a natural one-to-one correspondence with ordered pairs (λ, μ) of partitions such that $|\lambda| + |\mu| = n$. We call (λ, μ) a *double partition* of n, and following [13] denote the corresponding representation by $\{\lambda; \mu\}$. If $|\lambda| = k$ (so $|\mu| = n-k$), then $\deg\{\lambda; \mu\} = \binom{n}{k} f^\lambda f^\mu$, where $f^\nu = \deg\{\nu\}$, the degree of the irreducible representation $\{\nu\}$ of $\mathfrak{S}_{|\nu|}$. The trivial representation of B_n is given by $\{\varnothing; n\}$, while the sign representation (i.e., the representation of degree one whose value on each reflection is -1) is given by $\{1^n; \varnothing\}$. The elements of B_n may be identified in an obvious way with *barred permutations* of $[n]$, i.e., permutations $a_1 a_2 \cdots a_n$ of $[n]$ with bars over some of the a_i's. Now given two sequences $\lambda = (\lambda_1,..., \lambda_r)$ and $\mu = (\mu_1,..., \mu_s)$ of positive integers with $\Sigma\lambda_i + \Sigma\mu_j = n$, define a subgroup $B_{(\lambda,\mu)}$ of B_n by

$$B_{(\lambda,\mu)} \cong \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_r} \times B_{\mu_1} \times \cdots \times B_{\mu_s}$$

(obvious imbedding). Let $\Delta(\lambda; \mu)$ denote the induction of the trivial representation of $B_{(\lambda,\mu)}$ to B_n. The decomposition of $\Delta(\lambda; \mu)$ into irreducibles has been determined by Geissinger and Kinch [13, Theorem III.5]. In our notation, their result may be stated as follows.

6.1. Lemma. *Let λ and μ be as above, and let (α, β) be a double partition of n. The multiplicity of $\{\alpha; \beta\}$ in* $\Delta(\lambda, \mu)$ is given by

$$\sum_{\nu \leqslant \lambda} K_{\alpha\nu} K_{\beta,(\lambda-\nu)\cup\mu}, \tag{15}$$

where ν ranges over all sequences $(\nu_1,\ldots,\nu_r) \in \mathbb{N}^r$ with $\nu_i \leqslant \lambda_i$, and where $(\lambda-\nu)\cup\mu$ denotes the sequence $(\lambda_1-\nu_1,\ldots,\lambda_r-\nu_r,\mu_1,\ldots,\mu_s)$. ∎

We wish to give a combinatorial description of (15) analogous to the description of $K_{\lambda\mu}$ in Proposition 4.1, in the case where μ has only one part. Let (α,β) be a double partition of n. A *double standard Young tableau* (DSYT) of shape (α,β) is an ordered pair (π_1,π_2) of RCSPPs, such that π_1 has shape α, π_2 has shape β, and every integer $i \in [n]$ appears exactly once in $\pi_1 \cup \pi_2$. The *descent set* $D(\pi_1,\pi_2)$ of (π_1,π_2) is defined by

$$\begin{aligned} D(\pi_1,\pi_2) = \{i : {} & i \text{ and } i+1 \text{ both appear in the same } \pi_j, \text{ and} \\ & i+1 \text{ appears in a lower row than } i\} \\ \cup\, \{i : {} & i \text{ appears in } \pi_1 \text{ and } i+1 \text{ in } \pi_2, \text{ or} \\ & i=n \text{ and } n \text{ appears in } \pi_1\}. \end{aligned}$$

For instance, the DSYT

$$\begin{array}{ccc} 2\ 4\ 7 & \quad & 1\ 6 \\ 5 & & 3\ 8 \end{array}$$

has descent set $\{2,4,5,7\}$, while

$$\begin{array}{ccc} 1\ 4\ 5 & \quad & 2\ 7 \\ 6\ 9 & & 3 \\ & & 8 \end{array}$$

has descent set $\{1,2,5,6,7,9\}$.

6.2. Proposition. *Let $\lambda=(\lambda_1,\ldots,\lambda_r)$, μ, α,β be as in Lemma 6.1. Assume $\mu=(m)$, the partition with one part equal to m. Then the multiplicity of $\{\alpha;\beta\}$ in $\Delta(\lambda,\mu)=\Delta(\lambda,m)$ is equal to the number of DSYTs of shape (α,β) whose descent set is contained in the set $S=\{\lambda_1,\lambda_1+\lambda_2,\ldots,\lambda_1+\lambda_2+\cdots+\lambda_r\}$.*

Proof. The term $K_{\alpha\nu}K_{\beta,(\lambda-\nu)\cup\mu}$ in (15) counts the number of pairs (π_1,π_2) of RCSPPs such that π_1 has shape α and type ν, and π_2 has shape β and type $(\lambda-\nu)\cup\mu$. Given such a pair (π_1,π_2) and assuming $\mu=(m)$, replace the λ_i i's which appear in π_1 and π_2 by the numbers $\lambda_1+\lambda_2+\cdots+\lambda_{i-1}+1$, $\lambda_1+\lambda_2+\cdots+\lambda_{i-1}+2,\ldots$, $\lambda_1+\lambda_2+\cdots+\lambda_i$, by first filling in $\lambda_1+\cdots+\lambda_{i-1}+1,\ldots$, $\lambda_1+\cdots+\lambda_{i-1}+\lambda_i-\nu_i$ from left to right in π_2 and then $\lambda_1+\cdots+\lambda_i-\nu_i+1,\ldots$, $\lambda_1+\cdots+\lambda_i$ from left to right in π_1. Then replace the m occurences of $r+1$ in π_2 by the numbers $\lambda_1+\cdots+\lambda_r+1,\ldots,$

$\lambda_1 + \cdots + \lambda_r + m$ from left to right. For instance, if $\lambda = (3, 1, 3, 1, 2)$, $\mu = (2)$, $\alpha = (3, 3, 1)$, $\beta = (2, 2, 1)$, and (π_1, π_2) is given by

$$\begin{array}{lll} 1 & 1 & 3 \\ 2 & 3 & 5 \\ 4 & & \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 5 & 6 \\ 6 & \end{array}$$

then we obtain the arrays

$$\begin{array}{lll} 2 & 3 & 7 \\ 4 & 6 & 10 \\ 8 & & \end{array} \qquad \begin{array}{ll} 1 & 5 \\ 9 & 12 \\ 11 & \end{array}$$

In general we obtain a DSYT (τ_1, τ_2) of shape (α, β). Since $\nu \leqslant \lambda$, it follows that $D(\tau_1, \tau_2) \subset S$. This correspondence can easily be reversed, so the proof follows. ∎

EXAMPLE. The multiplicity of $\{31; 1^3\}$ in $\Delta(1131, 1)$ is 5, corresponding to the DSYTs

$$\left|\begin{array}{llll} 1 & 4 & 5 & 3 \\ 2 & & & 6 \\ & & & 7 \end{array}\right| \begin{array}{llll} 1 & 4 & 5 & 2 \\ 6 & & & 3 \\ & & & 7 \end{array} \left|\begin{array}{llll} 1 & 2 & 5 & 3 \\ 4 & & & 6 \\ & & & 7 \end{array}\right| \begin{array}{llll} 1 & 5 & 6 & 2 \\ 4 & & & 3 \\ & & & 7 \end{array} \left|\begin{array}{llll} 1 & 4 & 5 & 2 \\ 3 & & & 6 \\ & & & 7 \end{array}\right|$$

Just as for the case $\mathfrak{S}_n$, we may ask whether there is a version of Proposition 6.2 which deals directly with the elements of B_n themselves. In other words, we seek an analogue $\bar{\Sigma}$ of Schensted's correspondence Σ for which Proposition 4.2 exends to B_n.

