SOME ASPECTS OF THE ANALYSIS OF FACTORIAL EXPERIMENTS IN A COMPLETELY RANDOMIZED DESIGN[1]

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1. Introduction. This paper is concerned with some aspects of the statistical analysis of factorial experiments carried out according to a completely randomized design, and is one of the joint portions of an investigation into the role and meaning of linear statistical models in the analysis of randomized experiments.

There are essentially two ways of obtaining the analysis of data obtained in a comparative experiment. One way, which is given in standard texts, is to write down a model of the type

$$y_{ijk\cdots} = \mu + a_i + b_j + \cdots \text{ etc.},$$

where $y_{ijk\cdots}$ is the observation and the terms on the right-hand side are fixed unknown constants or random variables with specified properties. The above equation with a complete statement of all the properties of the quantities contained in it is usually called *the* model for the experiment. The texts and the literature are to the best of our knowledge, with a few exceptions to be mentioned later, bare with regard to how one determines the model, how one answers a question such as "Why not a multiplicative model?" or "Why are the a's fixed and the b's random?" The other way is that practiced intuitively by many experimental statisticians and described most aptly by Fisher ([3], [4], [5], [6]) in which (a) one envisages an analysis of variance of the observations from the point of view of topography, apart from treatment, such as for instance in a field experiment by rows, columns, plots within row-column cells, etc.; (b) one envisages an analysis of variance by treatments; (c) one notes how the treatments have been assigned to the experimental material, such as, for instance, factor $\mathfrak{A}$ to rows; and (d) one therefore sees with which part of the topographical analysis any particular component of the treatment breakdown should be associated.

The second procedure cannot be regarded as fully specified by what is said above. The first procedure can only be regarded as arbitrary unless some logical basis can be given for it. It is to the problem implied in the last sentence which we have addressed our work.

In preparing this paper for publication we have had the benefit of specific

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and general criticism and suggestions from Professor John Tukey, whose assistance and advice it is a pleasure to acknowledge.

2. Relation to other work. The general history of the line of attack is given by Wilk and Kempthorne [15]. Since that time Smith [10], Scheffe [8], and Cornfield and Tukey [1] have worked on the general problems indicated above. Cornfield and Tukey [1] also discuss relations between approaches to the problem.

3. Some fundamental concepts. The concepts implied in the words "treatment" (or "factor level"),[4] "experimental unit" and "true response" enter importantly into the developments in the sequel. We shall attempt to convey the general meaning which these terms have for us.

While recognizing that the term treatment generally (operationally) designates a *category* of entities or operations, we shall use it as synonymous with "ideal treatment" or "typical treatment." An example of treatment as a category is a variety of, say, corn with operational representations as individual seeds, so that the treatment may be thought of as having a nested structure. The conception of a treatment such as "a temperature of 45°C" is often different. Even if temperature control is difficult, so that in an actual trial one uses $(45° + \epsilon)$C with ϵ unknown, one usually feels that it is reasonable to conceive, at least on a macroscopic scale, of a "true or ideal treatment" of 45°C, in the attainment of which we are frustrated by physical difficulties.

In most cases it is useful to introduce explicitly the notion of a "treatment error" which will reflect the difficulty in attaining or reproducing a conceptually meaningful ideal. In this paper we shall take such a view. The case when the treatment should properly be regarded as a member of a well-defined population will be given in a later paper.

A reasonable operational definition of experimental units, though circular to some extent, is "those entities in an experiment to which treatments are assigned at random." It is often possible and useful to think of experimental units as physical entities such as plots of land or individual animals, but in many cases such a view is misleadingly naive. Extensions of the term to include periods of time, states of mind, and other ill-defined complexes of conditions are needed. In an agronomic experiment we would regard the unit not simply as a plot of ground, but rather as the plot plus weather and other conditions not subject to test. In specific instances such a view involving "ultimate identification" of experimental units may be too restrictive and could be meaningfully and usefully relaxed. In the formal developments in the sequel, we shall be operationally deterministic in that we shall regard an experimental unit to be conceptually entirely identified so that a given stimulus would produce a definite response. This should not be construed to mean that every situation must be fitted exactly into such a context for the analysis to be useful.

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[4] In general a "treatment" is partially specified by a "factor level." However, most of our remarks can be read substituting "factor level" for "treatment."

The notion of "true" or "typical" response seems readily meaningful at least superficially, and deeper analysis immediately involves one in philosophical discussion which is unnecessary in most experimental contexts.

As regards "experimental error" it may be useful to distinguish between "physical errors" and "sampling errors"; and in the first category to distinguish the experimenter's concern with "systematic errors" from the statistician's treatment which usually revolves around an assumption of "random errors." Some obvious categories of physical errors, with respect to subjecting a given experimental unit to a given treatment and observing the response, are errors of measurement of the response, errors of treatment application, and, from some points of view, errors dependent on the "physical state" of the experimental unit. As we shall see, certain sampling errors can be controlled, in a statistical sense, by the device of randomization. In the analysis of other errors the statistician and experimenter must rely on judicious assumption based on insight and experience.

4. The experimental situation and design; basic notation. The essence of the completely randomized design is that no attempt is made to structure the experimental units; or from another, more accurate viewpoint, no restrictions are imposed in the random assignment of treatments to available experimental units.

We shall describe in detail a situation in which treatment combinations of interest may be classified according to the "levels" of three "factors." This will provide enough generality to indicate extension of the methods and results. The case of two factors can be obtained formally by considering one factor to have only one level.

The factors (e.g., temperature, varieties, types of acid, etc.) will be denoted by script letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. The number of levels of each factor, in the experimental population, will be denoted by the corresponding capital letters A, B, C. We suppose, for purpose of reference only, that the levels of each factor are ordered (arbitrarily) and let $i = 1, 2, \cdots, A; j = 1, 2, \cdots, B; k = 1, 2, \cdots, C$, denote the various levels in the populations of levels of factors $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, respectively.