We proceed to define the correspondence $\bar{\Sigma}$. Let $\pi = \tilde{a}_1 \tilde{a}_2 \cdots \tilde{a}_n \in B_n$, where the overhead wiggles may be either blanks or bars. Let $a_{i_1} a_{i_2} \cdots a_{i_r}$ be the subsequence of unbarred elements. Apply Schensted's correspondence to the two–line array (or "generalized permutation") $\left(\begin{smallmatrix} i_1 & i_2 & \cdots & i_r \\ a_{i_1} & a_{i_2} & \cdots & a_{i_r} \end{smallmatrix}\right)$. This yields a pair (σ_1, τ_1) of RCSPPs of the same shape. The entries of σ_1 consist of $a_{i_1}, \ldots, a_{i_r}$, while the entries of τ_1 consist of $i_1, \ldots, i_r$. Similarly if $a_{j_1} a_{j_2} \cdots a_{j_s}$ is the subsequence of barred elements, apply Schensted's correspondence to $\left(\begin{smallmatrix} j_1 & j_2 & \cdots & j_s \\ a_{j_1} & a_{j_2} & \cdots & a_{j_s} \end{smallmatrix}\right)$ to obtain a pair (σ_2, τ_2). Finally define $\bar{\Sigma}(\pi) = (\sigma, \tau)$, where $\sigma = (\sigma_1, \sigma_2)$ and $\tau = (\tau_1, \tau_2)$. For instance, if $\pi = \bar{4}\ 1\ 7\ \bar{3}\ \bar{6}\ 9\ 2\ 8\ \bar{5}$, then

$$\Sigma \begin{pmatrix} 2 & 3 & 6 & 7 & 8 \\ 1 & 7 & 9 & 2 & 8 \end{pmatrix} = \left(\begin{array}{lll} 1 & 2 & 8 \\ 7 & 9 & \end{array} \quad \begin{array}{lll} 2 & 3 & 6 \\ 7 & 8 & \end{array}\right) = (\sigma_1, \tau_1),$$

$$\Sigma \begin{pmatrix} 1 & 4 & 5 & 9 \\ 4 & 3 & 6 & 5 \end{pmatrix} = \left(\begin{array}{ll} 3 & 5 \\ 4 & 6 \end{array} \quad \begin{array}{ll} 1 & 5 \\ 4 & 9 \end{array}\right) = (\sigma_2, \tau_2).$$

Hence $\bar{\Sigma}(\pi) = (\sigma, \tau)$, where

$$\sigma = \begin{pmatrix} 1 & 2 & 8 & 3 & 5 \\ 7 & 9 & & 4 & 6 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 2 & 3 & 6 & 1 & 5 \\ 7 & 8 & & 4 & 9 \end{pmatrix}.$$

Clearly this is a one-to-one correspondence, since given (σ, τ) where $\sigma = (\sigma_1, \sigma_2)$ and $\tau = (\tau_1, \tau_2)$, we can reconstruct uniquely the two-line arrays $\Sigma^{-1}(\sigma_1, \tau_1)$ and $\Sigma^{-1}(\sigma_2, \tau_2)$.

Now given $\pi \in B_n$, we first give an *ad hoc* definition of the descent set $D(\pi)$. Namely, if $\pi = \tilde{a}_1 \tilde{a}_2 \cdots \tilde{a}_n$ as above, then define

$$\begin{aligned} D(\pi) = \{i : \bar{a}_i > \bar{a}_{i+1} \quad &\text{or} \quad a_i > a_{i+1} \quad \text{or} \\ a_i < \bar{a}_{i+1} \quad &\text{or} \quad a_i > \bar{a}_{i+1} \quad \text{or} \\ i = n \quad &\text{and} \quad a_n \text{ is unbarred}\}. \end{aligned} \tag{16}$$

Here a notation such as $a_i < \bar{a}_{i+1}$ means that $a_i < a_{i+1}$ (as integers), a_i is unbarred, and a_{i+1} is barred. Thus for instance $D(\bar{4}\ 1\ 7\ \bar{3}\ \bar{6}\ 9\ 2\ 8\ \bar{5}) = \{3, 6, 8\}$.

6.3. PROPOSITION. *If $\bar{\Sigma}(\pi) = (\sigma, \tau)$, then $D(\pi) = D(\tau)$.*

Proof. Let $\tau = (\tau_1, \tau_2)$. By the ordinary Schensted correspondence, we have:

(a) $i + 1$ lies below i in τ_1 if and only if $a_{i+1} > a_i$ (both unbarred),

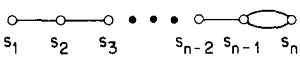
(b) $i + 1$ lies below i in τ_2 if and only if $\bar{a}_{i+1} > \bar{a}_i$,

(c) i appears in τ_1 and $i + 1$ in τ_2 if and only if a_i is unbarred and a_{i+1} is barred, and

(d) n appears in τ_1 if and only if a_n is unbarred.

The proof follows from the definition of $D(\tau_1, \tau_2)$. ∎

The definition (16) of $D(\pi)$ can be made less *ad hoc*. The group B_n is generated by simple reflections $s_1, ..., s_n$ corresponding to the Dynkin diagram:



Multiplication can be defined as follows: If $1 \leqslant i \leqslant n-1$ and $\pi = {}^{1}_{a_1}\,{}^{2}_{a_2} \cdots {}^{n}_{a_n}$, where the j above a_j is either blank or a bar, then $s_i \pi = {}^{1}_{a_1}\,{}^{2}_{a_2} \cdots {}^{i+1}_{a_{i+1}}\,{}^{i}_{a_i} \cdots {}^{n}_{a_n}$, if $1 \leqslant i \leqslant n-1$. Moreover, $s_n \pi$ replaces a bar over a_n by a blank or *vice versa*. If now $\pi \in B_n$, recall that the *length* $l(\pi)$ is the least integer l for which one can write $\pi = s_{i_1} s_{i_2} \cdots s_{i_l}$. It is then easy to check that $i \in D(\pi)$ if and only if $l(s_i\pi) > l(\pi)$. Equivalently, $s_i\pi > \pi$ in the *Bruhat order* of B_n, as defined, e.g., in [33].

Note. The corresponding result for $\pi \in \mathfrak{S}_n$ states that $i \in D(\pi)$ if and only if $s_i\pi < \pi$ in the Bruhat order of $\mathfrak{S}_n$. The reason for the reversed inequality (i.e., $s_i\pi > \pi$ in B_n and $s_i\pi < \pi$ in $\mathfrak{S}_n$) arises from our desire to adhere to the notation of [13].

It is now easy to determine the analogue of Theorem 4.3 for the cross-polytopal lattice $\mathscr{C}_n$. Since $\Delta(\mathscr{C}_n)$ is just the first barycentric subdivision of the boundary complex of the cross-polytope K_n, it follows that $\mathscr{C}_n$ is Cohen–Macaulay. Let $S = \{n_1, n_2, ..., n_k\}_< \subset [n]$. Set $\lambda = (n_1, n_2 - n_1, ..., n_k - n_{k-1})$ and $m = n - n_k$. Then the subgroup of B_n fixing a maximal chain in $(\mathscr{C}_n)_S$ is just $B_{(\lambda, m)}$. Hence α_S is just the representation $\Delta(\lambda, m)$ of B_n.

6.4. Theorem. *Let $P = \mathscr{C}_n$ and $S \subset [n]$. Let $G = B_n$ be the group of automorphisms of P. Then the multiplicity $b_S(\alpha, \beta)$ of the representation $\{\alpha; \beta\}$ of B_n in the representation β_S is equal to the number of* DSYTs *of shape (α, β) whose descent set is equal to S. Equivalently, $f^{(\alpha,\beta)} b_S(\alpha, \beta)$ is equal to the number of elements $\pi \in B_n$ such that $D(\pi) = S$ and such that $\bar{\Sigma}_2(\pi)$ has shape (α, β). Here $f^{(\alpha,\beta)} = \deg\{\alpha; \beta\}$.*

Proof. The proof parallels that of Theorem 4.3. Let $a'_S(\alpha, \beta)$ (resp. $b'_S(\alpha, \beta)$) be the number of DSYT τ of shape (α, β) with $D(\tau) \subset S$ (resp. $D(\tau) = S$). By definition,

$$a'_S(\alpha, \beta) = \sum_{T \subset S} b'_T(\alpha, \beta), \qquad \text{for all } S \subset [n].$$

Let $a_S(\alpha, \beta)$ be the multiplicity of $\{\alpha; \beta\}$ in α_S. If $S = \{n_1, n_2, ..., n_k\}_<$ then set $\lambda = (n_1, n_2 - n_1, ..., n_k - n_{k-1})$ and $m = n - n_k$. Since $\alpha_S = \Delta(\lambda, m)$, by Proposition 6.3 we have $a_S(\alpha, \beta) = a'_S(\alpha, \beta)$. By Theorem 1.1,

$$a_S(\alpha, \beta) = \sum_{T \subset S} b_T(\alpha, \beta), \qquad \text{for all } S \subset [n].$$

It follows that $b_S(\alpha, \beta) = b'_S(\alpha, \beta)$ for all $S \subset [n]$. ∎

Note that Theorem 2.4 applies to $\mathscr{C}_n$ since, as we've already remarked, $\Delta(\mathscr{C}_n)$ is isomorphic to the first barycentric subdivision of the boundary of the cross-polytope K_n. Since the sign representation of B_n is $\{1^n; \varnothing\}$, we conclude $\beta_{\bar{S}} = \{1^n; \varnothing\} \beta_S$, where $\bar{S} = [n] \backslash S$. This checks with Theorem 6.4, since $\{1^n; \varnothing\}\{\alpha; \beta\} = \{\beta'; \alpha'\}$ (where $'$ denotes conjugate partition), and if (τ_1, τ_2) is a DSYT then $D(\tau_1, \tau_2) = [n] \backslash D(\tau_2^t, \tau_1^t)$ (where ^{t} denotes transpose).

Note. While the generalization $\bar{\Sigma}$ of Schensted's correspondence which we have defined is certainly adequate for our purposes, it fails to possess a crucial property of interest for other applications. Namely, it is not true that two elements π, σ of B_n satisfying $\bar{\Sigma}_1(\pi) = \bar{\Sigma}_1(\sigma)$ lie in the same cell of B_n, as defined in [13].

7. The Lattice of Partitions of an n-Set

Let Π_n denote the poset (actually a lattice) of all partitions of $[n]$, ordered by refinement. Thus the elements of Π_n are sets $\beta = \{B_1, ..., B_k\}$, where the B_i's are pairwise-disjoint nonvoid subsets of $[n]$ with union $[n]$. Moreover, $\{B'_1, ..., B'_j\} \leqslant \{B_1, ..., B_k\}$ in Π_n if and only if every B'_r is contained in some B_s. Π_n is well known to be a geometric lattice of rank $n-1$ [5, p. 95], and therefore by the sentence preceding Theorem 3.1 is Cohen–Macaulay. The symmetric group $\mathfrak{S}_n$ acts on Π_n in an obvious way, and we can ask for the decomposition of the representations α_S and β_S for $S \subset [n-2]$. In particular, it is again well known [5, p. 93, Ex. 7; 25, p. 359] that in Π_n, $\mu(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$. Thus by (11), $\beta_{[n-2]}$ is a representation of $\mathfrak{S}_n$ of degree $(n-1)!$, and we can try to describe it more explicitly.

The crucial fact we need is the following recent result of Hanlon [16, Theorem 4].

7.1. Lemma. *Let $\pi \in \mathfrak{S}_n$, and let Π_n^π denote the sublattice of Π_n fixed pointwise by π. Let μ_π denote the Möbius function of Π_n^π. Then*

$$\begin{aligned} \mu_\pi(\hat{0}, \hat{1}) &= (-1)^{d-1} \mu(n/d)(d-1)!\,(n/d)^{d-1}, && \textit{if } \pi \textit{ is a product of } d \textit{ cycles of length } n/d \\ &= 0, && \textit{otherwise.} \end{aligned}$$

(Here $\mu(n/d)$ notes the usual number-theoretic Möbius function.) ∎

Remark. Hanlon actually computes $\mu_\pi(x_\pi, \hat{1})$, where x_π is the meet of the coatoms of Π_n^π. It is easy to see (alternatively, this follows from [16, Lemma 2]) that $x_\pi = \hat{0}$ if and only if all cycles of π have the same length. Since in any (finite) lattice $\mu(\hat{0}, \hat{1}) = 0$ unless $\hat{0}$ is a meet of coatoms [25, p. 349], Lemma 7.1 follows from Hanlon's results.