Suppose there are P experimental units with respect to which we wish to study comparatively the various treatment combinations. (The details of *what* we may be interested in doing will vary with the specific physical situation, but some general statistical aspects of what the bare situation and design enable us to do remain the same.) Again we suppose, for formal reference, that the units are ordered, and let $m = 1, 2, \cdots, P$ denote the unit in the population of units.

The experimental design is now defined as follows:

(i) Select a levels from A of factor $\mathcal{A}$ at random.

(ii) Select b levels from B of factor $\mathcal{B}$ at random.

(iii) Select c levels from C of factor $\mathcal{C}$ at random.

(We will use the notation $i^* = 1, 2, \cdots, a; j^* = 1, 2, \cdots, b; k^* = 1, 2, \cdots, c$ to denote the randomly selected levels of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ respectively *in order of* *their selection.* Thus, for example, $i^* = 1$ corresponds with probability $1/A$ to *any* designated value of i.)

(iv) Select p experimental units at random from P, where

$$p = \sum_{i^*=1}^{a} \sum_{j^*=1}^{b} \sum_{k^*=1}^{c} n_{i^*j^*k^*}, \qquad \text{and all } n_{i^*j^*k^*} \geqq 1.$$

The values of the $n_{i^*j^*k^*}$ are treated as known prechosen fixed numbers. (Further explanation of this is given at the end of this section.)

(v) Apply selected factor levels to selected experimental units at random but so that selected treatment combination $(i^*j^*k^*)$ appears on $n_{i^*j^*k^*}$ of the selected experimental units.

Some purely formal difficulties can arise in this general exposition for the case of, say, $a = A$. According to our description above, the identification of levels of $\mathcal{A}$ by i^* would be a random arrangement of that effected by i. In dealing with symmetric functions, clearly no difficulties arise. The whole matter can be handled simply by a convention that, for example, when $A = a$ we take i and i^* to be identical indices; or, where non-symmetric functions arise, it can be handled by an extended notation, as will be seen in the sequel.

It is most natural to think of the design as being imposed upon given background populations of levels of factors and of experimental units, but it should be pointed out that it is in fact our procedure in the design which determines the relevant (statistically) population of treatments and units to which our experiment applies. Some further discussion of consequences of this point is given below.

The description above is intended to be general. Cases of fixed, mixed, and random model situations are included as special cases. The possibility of equal, proportional, or unequal numbers in the "subclasses" of the observations is allowed for. In the described set-up the number of observations associated with a treatment combination depends on the actual realization of the experiment, that is, on the outcome of the random selections, and not, in general, on the population of treatments. An important exception to this is the case of fixed factors. Thus, we specify that the *selected* treatment combination $(i^*j^*k^*)$ appear $n_{j^*j^*k^*}$ times; but, in general, the association of $(i^*j^*k^*)$ with values of (ijk) will depend on the random selection process.

5. Some discussion. In the formal description of the experimental situation and design in the preceding section, the role of experimental units in the experimental situation and the relation of the sample of units to the population are specified explicitly.

The population of units defines, in a sense, our experimental milieu or background. Even if all units can be thought of as identical (a rare event) many background influences (not under direct study) are being "held constant." For example, 10 cc samples from a well-mixed, non-volatile solution may well be considered (aside from pipetting errors) essentially identical. But if, in a two-factor experiment, one factor consists of levels of concentration of a reagent and another is time allowed for reaction, then it is part of the relevant background for inference, tied up with the conception of experimental unit, to describe (or at least keep in mind the existence of) such influences as ambient temperature, barometric pressure, type and shape of container, etc. Thus our inferences must always be interpreted with respect to some population of experimental units, even though in specific instances we may be quite certain of the absence of influence of certain aspects of the experimental milieu.

The emphasis on the relation of sample to population is a fundamental contribution of procedures of modern statistical inference toward scientific objectivity. In spite of the wide acceptance which, we believe, the preceding sentence would find there appears to be some tendency among statisticians to think of the population to which *statistical* inferences are to be made to be not that from which the random sample is obtained but rather one which is indicated by their "interest." The key to this difficulty may lie in the failure to recognize any distinctions between "empirical inferences" based on statistical techniques and "scientific inferences" based on theories of mechanism, mechanical analogies, intuition, etc., in *addition* to statistical inferences.

For example, in a two-factor experiment involving specific insecticides tested with respect to a random selection of 15 types of insects from a population of 200 types of insects we would recognize the statistical validity of two viewpoints in evaluating the comparative utility of the insecticides: (i) relative to the entire population of 200 types of insects from which we have a random sample; (ii) relative to the 15 types of insects actually tested (i.e., the ones which appeared in our random selection). There does not appear to be any *general* justification in attempting, on the basis of data relating to 15 non-randomly selected types of insects (as, for example, those prominent in a certain region), to extend the statistical (empirical) inference to some broader, undefined, population of insect types. There can be no question as to the need or importance of making such an extension, but such extension is essentially non-statistical and must be based on subject matter knowledge and intuition.

6. A conceptual framework for analyses; the population model. In the previous sections we have described an experimental situation and procedure which, at least formally (and granted agreement on the meaning and necessary procedure implied by "random"), is non-controversial. We propose now to provide a conceptual framework for the statistical analysis. This will naturally require some assumptions, all of which we will attempt to make elementary, in the sense that their implications are easily appreciated, and explicit.

We postulate the existence of a real (unknown number Y_{ijkm} which represents the "true" (or "typical") response if unit m is subjected to the treatment combination consisting of the ith level of $\mathcal{A}$, jth level of $\mathcal{B}$, and kth level of $\mathcal{C}$; and we take as our immediate framework of statistical concern the conceptual set $\{Y_{ijkm}\}$, and more particularly certain functions defined on the elements of this set. Several presumptions are implicit in the preceding sentences. First, the scale of observation is considered as "given," though our succeeding discussion could proceed equally well in terms of any function of Y. This is *not* to imply that any scale for analysis is as informative as any other. Second, for the quantity Y_{ijkm} to have meaning by itself it is a necessary assumption that the response from given treatment on given unit be dependent only on that treatment and unit alone, and not on the over-all configuration of other treatments and other units; this excludes certain experiments such as those involving competition in animal-feeding trails. Third, we assume that the notion of "true" or "typical" response can be given an objective meaning in the given situation.