Now let C_n be a cyclic subgroup of $\mathfrak{S}_n$ of order n generated by an n-cycle σ. Let $\zeta = e^{2\pi i/n}$, and (by slight abuse of notation) identify ζ with the representation (or character) of C_n whose value $\zeta(\sigma)$ at σ is ζ. Define the induced representation

$$\psi_n = \operatorname{ind}_{C_n}^{\mathfrak{S}_n}(\zeta). \tag{17}$$

7.2. LEMMA. *Let $\pi \in \mathfrak{S}_n$. Then*

$$\langle \psi_n, \pi \rangle = \mu(n/d)(d-1)!\,(n/d)^{d-1}, \qquad \textit{if } \pi \textit{ is a product of } d \textit{ cycles of length } n/d$$
$$= 0, \qquad \textit{otherwise.}$$

Proof. The usual formula (e.g., [15, Theorem 16.7.2]) for the character of an induced representation yields

$$\langle \psi_n, \pi \rangle = \frac{(n-1)!}{|C_\pi|} \sum_{\sigma \in C_\pi \cap C_n} \zeta(\sigma),$$

where C_π is the conjugacy class of $\mathfrak{S}_n$ containing π. Hence $\langle \psi_n, \pi \rangle = 0$ unless $d \mid n$ and π has d cycles of length n/d. In this case $\zeta(\sigma)$ runs through all primitive (n/d)th roots of unity and $|C_\pi| = n!/(n/d)^d\, d!$. Since the sum of the primitive (n/d)th roots of unity is $\mu(n/d)$ (e.g., [17, (16.64)]), the proof follows. ∎

7.3. THEOREM. *Let $G = \mathfrak{S}_n$ act on $P = \Pi_n$ in the obvious way. Then $\beta_{[n-2]} = (\mathrm{sgn})\, \psi_n$. (Note that if $n \not\equiv 2 \pmod 4$, then either $(-1)^{n-d} = 1$ or $\mu(n/d) = 0$ for all $d \mid n$. Thus in this case $\beta_{[n-2]} = \psi_n$ and $\beta_{[n-2]}$ is "self-conjugate.")*

Proof. Let $\pi \in \mathfrak{S}_n$. By (4), we have

$$(-1)^{n-1} < \beta_{[n-2]}, \pi \rangle = \tilde{\Lambda}_{[n-2]}(\pi) = \tilde{\chi}(\Delta(\Pi_n)^\pi) = \tilde{\chi}(\Delta(\Pi_n^\pi)).$$

Thus by (8),

$$\langle \beta_{[n-2]}, \pi \rangle = (-1)^{n-1} \mu_\pi(\hat{0}, \hat{1}),$$

and the proof follows by comparing Lemmas 7.1 and 7.2. ∎

7.4. COROLLARY. *Let p be an odd prime. Then the representation $\beta_{[p-2]}$ of $\mathfrak{S}_p$ given in Theorem 7.3 is equal to "$(1/p)$th of the regular representation" of $\mathfrak{S}_p$. More precisely, if $\lambda \vdash p$ then the multiplicity of $\{\lambda\}$ in $\beta_{[p-2]}$ is equal to $\langle f^\lambda/p \rangle$, where $\langle x \rangle$ denotes the nearest integer to x.*

Proof. Let χ^λ be the character of $\{\lambda\}$, and let σ be an n-cycle. Since $\beta_{[p-2]} = \psi_p$, by Frobenius reciprocity the desired multiplicity is given by

$$\frac{1}{p} \sum_{j=0}^{p-1} \chi^\lambda(\sigma^j)\, e^{-2\pi i j/p} = \frac{1}{p}\,(f^\lambda - \chi(\sigma)).$$

It is well known (e.g., [37, Lemma 4.11]) that

$$\chi^\lambda(\sigma) = (-1)^{i-1}, \qquad \text{if} \quad \lambda = (p-i, \underbrace{1, 1, \ldots, 1}_{i}), 0 \leqslant i < p$$

$$= 0, \qquad \text{otherwise.}$$

From this the proof is immediate. ∎

7.5. Corollary. *For any $n \geqslant 1$, the representation ψ_n of $\mathfrak{S}_n$ can be realized by matrices whose entries are all equal to* 0, +1, *or* −1.

Proof. Follows from Theorems 3.1 and 7.3, and the fact that Π_n is a geometric lattice. ∎

7.6. Corollary. *The group $G = \mathfrak{S}_{n-1}$ acts on $[n]$ by permuting $[n-1]$ and fixing n. Extend this action to Π_n. Then the corresponding representation $\beta_{[n-2]}$ of $\mathfrak{S}_{n-1}$ is isomorphic to the regular representation.*

Proof. By Theorem 7.3, the representation $\beta_{[n-2]}$ is the restriction of $(\mathrm{sgn})\psi_n$ to $\mathfrak{S}_{n-1}$. By Lemma 7.2, it follows that $\langle \beta_{[n-2]}, \pi \rangle = 0$ unless $\pi = 1$. Since $\deg \beta_{[n-2]} = (n-1)!$, the proof follows. ∎

Theorem 7.3 completely determines $\beta_{[n-2]}$ for the action of $\mathfrak{S}_n$ on Π_n. We have been unable to find an analogous description of β_S for arbitrary $S \subset [n-2]$. We have, however, obtained some information on the problem of computing the multiplicity (which we will denote by $b_S = b_S(n)$) of the trivial representation $\{n\}$ in β_S. Let $a_S = a_S(n)$ denote the multiplicity of $\{n\}$ in α_S. Thus $a_S = |(\Pi_n)_S / \mathfrak{S}_n|$, the number of orbits of maximal chains of $(\Pi_n)_S$ under the action of $\mathfrak{S}_n$. I am grateful to I. Gessel and J. Shearer for supplying (independently) a proof of the following result.

7.7. Theorem. *Define integers A_n by*

$$\sum_{n \geqslant 0} A_n x^n / n! = \tan x + \sec x.$$

Then $a_{[n-2]}(n) = A_{n-1}$.

Proof. We seek the number of orbits of maximal chains of Π_n under the action of $\mathfrak{S}_n$. For convenience denote this number by c_n. We obtain exactly two representatives $\hat{0} = x_0 < x_1 < \cdots < x_{n-1} = \hat{1}$ of each orbit as follows: Choose $1 \leqslant i \leqslant n-1$. Then choose the partition x_{n-2} of $[n]$ to consist of two specified blocks B_1 and B_2, where $|B_1| = i$ and $|B_2| = n-i$. (We get *two* orbit representatives because B_1 and B_2 could be interchanged.) Next choose representatives of the orbits of maximal chains of $\Pi(B_1)$ and $\Pi(B_2)$ under $\Pi(B_1)$ and $\Pi(B_2)$ in $c_i c_{n-i}$ ways. Finally "interweave" these two maximal chains in $\binom{n-2}{i-1}$ ways to obtain a maximal chain of Π_n containing x_{n-2}. It follows that

$$2c_n = \sum_{i=1}^{n-1} \binom{n-2}{i-1} c_i c_{n-i},\ n \geqslant 2,\ c_1 = 1.$$

Comparing with the well-known recurrence [8, p. 258]

$$2A_{n+1} = \sum_{i=0}^{n} \binom{n}{i} A_i A_{n-i}, \qquad n \geqslant 0, A_0 = 1,$$

yields the proof. ∎

By Theorem 1.1, we have

$$\sum_{S \subset [n-2]} b_S(n) = A_{n-1}. \tag{18}$$

Since Π_n is Cohen–Macaulay, each $b_S(n) \geqslant 0$. Moreover, A_n is equal to the number of *alternating permutations* in $\mathfrak{S}_n$, i.e., permutations $\pi \in \mathfrak{S}_n$ satisfying $D(\pi) = \{1, 3, 5, ..., 2[n/2] - 1\}$ (e.g., [8, p. 258]). This suggests the following problem:

Problem. Find a combinatorial interpretation of $b_S(n)$ as the number of alternating permutations in $\mathfrak{S}_{n-1}$ with a certain property (parametrized by $S \subset [n-2]$), so that (18) is evident.

It is easy to see that if $1 \leqslant i \leqslant n-2$, then $b_{[i]}(n) = p_{n-i}(n) - 1$, where $p_k(n)$ denotes the number of partitions of n into k parts. We also have the following result.

7.8. Proposition. *Let* $1 \leqslant i \leqslant n-2$. *Then* $b_{[i]}(n) = 0$.

Proof. If $i = n-2$, then by Theorem 7.3, $b_{[n-2]}(n)$ is equal to the multiplicity of $\{n\}$ in $(\mathrm{sgn})\,\psi_n$. Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. By Frobenius reciprocity,

$$\begin{aligned} b_{[n-2]}(n) &= \frac{1}{n} \sum_{j=0}^{n-1} e^{-2\pi i j/n} \operatorname{sgn}(\sigma^j) \\ &= \frac{1}{n} \sum_{j=0}^{n-1} e^{-2\pi i j/n} (-1)^{(n-1)j} = 0. \end{aligned}$$

If $1 \leqslant i \leqslant n-3$, then the result $\beta_{[i]}(n) = 0$ is due to P. Hanlon (private communication), using the case $i = n-2$ to deduce the general result. Hanlon's proof will appear elsewhere. ∎

Even if the numbers $b_S(n)$ cannot be determined explicitly, it may still be possible to say when $b_S(n) = 0$. Table 1 shows that the sufficient condition of the previous proposition is not necessary. In regard to this table, we remark that it is not difficult to see that if $S \subset [m]$ and if $n \geqslant 2m$, then $b_S(n)$ depends only on S, not on n.

There is an alternative method of computing $a_{[n-2]}(n)$ which leads to a curious combinatorial identity.

7.9. PROPOSITION. *Define an array $g(k, n)$ of rational numbers as follows*:

$$g(0,0) = 1, \qquad g(k,n) = 0 \qquad \text{if} \quad k < 0 \text{ or } n < 0,$$

$$g(k,n) = \tfrac{1}{2}(n - 2k + 2)\, g(k-1, n-1) + \tfrac{1}{2}(n - 2k)\, g(k, n-1), \qquad \text{if } k, n \geqslant 0,\ (k,n) \neq (0,0).$$

Then $g(k, n) = 0$ if $k > [\frac{1}{2}(n+1)]$, and

$$\sum_{k=0}^{[(n+1)/2]} g(k,n) = A_n.$$

TABLE I

Some Values of $b_S(n)$

S	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n\geqslant 8$
∅	1	1	1	1	1	1
1	0	0	0	0	0	0
2		1	1	1	1	1
3			1	2	2	2
4				2	3	4
12		0	0	0	0	0
13			1	1	1	1
14				1	2	2
23			1	1	1	1
24				3	5	5
34				1	2	2
123			0	0	0	0
124				0	0	0
134				2	3	3
234				1	1	1
1234				0	0	0

Proof. By Burnside's lemma (e.g., [8, p. 249]),

$$a_{[n-1]}(n+1)=\frac{1}{(n+1)!}\sum_{\pi\in\mathfrak{S}_{n+1}}|\mathrm{Fix}(\pi)|,$$

where $\mathrm{Fix}(\pi)$ is the set of maximal chains C of Π_{n+1} fixed by π. Since a maximal chain C of Π_{n+1} is obtained from the partition into n singletons by merging two blocks at a time, we easily see that $\mathrm{Fix}(\pi)=\varnothing$ unless $\pi^2=1$. Let $f(k,n)=|\mathrm{Fix}(\pi)|$, where π contains k 2-cycles and $n+1-2k$ fixed points. There are $(n+1)!/2^k k!\,(n+1-2k)!$ such π, so

$$a_{[n-1]}(n+1)=\sum_{k=0}^{[(n+1)/2]}\frac{f(k,n)}{2^k k!\,(n+1-2k)!}. \tag{20}$$

Suppose π has k 2-cycles and $n+1-2k$ fixed points, and consider a maximal chain $\hat{0}<x_1<\cdots<x_{n-1}<\hat{1}$ in Π_{n+1} fixed by π. The partition x_1 has a unique two-element block $\{a,b\}$, and we have the following two alternatives:

(i) (a,b) is a 2-cycle of π (k possibilities).

(ii) a and b are fixed points of π ($\binom{n+1-2k}{2}$ possibilities).

In the first case, there are $f(k-1,n-1)$ ways to choose $x_2,x_3,\ldots,x_{n-1}$. In the second case, there are $f(k,n-1)$ ways to choose $x_2,x_3,\ldots,x_{n-1}$. Hence

$$f(0,0)=1,\qquad f(k,n)=0\qquad\text{if}\quad k<0\text{ or }n<0,$$

$$f(k,n)=kf(k-1,n-1)+\binom{n+1-2k}{2}f(k,n-1),$$

$$\text{if}\quad k,n\geqslant 0,\ (k,n)\neq(0,0). \tag{21}$$

Now set

$$g(k,n)=f(k,n)/2^k k!(n+1-2k)!.$$

It is immediate from (21) that $g(k,n)$ satisfies (19). Comparing Theorem 7.7 and (20) yields the proof. ∎

Ira Gessel has found a proof of Proposition 7.9 which avoids Burnside's lemma or other group-theoretical considerations.