Proceeding now on the basis of the previous paragraph, we know that if we actually subjected unit m to factor combination (ijk), we would not in general observe the true response, Y_{ijkm}, owing to inevitable errors in treatment application, in response measurement, and variations for a given unit owing to its "physical state". These types of errors we refer to as technical errors. These technical errors have no relation to the formal randomization procedure but belong to the conceptual framework. Consequently, in a general study of this sort we have three alternatives with respect to technical errors: (i) To deal with the "ideal" case where such errors are not considered, with the understanding that the application of the method and results in specific situations would require some extensions, depending on "reasonable" assumptions in the specific case. (ii) To employ simple assumptions, which are popular, easily understood, and often reasonable, again with the understanding that adjustment may be necessary to meet specific situations. (iii) To attempt to carry technical errors with some sort of "maximum generality." Procedure (ii) appeared to us to be the most useful.

Accordingly, we will assume that if combination (ijk) were applied to unit m, then we would observe

$$y_{ijkm} = Y_{ijkm} + \epsilon_{ijkm},$$

where the ϵ_{ijkm}, representative of combined technical errors, can be treated as random variables which are mutually uncorrelated with mean 0 and common variance σ^2.

Some directions of increasing generality of assumptions would be (i) relaxing the homogeneity assumptions to, say, variance $(\epsilon_{ijkm}) = \sigma_m^2$; (ii) relaxing the homogeneity assumptions to, say, variance $(\epsilon_{ijkm}) = \sigma_i^2$; (iii) y_{ijkm} follows some distribution $F_{ijkm}(y)$ of which Y_{ijkm} is some parameter. It is easy to see that the results we shall give are in fact essentially valid if generalization (i) above is permitted; we have not built it in explicitly to simplify the presentation and lay clear some aspects of the results. Furthermore, the results on ems (expectation of mean squares) are essentially valid if generalization (ii) is allowed.

Anticipating its utility in the succeeding section we can now write down the *population model* as

$$y_{ijkm} = \mu + a_i + b_j + c_k + (ab)_{ij} + (ac)_{ik} + (bc)_{jk} + (abc)_{ijk} + p_m + q_{ijkm} + \epsilon_{ijkm}.$$

No further assumptions are involved in this decomposition, which is based on an algebraic identity involving means and deviations over the array $\{Y_{ijkm}\}$. The explicit definitions and physical interpretations of the components of population model are delayed to Section 12 below. We note here that while the detail of the population model depends only on the experimental situation, the specific breakdown which we employ is determined by the design, since it will turn out that certain of the components of the model are estimable.

7. The statistical model; the function of randomization. We turn now to a consideration of the actual experimental observables. Let $x_{i^*j^*k^*f}$ denote the fth replicate observation obtained from selected factor combination $(i^*j^*k^*)$, where $f = 1, 2, \cdots, n_{i^*j^*k^*}$. Since $x_{i^*j^*k^*f}$ is obtained from some one experimental unit, each $(i^*j^*k^*f)$ corresponds to some value of m, the experimental unit index. Against the background of the previous section we may regard the statistical effect of our experiment as giving a random (within the well-defined restrictions of the experimental design) sample, the $\{x_{i^*j^*k^*f}\}$, from the set of random variables $\{y_{ijkm}\}$; i.e., a restricted random sample of size $\sum_{i^*j^*k^*} n_{i^*j^*k^*}$ from the $ABCP$ populations specified by the random variables $\{y_{ijkm}\}$.

It is appropriate to discuss here the function of randomization in this experimental design. Clearly, if we could observe the entire set $\{Y_{ijkm}\}$, we would know everything (empirically) possible about the experimental situation under consideration. Alternatively, if we could obtain observations on each member of the set $\{y_{ijkm}\}$, then only the technical errors $\{\epsilon_{ijkm}\}$ would be involved in our inferences about functions defined on elements of the set $\{Y_{ijkm}\}$. However, we are in general able to observe only a subset of the $\{y_{ijkm}\}$, and hence our inferences will be influenced by additional variabilities. The function of randomization is to attempt to control, in a statistical sense, these additional variabilities, and to enable us, perhaps, to obtain valid estimates of the uncertainties of inferences.

We incorporate the restrictions of the experimental design with the population model to obtain a statistical model for the observations, $\{x_{i^*j^*k^*f}\}$, in terms of parameters defined on elements of the set $\{Y_{ijkm}\}$ and of random variables which reflect (and define) the restrictions of the design. This statistical model has the advantage that it, together with the properties of its components, summarizes sufficiently all the relevant statistical knowledge and assumptions for the experiment. In addition certain results on linear estimation, variances of estimates, and expectations of analysis of variance mean squares may be derived by elementary algebraic operations using the statistical models. Furthermore there would be nothing more difficult than heavy algebra involved in obtaining more complex results, such as variances of mean squares, using the statistical model. It is to be expected, however, that more purely combinatorial arguments will shorten the process with regard to particular attributes (cf. Tukey [11] and Hooke [7]).

Full detail on the necessary additional notation and definitions needed to write down the statistical model is delayed till Section 12. At this point we note that the statistical model takes the form

$$x_{i^*j^*k^*f} = \mu + a^*_{i^*} + b^*_{j^*} + c^*_{k^*} + (ab)^*_{i^*j^*} + (ac)^*_{i^*k^*} + (bc)^*_{j^*k^*} + (abc)^*_{i^*j^*k^*} + p^*_{i^*j^*k^*f} + q^*_{i^*j^*k^*f} + \epsilon^*_{i^*j^*k^*f},$$

where, for example, $a^*_{i^*} = \sum_{i=1}^{A} \alpha_i^{i^*} a_i$, the a_i being parameters from the population model, the $\alpha_i^{i^*}$ being random variables which take on values zero or one with joint probability distribution specified by the experimental design. (In particular, for $a^*_{i^*}$ the relevant item in the design is the random selection of a levels of factor $\mathcal{A}$ from the population of A levels.)

It is apparent from the subscripts in the above model that the last three components are mutually confounded, but their separation in the model is of importance because their statistical properties and experimental content are not alike.

The formal resemblance of the above statistical model (which may be appropriately called a definitional type model) to the usual "assumed linear models said to underly the analysis of variance" will be apparent and is not fortuitous. We note for emphasis that the model above depends only on the assumptions given in Section 6 above and not on any detailed knowledge or assumption concerning the mechanism (behaviour) of the experimental factors or units.

An extension of the application of the statistical model which we shall consider in this paper only very superficially would be to deduce certain elementary properties of the terms on the right-hand side (e.g., means and variances) and employ these with sufficient homogeneity and distributional assumptions to suggest a modified mathematical model which is more tractable from the point of view of "exact" distribution theory (cf. Scheffé [8]).

8. Succeeding sections. We invert the logical order of development by giving, in succeeding sections, results on expectations of analysis of variance mean squares (ems) in advance of definitions, notation, and derivations underlying these results. This is done because many who may be interested in the structure of these results will have much less concern with the detail of their derivation.

In Section 9 we deal with the case of proportional numbers (defined below) and on orthogonal[5] analysis of variance based on weighted cell means; in Section 10 we consider the case of general numbers and a nonorthogonal analysis based on unweighted cell means; Section 11 deals with the special case of one factor. In addition to general formulae for expectations of mean squares, some questions of estimability of components of variation and of "proper error terms" are taken up.

In Section 12 we give details concerning the population and statistical models, explicit definitions of the components of variation, an example of the use of the statistical model in deriving ems,[6] and discussion of various complements such

[5] We use this term to refer to a decomposition in which the individual sums of squares sum to the so-called total sum of squares.

[6] Expectations of mean squares.

as the physical interpretation of the parameters of the population model, relation of non-additivity to scale of observation, etc.

In Section 13 we describe briefly a more symmetric form for the results on ems which makes the extension to four or more factors very simple indeed. (This general pattern has been extended to include other experimental designs and its over-all structure will be described in later communications.)

Section 14 deals illustratively with problems of linear estimations, errors of estimates, and estimation of these errors, using the statistical model for these considerations.

Certain problems connected with the different roles of fixed and random factors and the need for functional structure analysis in the former are discussed by Wilk and Kempthorne [15] and will not be treated here.

9. The case of three factors, proportional numbers, no additivity assumptions. We present in this section results on expectations of analysis of variance mean squares (henceforth referred to as ems) for the experimental situation and design given in Section 4, employing the conceptional framework described in Section 6, under the restriction that the number of observations in the subclasses fulfill the condition that

$$n_{i^*j^*k^*} = ru_{i^*}v_{j^*}w_{k^*},$$

where r is the highest common factor of the $\{n_{i^*j^*k^*}\}$. Such a condition is often known as that of "proportional numbers."

Under these conditions an orthogonal analysis of variance, based on weighted means, exists. A case of "proportional numbers" can arise quite naturally when there are unequal numbers of observations corresponding to only one factor of classification.

The algebraic structure of the mean squares for such an analysis is well known; for example,

$$A^* = \frac{1}{(a-1)} \sum_{i^*j^*k^*f} (x_{i^*\cdots} - x_{\cdots\cdot})^2$$

$$I_{AB}^* = \frac{1}{(a-1)(b-1)} \sum_{i^*j^*k^*f} (x_{i^*j^*\cdot\cdot} - x_{i^*\cdots} - x_{\cdot j^*\cdot\cdot} + x_{\cdots\cdot})^2,$$

where the usual dot convention is used to denote means.

We shall have use for the following notation:

$$U = \sum_{i^*} u_{i^*}; \quad V = \sum_{j^*} v_{j^*}; \quad W = \sum_{k^*} w_{k^*};$$

$$U^* = \sum_{i^*} u_{i^*}^2/U^2; \quad V^* = \sum_{j^*} v_{j^*}^2/V^2; \quad W^* = \sum_{k^*} w_{k^*}^2/W^2.$$

(Note that for the case of equal numbers $U = a$, $V = b$, $W = c$, $U^* = 1/a$, $V^* = 1/b$, $W^* = 1/c$. Of course, in general, $U^* \leqq 1$, $U^* \geqq 1/a$.) Employing this notation, and recalling that f has range $1, 2, \cdots, ru_{i^*}v_{j^*}w_{k^*}$, we obtain

$$A^* = \frac{1}{a-1} rVW \sum_{i*} u_{i*}(x_{i*...} - x_{....})^2,$$

$$I_{AB}^* = \frac{1}{(a-1)(b-1)} rW \sum_{i*j*} u_{i*} v_{j*}(x_{i*j*..} - x_{i*...} - x_{\cdot j*..} + x_{....})^2$$

General results on ems for this analysis are given in Table 1. The definitions of the components of variation[7] which appear in the table are given in Section 12. For all the σ^2's and Q^2's with the exception of σ_q^2 the definition is such that they are a sum of squares of quantities divided by the number of linearly independent relations among these quantities. The subscript notation is intended to be suggestive; for example, σ_a^2 is a measure of dispersion of the population parameters $\{a_i\}$ which are the "main effects" of the levels of $\mathcal{A}$; σ_{ab}^2 reflects the dispersion of the population of interactions of levels of $\mathcal{A}$ with levels of $\mathcal{B}$; Q_{ap}^2 reflects the dispersion of the interactions of levels of A with experimental units; etc. (See Section 12 for further detail.) The definition of σ_q^2 requires a little comment. It is defined as

$$\sigma_q^2 = \frac{1}{ABC(P-1)} \sum_{ijkm} q_{ijkm}^2,$$

while the number of linearly independent relations among the set $\{qijkm\}$ is $(ABC - 1)(P - 1)$. The reason for this definition is partly because σ_q^2 appears in the ems for the residual and partly to simplify the formulae in Table 1. (Later, when we put Table 1 in a more symmetric form in Section 13, this disturbance will be eliminated.) The only distinction between the Q^2's and the σ^2's is that the former all reflect interactions of treatments with experimental units. The distinctive notation was employed to make this readily apparent in the table.