8. Barycentric Subdivision

Let P be a finite poset with $\hat{0}$ and $\hat{1}$. The simplicial complex $\Delta(P)$ may itself be regarded as a poset, with its faces being ordered by inclusion. Thus

$\Delta(P)$ has a $\hat{0}$ consisting of the void face $\varnothing$. Let $\hat{\Delta}(P)$ denote $\Delta(P)$ with a $\hat{1}$ adjoined. If P is graded of rank n, then so is $\hat{\Delta}(P)$. In fact, $\Delta(\hat{\Delta}(P))$ is just the first barycentric subdivision of $\Delta(P)$. Thus $\hat{\Delta}(P)$ is Cohen–Macaulay if and only if P is, and any group $G \subset \operatorname{Aut} P$ also acts in an obvious way on $\hat{\Delta}(P)$. In this section we will compute $\alpha_S^{\hat{\Delta}(P)}$ and $\beta_S^{\hat{\Delta}(P)}$ in terms of α_T^P and β_T^P.

8.1. Proposition. *Let P and G be as above, and let $S = \{s_1, ..., s_k\}_< \subset [n-1]$. Then*

$$\alpha_S^{\hat{\Delta}(P)} = \binom{s_k}{s_1, s_2 - s_1, ..., s_k - s_{k-1}} \sum_{\substack{T \subset [n-1] \\ |T| = s_k}} \alpha_T^P.$$

Proof. Let $\hat{0} < x_1 < \cdots < x_k < \hat{1}$ be a maximal chain C in $\Delta(P)_S$. Thus x_i is a chain in $P - \{\hat{0}, \hat{1}\}$ with s_i elements, and $x_{i-1} \subset x_i$ (as subsets of P). If $g \in G$, then g fixes C if and only if it fixes x_k. It follows that a maximal chain of $\hat{\Delta}(P)_S$ fixed by g is obtained by: (a) choosing a subset T of $[n-1]$ with $|T| = s_k$, (b) choosing a maximal chain x_k of P_T in $\langle \alpha_T^P, g \rangle$ ways, and (c) choosing subchains $x_1 \subset \cdots \subset x_{k-1} \subset x_k$ of x_k with $|x_i| = s_i$ in $\binom{s_k}{s_1, s_2 - s_1, ..., s_k - s_{k-1}}$ ways. From this the proof follows. ∎

8.2. Lemma. *Let $T \subset [n-1]$ and $0 \leqslant i < n$. Define $H_i(T)$ to be the number of permutations $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ with descent set $D(\pi) = T$ and with $a_n = n - i$. If $S = \{s_1, ..., s_k\}_< \subset [n-1]$, then*

$$\sum_{T \subset S} H_i(T) = \binom{s_k}{s_1, s_2 - s_1, ..., s_k - s_{k-1}} \binom{n-i-1}{n-s_k-1}.$$

Proof. If $D(\pi) \subset S$ and $a_n = n - i$, then $1 \leqslant a_{s_k+1} < a_{s_k+2} < \cdots < a_{n-1} \leqslant n - i - 1$. There are therefore $\binom{n-i-1}{n-s_k-1}$ ways to choose $a_{s_k+1}, ..., a_{n-1}$. The remaining a_j's must satisfy $a_1 < a_2 < \cdots < a_{s_1}, a_{s_1+1} < a_{s_1+2} < \cdots < a_{s_2}, ..., a_{s_{k-1}+1} < \cdots < a_{s_k}$ and can therefore be chosen in $\binom{s_k}{s_1, s_2 - s_1, ..., s_k - s_{k-1}}$ ways. From this the proof follows. ∎

8.3. Theorem. *Let P and G be as above, and let $S \subset [n-1]$. Set*

$$\delta_i = \sum_{\substack{T \subset [n-1] \\ |T| = i}} \beta_T^P.$$

Then

$$\beta_S^{\hat{\Delta}(P)} = \sum_{i=0}^{n-1} H_i(S)\, \delta_i, \tag{22}$$

where $H_i(S)$ is defined in Lemma 8.2.

Proof. Let ζ_S denote the right-hand side of (22). By Theorem 1.1, it suffices to show that

$$\sum_{T \subset S} \zeta_T = \alpha_S^{\hat{\Delta}(P)},$$

where $\alpha_S^{\hat{\Delta}(P)}$ is given by Proposition 8.1. Let $S = \{s_1, ..., s_k\}_<$. Now

$$\begin{aligned}
\sum_{T \subset S} \zeta_T &= \sum_{T \subset S} \sum_{i=0}^{n-1} H_i(T) \sum_{|V|=i} \beta_V^P \\
&= \sum_{T \subset S} \sum_i H_i(T) \sum_{|V|=i} \sum_{W \subset V} (-1)^{|V-W|} \alpha_W^P \\
&= \sum_W \alpha_W^P \left(\sum_{V \supset W} (-1)^{|V-W|} \sum_{T \subset S} H_{|V|}(T) \right) \\
&= \sum_{\substack{W \\ (|W|=w)}} \alpha_W^P \sum_{i \geqslant w} (-1)^{i-w} \binom{n-1-w}{i-w} \sum_{T \subset S} H_i(T) \\
&= \binom{s_k}{s_1, s_2 - s_1, ..., s_k - s_{k-1}} \sum_W \alpha_W^P \sum_{i \geqslant w} \\
&\quad \times (-1)^{i-w} \binom{n-1-w}{i-w} \binom{n-i-1}{n-s_k-1},
\end{aligned}$$

the latter step by the previous lemma. A well-known combinatorial identity is equivalent to

$$\begin{aligned}
\sum_{i \geqslant w} (-1)^{i-w} \binom{n-1-w}{i-w} \binom{n-i-1}{n-s_k-1} &= 1, \qquad \text{if} \quad s_k = w \\
&= 0, \qquad \text{if} \quad s_k \neq w.
\end{aligned}$$

(This may be proved, e.g., by multiplying the two power series $(1-x)^{n-w}$ and $x^{w-s_k}(1-x)^{-(n-w)}$.) It follows that

$$\sum_{T \subset S} \zeta_T = \binom{s_k}{s_1, s_2 - s_1, ..., s_k - s_{k-1}} \sum_{|W|=s_k} \alpha_W^P = \alpha_S^{\hat{\Delta}(P)},$$

as was to be proved. ∎

9. Unimodality

A sequence $a_0, a_1, ..., a_n$ of real numbers is called *symmetric* if $a_i = a_{n-i}$, and *unimodal* if $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_j \geqslant a_{j+1} \geqslant \cdots \geqslant a_n$ for some j. Given n, a polynomial $a_0 + a_1 q + \cdots + a_n q^n$ is said to be symmetric or unimodal if the sequence $a_0, a_1, \ldots, a_n$ of coefficients is symmetric or unimodal, respectively. We will show how the theory of groups acting on posets can be used to show that certain sequences are unimodal. In particular, we obtain what seems to be the simplest known proof that the q-binomial coefficient $\left[{n \atop k} \right]$ is unimodal; and we obtain some new examples of unimodal sequences.