The results of Table 1 indicate that, in general, unbiased estimates of σ_a^2, σ_b^2, σ_c^2, σ_{ab}^2, etc., cannot be obtained from the analysis of variance mean squares if unit-treatment interactions are not negligible.[8] The corresponding statement for the appropriate denominator in a test of significance criterion is complicated by possible ambiguity with respect to the null hypothesis of concern. But it is apparent that in a test of significance concerning, for example, the main effects of levels of $\mathcal{A}$ (see definitions of Section 12), we cannot in general find a "denominator" whose expectation is

$$E\left(A^* - rUVW \frac{(1 - V^*)}{(a-1)} \sigma_a^2\right).$$

The question may arise as to whether it is in fact components such as σ_a^2

[7] We refer to these quantities as "components of variation" rather than as "components of variance" to avoid possible ambiguity, since they are in fact measures of dispersion for the population of quantities on which they are defined, and are *not*, in the usual meaning of the word, variances of random variables.

[8] This "bias" in the analysis of variance is the generalization of a similar result for a simpler situation, given by Wilk [12], [13].

TABLE 1

Expectations of mean squares for orthogonal analysis of variance

Mean squares	Expectation
A^*	$\sigma^2 + \sigma_q^2 + \sigma_p^2 + rUVW \frac{(1-U^*)}{(a-1)} \left\{ \left(V^* - \frac{1}{B}\right)\left(W^* - \frac{1}{C}\right)\left[\sigma_{abc}^2 - \frac{1}{P} Q_{abcp}^2\right] + \left(W^* - \frac{1}{C}\right)\left[\sigma_{ac}^2 - \frac{1}{P} Q_{acp}^2\right] + \left(V^* - \frac{1}{B}\right)\left[\sigma_{ab}^2 - \frac{1}{P} Q_{abp}^2\right] + \left[\sigma_a^2 - \frac{1}{P} Q_{ap}^2\right] \right\}$
B^*	$\sigma^2 + \sigma_q^2 + \sigma_p^2 + rUVW \frac{(1-V^*)}{(b-1)} \left\{ \left(U^* - \frac{1}{A}\right)\left(W^* - \frac{1}{C}\right)\left[\sigma_{abc}^2 - \frac{1}{P} Q_{abcp}^2\right] + \left(W^* - \frac{1}{C}\right)\left[\sigma_{bc}^2 - \frac{1}{P} Q_{bcp}^2\right] + \left(U^* - \frac{1}{A}\right)\left[\sigma_{ab}^2 - \frac{1}{P} Q_{abp}^2\right] + \left[\sigma_b^2 - \frac{1}{P} Q_{bp}^2\right] \right\}$
C^*	$\sigma^2 + \sigma_q^2 + \sigma_p^2 + rUVW \frac{(1-W^*)}{(c-1)} \left\{ \left(U^* - \frac{1}{A}\right)\left(V^* - \frac{1}{B}\right)\left[\sigma_{abc}^2 - \frac{1}{P} Q_{abcp}^2\right] + \left(V^* - \frac{1}{B}\right)\left[\sigma_{bc}^2 - \frac{1}{P} Q_{bcp}^2\right] + \left(U^* - \frac{1}{A}\right)\left[\sigma_{ac}^2 - \frac{1}{P} Q_{acp}^2\right] + \left[\sigma_c^2 - \frac{1}{P} Q_{cp}^2\right] \right\}$
I_{AB}^*	$\sigma^2 + \sigma_q^2 + \sigma_p^2 + rUVW \frac{(1-U^*)(1-V^*)}{(a-1)(b-1)} \cdot \left\{ \left(W^* - \frac{1}{C}\right)\left[\sigma_{abc}^2 - \frac{1}{P} Q_{abcp}^2\right] + \left[\sigma_{ab}^2 - \frac{1}{P} Q_{abp}^2\right] \right\}$
I_{AC}^*	$\sigma^2 + \sigma_q^2 + \sigma_p^2 + rUVW \frac{(1-U^*)(1-W^*)}{(a-1)(c-1)} \cdot \left\{ \left(V^* - \frac{1}{B}\right)\left[\sigma_{abc}^2 - \frac{1}{P} Q_{abcp}^2\right] + \left[\sigma_{ac}^2 - \frac{1}{P} Q_{acp}^2\right] \right\}$
I_{BC}^*	$\sigma^2 + \sigma_q^2 + \sigma_p^2 + rUVM \frac{(1-V^*)(1-W^*)}{(b-1)(c-1)} \cdot \left\{ \left(U^* - \frac{1}{A}\right)\left[\sigma_{abc}^2 - \frac{1}{P} Q_{abcp}^2\right] + \left[\sigma_{bc}^2 - \frac{1}{P} Q_{bcp}^2\right] \right\}$
I_{ABC}^*	$\sigma^2 + \sigma_q^2 + \sigma_p^2 + rUVW \frac{(1-U^*)(1-V^*)(1-W^*)}{(a-1)(b-1)(c-1)} \left[\sigma_{abc}^2 - \frac{1}{P} Q_{abcp}^2\right]$
R^*	$\sigma^2 + \sigma_q^2 + \sigma_p^2$

which are of interest rather than the linear combination $[\sigma_a^2 - (1/P)Q_{ap}^2]$. The answer to this lies in an examination of the quantities $\{a_i\}$ whose dispersion make up σ_a^2. By definition $a_i = Y_{i\ldots} - Y_{\ldots}$ and is thus the deviation of the average "true" response from level i of $\mathcal{A}$, in combination with all levels of all other factors and all experimental units, from the over-all average from all levels of all factors on all units. We refer to a_i as the main effect of the ith level of $\mathcal{A}$. The difference between two such main effects $a_i - a_{i'}$ measures the difference (averaged over all levels of all other factors and all units) between operating at level i of $\mathcal{A}$ and level i' of $\mathcal{A}$. On the other hand the combination

$$[\sigma_a^2 - (1/P)Q_{ap}^2]$$

is not always necessarily positive (though it will be in most cases of practical

TABLE 2

Error terms

Classification	Error terms
$\mathcal{A}$	$V_A = R^* + \frac{(b-1)}{(1-V^*)}\left(V^* - \frac{1}{B}\right)(I_{AB}^* - R^*) + \frac{(c-1)}{(1-W^*)}\left(W^* - \frac{1}{C}\right)(I_{AC}^* - R^*) - \frac{(b-1)(c-1)}{(1-V^*)(1-W^*)}\left(V^* - \frac{1}{B}\right)\left(W^* - \frac{1}{C}\right)(I_{ABC}^* - R^*)$
$\mathcal{B}$	$V_B = R^* + \frac{(a-1)}{(1-U^*)}\left(U^* - \frac{1}{A}\right)(I_{AB}^* - R^*) + \frac{(c-1)}{(1-W^*)}\left(W^* - \frac{1}{C}\right)(I_{BC}^* - R^*) - \frac{(a-1)(c-1)}{(1-U^*)(1-W^*)}\left(U^* - \frac{1}{A}\right)\left(W^* - \frac{1}{C}\right)(I_{ABC}^* - R^*)$
$\mathcal{C}$	$V_C = R^* + \frac{(a-1)}{(1-U^*)}\left(U^* - \frac{1}{A}\right)(I_{AC}^* - R^*) + \frac{(b-1)}{(1-V^*)}\left(V^* - \frac{1}{B}\right)(I_{BC}^* - R^*) - \frac{(a-1)(b-1)}{(1-U^*)(1-V^*)}\left(U^* - \frac{1}{A}\right)\left(V^* - \frac{1}{B}\right)(I_{ABC}^* - R^*)$
$\mathcal{A} \times \mathcal{B}$	$V_{AB} = R^* + \frac{(c-1)}{(1-W^*)}\left(W^* - \frac{1}{C}\right)(I_{ABC}^* - R^*)$
$\mathcal{A} \times \mathcal{C}$	$V_{AC} = R^* + \frac{(b-1)}{(1-V^*)}\left(V^* - \frac{1}{B}\right)(I_{ABC}^* - R^*)$
$\mathcal{B} \times \mathcal{C}$	$V_{BC} = R^* + \frac{(a-1)}{(1-U^*)}\left(U^* - \frac{1}{A}\right)(I_{ABC}^* - R^*)$
$\mathcal{A} \times \mathcal{B} \times \mathcal{C}$	$V_{ABC} = R^*$

interest) and hence is not a measure of dispersion of *any* quantities defined on the basic population of true responses $\{Y_{ijkm}\}$.

Two factors tend to decrease the importance of this "bias" in the analysis of variance due to interactions of treatments with experimental units. First, the quantity confounded with the component of variation of interest enters in the ems with coefficient $1/P$. Thus if P, the number of experimental units is large, then the effect of the confounded term will usually be small. The origin of the confounded term is the negative correlation induced on observed responses for a given treatment combination owing to the random assignment of units from a finite population. As the size of the population of units increases, this correlation goes to zero. Secondly, each Q^2 quantity represents a higher order interaction term than the component with which it is associated, and it is often true that the higher the order of the interaction the smaller it will be. The size of unit treatment interactions depends somewhat independently on two considerations, namely, the scale of measurement of the responses and the heterogeneity of the experimental units. Of course, homogeneous experimental units will mean additivity of units and treatments on any scale.

Under the assumption that all unit-treatment interactions are zero (i.e., that $q_{ijkm} = 0$) so-called proper error terms would exist. Table 2 lists error terms for each classification of the design. The bias in using these error terms when unit-treatment interactions are not negligible is exemplified by

$$[-rUVW(1 - U^*) / (a - 1)(1/P)Q_{ap}^2],$$

which is the bias in using V_A as an error term for $\mathcal{A}$ main effects.

As we shall see in a later section, the device of randomization is fully effective in allowing unbiased linear estimation of treatment effects. But essentially unbiased error terms will be obtainable from the analysis of variance, in general, only when the experimental units are not too heterogeneous or the size, P, of the population of units is large, or the scale is such that units and treatment combinations are additive (in the sense that their interactions on that scale are zero.) There does not appear to be any simple statistical method to overcome this confounding which is due to the "fractional replication" which is imposed by the restriction that each unit can be "used only once."

We close this section with a discussion of three special cases which have been given much attention in the past. For simplicity we reduce our consideration to those involving two factors, $\mathcal{A}$ and $\mathcal{B}$, putting $C = c = 1$, and shall take P as "very large." The cases we detail are the so-called "fixed," "mixed," and "random" model situations. The results on ems are then those of Table 3: $\sigma_0^2 = \sigma_p^2 + \sigma_q^2$.

The following points from Table 3 are worthy of note: If the numbers of observations in each "cell" are equal, then $U^* = 1/a$ and $V^* = 1/b$ and then the component σ_{ab}^2 vanishes from $E(A^*)$ and from $E(B^*)$ for the fixed case; and from $E(B^*)$ in the mixed case, where $\mathcal{A}$ is the "fixed" factor, but not from $E(A^*)$, where $\mathcal{B}$ is the random factor. If the numbers are proportional and not equal