Let $\Gamma = (X, Y, E)$ be a finite *bipartite graph*, i.e., X and Y are finite disjoint sets of *vertices*, and $E \subset X \times Y$ is a set of *edges*. The *adjacency matrix* A_Γ of Γ is the matrix whose rows are indexed by X and columns by Y, and whose (v, w)-entry is 1 if $(v, w) \in E$, and 0 if $(v, w) \notin E$. An *automorphism* g of Γ is a bijection $g: X \cup Y \to X \cup Y$ such that $(v, w) \in E \Leftrightarrow (gv, gw) \in E$. (In particular, $gX = X$, $gY = Y$.) If ϕ_1 and ϕ_2 are (complex, linear) representations of a finite group G, write $\phi_1 \leqslant \phi_2$ to mean that the virtual representation $\phi_2 - \phi_1$ is a genuine representation, i.e., if $m_i(\rho)$ is the multiplicity of the irreducible representation ρ in ϕ_i, then $m_1(\rho) \leqslant m_2(\rho)$ for all ρ.

The following lemma is well-known (e.g., [9, p. 22]), but since it is usually stated in a more general form we will include a proof.

9.1. LEMMA. *Let Γ be a bipartite graph as above, and let G be a group of automorphisms of Γ. Let ϕ_X and ϕ_Y denote the permutation representations of G acting on X and Y, respectively. If* rank $A_\Gamma = |X|$, *then* $\phi_X \leqslant \phi_Y$.

Proof. Let $\mathbb{C}X$ and $\mathbb{C}Y$ denote the complex vector spaces with bases X and Y, respectively. Define a linear transformation $\phi: \mathbb{C}X \to \mathbb{C}Y$ by setting

$$\phi(v) = \sum_{\substack{w \in Y \\ (v, w) \in E}} w,$$

where $v \in X$. The matrix of ϕ with respect to the bases X and Y is A_Γ, so rank $\phi = |X|$. Hence ϕ is injective. On the other hand, since each $g \in G$ is an automorphism of Γ, it follows that ϕ is a G-homomorphism. From this the proof is immediate. ∎

Now let P be a graded poset of rank n with $\hat{0}$ and $\hat{1}$, and with rank function r. Let $P_i = \{x \in P \mid r(x) = i\}$. Let Γ_i denote the bipartite graph obtained by restricting P to $P_i \cup P_{i+1}$, $0 \leqslant i < n$. If rank $A_{\Gamma_i} = \min\{|P_i|, |P_{i+1}|\}$, for all $0 \leqslant i < n$, then we say that P is *ample*. If the sequence $|P_0|, |P_1|, \ldots, |P_n|$ is unimodal, then we say that P is *unimodal*.

9.2. THEOREM. *Let P (as above) be unimodal and ample. Suppose that $G \subset$ Aut P and that ρ is an irreducible representation of G. Let $m_i(\rho)$ denote the multiplicity of ρ in the action (permutation representation) ϕ_i of G on P_i. Then the sequence $m_0(\rho), m_1(\rho), \ldots, m_n(\rho)$ is unimodal. In particular (taking ρ*

to be trivial), if $m_i = |P_i/G|$, the number of orbits of ϕ_i, then the sequence $m_0, m_1, \ldots, m_n$ is unimodal.

Proof. If $|P_i| \leqslant |P_{i+1}|$, then by the previous lemma $\phi_i \leqslant \phi_{i+1}$ so $m_i(\rho) \leqslant m_{i+1}(\rho)$. All inequalities are reversed when $|P_i| \geqslant |P_{i+1}|$, so the proof follows. ▮

Though Theorem 9.2 is quite elementary and simple to prove, it does lead to some interesting consequences.

9.3. PROPOSITION. *The boolean algebra $\mathscr{B}_n$ is ample.* ▮

The are several elementary proofs of this result in the literature, e.g., [11, Lemma 5.1; 14, p. 13] (where an explicit right inverse to the adjacency matrices A_{Γ_i}, $0 \leqslant i < \frac{1}{2}n$, is constructed), and [18].

9.4. PROPOSITION. *Let G a group of permutations of the n-element set S. Let ρ be an irreducible representation of G, and let $m_i(\rho)$ be the multiplicity of ρ in the action of G on the set $\binom{S}{i} = \{T \subset S : |T| = i\}$. Then the sequence $m_0(\rho), m_1(\rho), \ldots, m_n(\rho)$ is symmetric and unimodal.*

Proof. Unimodality is an immediate consequence of Theorem 9.2, Proposition 9.3, and the fact that $\mathscr{B}_n$ is unimodal. Since the actions of G on $\binom{S}{i}$ and $\binom{S}{n-i}$ are isomorphic, we have $m_i(\rho) = m_{n-i}(\rho)$. ▮

In particular, taking ρ to be trivial in Proposition 9.4 we deduce the following result of White [35, Corollary 7].

9.5. COROLLARY. *Let G and S be as above. If m_i denotes the number $|\binom{S}{i}/G|$ of orbits of G on $\binom{S}{i}$, then the sequence $m_0, m_1, \ldots, m_n$ is symmetric and unimodal.* ▮

Following G. D. James, D. White and others, the wreath product $\mathfrak{S}_l \sim \mathfrak{S}_k$ acts (as its defining representation) on a $k \times l$ rectangle R of squares by permuting the elements of each row independently, and by permuting the k rows among themselves. Thus $\mathfrak{S}_l \sim \mathfrak{S}_k$ acts on the set $\binom{R}{i}$, and each orbit of this action contains a unique Ferrers diagram. (See, e.g., [1, p. 7] for the notion of a Ferrers diagram.) Hence if $p_{kl}(i)$ denotes the number of Ferrers diagrams fitting in a $k \times l$ rectangle, we deduce that the sequence

$$p_{kl}(0), p_{kl}(1), \ldots, p_{kl}(kl)$$

is symmetric and unimodal. It is well known [1, Theorem 3.1] that

$$\sum_i p_{kl}(i)\, q^i = \begin{bmatrix} k+l \\ k \end{bmatrix} = \frac{(1-q^{k+l})(1-q^{k+l-1}) \cdots (1-q^{l+1})}{(1-q^k)(1-q^{k-1}) \cdots (1-q)}, \tag{23}$$

the q-binomial coefficient. Hence we obtain what appears to be the simplest proof to date of the following corollary. See [32, p. 132] for a brief history of this result.