TABLE 3

Ems for special cases of a two-factor experiment

Mean square	1. Fixed: $A = a, B = b$	2. Mixed: $A = a, B \gg b$	3. Random: $A \gg a, B \gg b$
A^*	$\sigma_0^2 + rUV \frac{(1 - U^*)}{(a - 1)} \cdot \left[\left(V^* - \frac{1}{b}\right)\sigma_{ab}^2 + \sigma_a^2\right]$	$\sigma_0^2 + rUV \frac{(1 - U^*)}{(a - 1)} \cdot [V^*\sigma_{ab}^2 + \sigma_a^2]$	Same as 2.
B^*	$\sigma_0^2 + rUV \frac{(1 - V^*)}{(b - 1)} \cdot \left[\left(U^* - \frac{1}{a}\right)\sigma_{ab}^2 + \sigma_b^2\right]$	Same as 1.	$\sigma_0^2 + rUV \frac{(1 - V^*)}{(b - 1)} \cdot [U^*\sigma_{ab}^2 + \sigma_b^2]$
I_{AB}^*	$\sigma_0^2 + rUV \frac{(1 - U^*)}{(a - 1)} \cdot \frac{(1 - V^*)}{(b - 1)} \sigma_{ab}^2$	Same as 1.	Same as 1.
R^*	σ_0^2		

then, even for these special cases, we do not have simple comparability of the orthogonal analysis of variance mean squares, as has been pointed out by Smith [9]. The fact that, for the case of equal numbers in the mixed case, the component due to interaction remains associated with the fixed factor but not with the random factor is due, loosely speaking, to our having information on each observed "random" factor level in combination with every fixed factor level in our population; but for each fixed factor level we have only a random selection from the possible random factor levels. The crucial point is that for the case $\mathcal{A}$ fixed, $\mathcal{B}$ random, σ_a^2 reflects the dispersion of effects of levels of $\mathcal{A}$ averaged over all levels of $\mathcal{B}$ and similarily for σ_b^2; and while every level of $\mathcal{A}$ is used in the experiment, only a sample of levels of $\mathcal{B}$ are studied.

10. The case of three factors, general numbers, no additivity assumptions. In the event that no restrictions are placed on the numbers $n_{i^*j^*k^*}$, except that they be non-zero, an orthogonal analysis of variance, in which the various sums of squares all have a meaningful relationship to the experimental situation, for a multiple factor experiment will not, in general, exist. One can, however, make an analysis of variance based on cell means. The algebraic structure of such an analysis is exemplified as follows: Let A^{**} be the mean square associated with $\mathcal{A}$ main effects in this analysis, and let

$$\bar{x}_{i^*\cdots} = \frac{1}{bc} \sum_{j^*k^*} x_{i^*j^*k^*},$$

and

$$\bar{x}_{\cdots\cdot} = \frac{1}{abc} \sum_{i^*j^*k^*} x_{i^*j^*k^*};$$

then,

$$A^{**} = bc \sum_{i^*} (\bar{x}_{1^*\cdots} - \bar{x}_{\cdots\cdot})^2/(a - 1).$$

The table is completed by a line for residual mean square, R**, which reflects "within cell" deviations and is in fact identical with R^* of Section 9. This analysis is not orthogonal in the sense that the individual sums of squares will not. in general, sum to the so-called total sum of squares,

TABLE 4

Expected mean squares for non-orthogonal analysis of variance

Mean square	Expectation of mean square
A^{**}	$\frac{1}{n^*}(\sigma^2 + \sigma_q^2 + \sigma_p^2) + \frac{(B-b)}{B}\frac{(C-c)}{C}\left[\sigma_{abc}^2 - \frac{1}{P}Q_{abcp}^2\right]$
	$+ \frac{(C-c)}{C}b\left[\sigma_{ac}^2 - \frac{1}{P}Q_{acp}^2\right] + \frac{(B-b)}{B}c\left[\sigma_{ac}^2 - \frac{1}{P}Q_{abp}^2\right] + bc\left[\sigma_a^2 - \frac{1}{P}Q_{ap}^2\right]$
B^{**}	$\frac{1}{n^*}(\sigma^2 + \sigma_q^2 + \sigma_p^2) + \frac{(A-a)}{A}\frac{(C-c)}{C}\left[\sigma_{abc}^2 - \frac{1}{P}Q_{abcp}^2\right]$
	$+ \frac{(C-c)}{C}a\left[\sigma_{bc}^2 - \frac{1}{P}\sigma_{bcp}^2\right] + \frac{(A-a)}{A}c\left[\sigma_{ab}^2 - \frac{1}{P}Q_{abp}^2\right] + ac\left[\sigma_b^2 - \frac{1}{P}Q_{bp}^2\right]$
C^{**}	$\frac{1}{n^*}(\sigma^2 + \sigma_q^2 + \sigma_p^2) + \frac{(A-a)}{A}\frac{(B-b)}{B}\left[\sigma_{abc}^2 - \frac{1}{P}Q_{abcp}^2\right]$
	$+ \frac{(B-b)}{B}a\left[\sigma_{bc}^2 - \frac{1}{P}Q_{bcp}^2\right] + \frac{(A-a)}{A}b\left[\sigma_{ac}^2 - \frac{1}{P}Q_{acp}^2\right] + ab\left[\sigma_c^2 - \frac{1}{P}Q_{cp}^2\right]$
I_{AB}^{*}	$\frac{1}{n^*}(\sigma^2 + \sigma_q^2 + \sigma_p^2) + \frac{(C-c)}{C}\left[\sigma_{abc}^2 - \frac{1}{P}Q_{abcp}^2\right] + c\left[\sigma_{ab}^2 - \frac{1}{P}Q_{abp}^2\right]$
I_{AC}^{**}	$\frac{1}{n^*}(\sigma^2 + \sigma_q^2 + \sigma_p^2) + \frac{(B-b)}{B}\left[\sigma_{abc}^2 - \frac{1}{P}Q_{abcp}^2\right] + b\left[\sigma_{ac}^2 - \frac{1}{P}Q_{acp}^2\right]$
I_{BC}^{**}	$\frac{1}{n^*}(\sigma^2 + \sigma_q^2 + \sigma_p^2) + \frac{(A-a)}{A}\left[\sigma_{abc}^2 - \frac{1}{P}Q_{abcp}^2\right] + a\left[\sigma_{bc}^2 - \frac{1}{P}Q_{bcp}^2\right]$
I_{ABC}^{**}	$\frac{1}{n^*}(\sigma^2 + \sigma_q^2 + \sigma_p^2) + \left(\sigma_{abc}^2 - \frac{1}{P}Q_{abcp}^2\right)$
R^{**}	$(\sigma^2 + \sigma_q^2 + \sigma_p^2)$

$$\sum_{i^*j^*k^*f} (x_{*i*j*k*f} - x_{....})^2.$$

The only exception to this last statement (when dealing with two or more factors) is when the numbers $n_{i^*j^*k^*}$ are all equal.