9.6. COROLLARY. *For any integers $k, l \geqslant 0$, the q-binomial coefficient $\left[\begin{smallmatrix} k+l \\ k \end{smallmatrix}\right]$ is symmetric and unimodal (of degree kl).* ∎

9.7. COROLLARY. *For any integers k, $l \geqslant 0$, the polynomial $(1+q)^{kl} - \left[\begin{smallmatrix} k+l \\ k \end{smallmatrix}\right]$ is symmetric and unimodal.*

Proof. Let $G = \mathfrak{S}_l \sim \mathfrak{S}_k$ act on R as above, and let X^* be the set of all inequivalent non-trivial irreducible representations of G. By Proposition 9.4, the sequence $t_0, t_1, \ldots, t_n$, where $t_i = \sum_{\rho \in X^*} m_i(\rho)$, is symmetric and unimodal. Since $\Sigma t_i q^i = (1+q)^{kl} - \left[\begin{smallmatrix} k+l \\ k \end{smallmatrix}\right]$, the proof follows. ∎

We next give a q-analogue of Proposition 9.4. As in Section 5, let $\mathscr{B}_n(q)$ denote the lattice of all subspaces of an n-dimensional vector space $V_n(q)$ over $GF(q)$. The following result is due to Kantor [18].

9.8. PROPOSITION. *The lattice $\mathscr{B}_n(q)$ is ample.* ∎

9.9. COROLLARY. *Let G be a subgroup of $GL_n(q)$, the group of linear automorphisms of $V_n(q)$. Let ρ be an irreducible representation of G, and let $m_i(\rho)$ be the multiplicity of ρ in the natural action ϕ_i of G on the Grassmann variety G_{ni} of all i-dimensional subspaces of $V_n(q)$. Then the sequence $m_0(\rho), m_1(\rho), \ldots, m_n(\rho)$ is symmetric and unimodal.*

Proof. Unimodality follows from Theorem 9.2 and Proposition 9.8. Symmetry follows from the fact that the action of a finite group G on a vector space V over $\mathbb{R}$ is isomorphic to the action of G on the dual space $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. ∎

It seems difficult to find additional interesting examples of ample posets with non-trivial automorphisms. We will conclude with one very specialized result of this nature. Following [33], let $L(m, n)$ denote the poset (actually a distributive lattice) of all Ferrers diagrams fitting in an $m \times n$ rectangle, ordered by inclusion. Equivalently, $L(m, n)$ consists of all integer sequences $\mathbf{a} = (a_1, \ldots, a_m)$ satisfying $0 \leqslant a_1 \leqslant \cdots \leqslant a_m \leqslant n$, ordered component-wise (i.e., $\mathbf{a} \leqslant \mathbf{b}$ if $a_i \leqslant b_i$ for all i).

9.10. PROPOSITION. *$L(m, n)$ is ample.*

Proof. Let A_i denote the adjacency matrix of the bipartite graph Γ_i obtained by restricting $L(m, n)$ to ranks i and $i+1$, for $0 \leqslant i \leqslant mn$. By Pieri's formula in the Schubert calculus, A_i is the matrix of the linear transformation ω: $H^{2i}(G_{m+n,m}; \mathbb{C}) \to H^{2(i+1)}(G_{m+n,m}; \mathbb{C})$, with respect to the basis of Schubert cycles, obtained by multiplication by a hyperplane section ω. By the hard Lefschetz theorem, if $0 \leqslant i < \frac{1}{2}mn$ then the matrix $A_{n-i-1}A_{n-i-2} \cdots A_{i+1}A_i$ is invertible. Hence A_i has full rank, as desired. ∎

For further information on the concepts used in the above proof, see e.g., [33, especially pages 175–176]. A more elementary proof appears in [24].

9.11. Proposition. *Let n be a positive integer. Then the two polynomials*

$$P_1(q) = \begin{bmatrix} 2n \\ n \end{bmatrix} + (1+q)(1+q^3)(1+q^5) \cdots (1+q^{2n-1}),$$

$$P_2(q) = \begin{bmatrix} 2n \\ n \end{bmatrix} - (1+q)(1+q^3)(1+q^5) \cdots (1+q^{2n-1})$$

are symmetric and unimodal.

Proof. The group C_2 of order 2 acts on an $n \times n$ rectangle R by interchanging rows and columns, and thus on $L(n, n)$. Let ε denote the trivial character of C_2 and χ the irreducible character of order 2. An element Y of $L(n, n)$ (i.e., a Ferrers diagram fitting in R) is fixed by C_2 if and only if it is self-conjugate. Hence the number of orbits $m_i(\varepsilon)$ of C_2 acting on $L(n, n)_i$ is given by $\frac{1}{2}(p_{nn}(i) + c_n(i))$, where $p_{nn}(i) = |L(n, n)_i|$ and $c_n(i)$ is the number of self-conjugate Ferrers diagrams in R of cardinality i. By (23),

$$\sum_i p_{nn}(i)\, q^i = \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

On the other hand, it is well-known (e.g., [17, Theorem 347]) that

$$\sum_i c_n(i)\, q^i = (1+q)(1+q^3)(1+q^5) \cdots (1+q^{2n-1}).$$

Since $L(n, n)$ is ample (by Proposition 9.9) and unimodal (since $\left[\begin{smallmatrix} 2n \\ n \end{smallmatrix}\right]$ is a unimodal polynomial, e.g., by Corollary 9.6), it follows from Theorem 9.2 that $\frac{1}{2}P_1(q)$, and therefore $P_1(q)$, is unimodal.

Since $m_i(\varepsilon) + m_i(\chi) = |L(n, n)_i| = p_{nn}(i)$, we have

$$\sum_i m_i(\chi)\, q^i = \tfrac{1}{2}P_2(q).$$

Hence by Theorem 9.2, $P_2(q)$ is also unimodal. The symmetry of $P_1(q)$ and $P_2(q)$ is obvious. ∎

The fact that $P_1(q)$ is unimodal is hardly surprising. In fact, it follows from [23] that for some $k \geqslant 0$ (independent of n), if we set $(1+q)$

$(1+q^3)\cdots(1+q^{2n-1})=\sum_{i=0}^{n^2} c_n(i)\,q^i$, then the sequence $c_n(k), c_n(k+1),\ldots, c_n(n^2-k)$ is unimodal. Quite probably $k=3$ for n large, which should be provable by the methods of [23]. At any rate, since the sum of symmetric unimodal polynomials of the same degree is symmetric and unimodal, we conclude that "most of" $P_1(q)$ is unimodal. On the other hand, there seems to be no simple reason for expecting $P_2(q)$ to be unimodal.

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