The statistical model appropriate here is identical with that used for Section 9 and is developed in Section 12. Table 4 gives the ems for this analysis, with no additivity assumptions. We employ the notation

$$\frac{1}{n^*} = \frac{1}{abc} \sum_{i^*j^*k^*} \left(\frac{1}{n_{i^*j^*k^*}}\right) = \text{average value of elements of the set} \left\{\frac{1}{n_{i^*j^*k^*}}\right\}.$$

Definitions of components of variation are the same as in Section 9 and are detailed in Section 12.

The advantage attached to this analysis of variance is the simple structure of the expectations of mean squares, as opposed to the very complex relations exhibited in Table 1. In fact, if all mean squares in Table 4, except R^{**}, are adjusted by multiplying by n^* then, speaking rather loosely, this analysis may be superficially interpreted in a similar way to an analysis for a case with equal numbers in the cells. (For equal numbers, n^* becomes simply r.)

The discussion given in the preceding section in connection with difficulties when unit-treatment interactions are not negligible applies also to the non-orthogonal analysis. If unit treatment interactions are negligible, then one can obtain from linear combinations of the mean squares of the non-orthogonal analysis unbiased estimates of the various components of variation of interest. For example, with negligible unit treatment interactions an unbiased estimate of σ^2_{ab} is given by

$$\frac{1}{c}\left[I^{**}_{AB} - \frac{(C - c)}{C}\left(I^{**}_{ABC} - \frac{I}{n^*}R^{**}\right) - \frac{1}{n^*}R^{**}\right].$$

The relation of this to the selection of appropriate "error terms" to serve as denominators in F-type comparisons of mean squares will be apparent. The relation to the estimation of variances of linear estimates is no less immediate and is dealt with explicitly in Section 14.

If one has a situation involving proportional but unequal numbers, the question arises whether one should employ the orthogonal analysis based on weighted means or the non-orthogonal analysis based on unweighted means. In the present state of knowledge it appears to be a matter of taste, convenience, and opinion as to which analysis is more advantageous. (Some recent relevant references on this point are Cox [2] and Tukey [11]).

The non-orthogonal analysis has the advantages of wider generality, easier computations, simpler terms and more direct connection with the estimation of linear contrasts among treatment effects. Furthermore, speaking very loosely, the non-centrality enters into the mean squares of the non-orthogonal analysis in a more easily appreciated and more symmetric fashion than for the orthogonal analysis.

The questions of efficiency of estimation of components of variation and of sensitivity of significance, as regards these two analyses, are still open.

11. One factor, general numbers. The case of two factors may be obtained as a special case of the three-factor development by putting $C = c = 1$. We will not deal with it explicitly. The case of one factor can be obtained by putting $B = b = 1$, in addition. Because of some peculiarities in this situation we give some brief discussion below.

The one factor case corresponds to the within and between analysis of variance and has the associated property that an orthogonal analysis always exists whatever the numbers of observations corresponding to the various levels tested. On the other hand one still has the choice as to whether to analyze weighted or unweighted means of observation corresponding to the levels tested.

The residual mean square is the same for both analyses. For the proportional analysis

$$A^* = \frac{r}{(a-1)} \sum_{i^*} u_{i^*}(x_{i^*}. - x..)^2,$$

where

$$x_{i^*}. = \frac{1}{n_{i^*}} \sum_{f} x_{i^*f} \qquad \text{and} \qquad x.. = \frac{1}{\sum_{i^*} n_{i^*}} \sum_{i^*f} x_{i^*f} \, ;$$

$$E(A^*) = \sigma_0^2 + rU \frac{(1 - U^*)}{(a-1)} \left[\sigma_a^2 - \frac{1}{P} Q_{ap}^2 \right],$$

where $\sigma_0^2 = \sigma^2 + \sigma_p^2 + \sigma_q^2$, $n_{i^*} = ru_{i^*}$, $U = \sum_{i^*} u_{i^*}$, $U^* = \sum_{i^*} u_{i^*}^2 / U^2$. For the non-orthogonal analysis,

$$A^{**} = \frac{1}{(a-1)} \sum_{i^*} (x_{i^*}. - \bar{x}..)^2, \qquad \text{where } \bar{x}.. = \frac{1}{a} \sum_{i^*} x_{i^*}.;$$

$$E(A^{**}) = \frac{1}{n^*} \sigma_0^2 + \left[\sigma_a^2 - \frac{1}{P} Q_{ap}^2 \right], \qquad \text{where } \frac{1}{n^*} = \frac{1}{a} \sum_{i^*} \frac{1}{n_{i^*}}.$$

Thus, in the non-orthogonal analysis of variance equal weight is given to each observed level of the factor. In the case of a single factor there does not appear, offhand at least, to be any basis to suggest that one analysis will be, in general, superior to the other.

12. Derivation of models and ems. Our attention is directed in this section to the following main items: (i) definitions and physical interpretations of the parameters of the population model; (ii) the explicit development of a formal statistical model for the observations; (iii) definitions for the various components of variation; (iv) illustration of the use of the statistical model in the derivation of ems.

In Section 6 we gave a conceptional framework for the analysis of the general three-factor completely randomized experiment. This specified as the background population a set of ABCP (unknown) numbers, $\{Y_{ijkm}\}$, the "true" or "typical" responses. A useful and meaningful representation of these is the one implied by the definitions

$$\begin{aligned}
\mu &= Y_{\cdots}, \\
a_i &= Y_{i\cdots} - \mu, \\
b_j &= Y_{\cdot j\cdot\cdot} - \mu, \\
c_k &= Y_{\cdot\cdot k\cdot} - \mu, \\
(ab)_{ij} &= Y_{ij\cdot\cdot} - Y_{i\cdots} - Y_{\cdot j\cdot\cdot} + \mu, \\
(ac)_{ik} &= Y_{i\cdot k\cdot} - Y_{i\cdots} - Y_{\cdot\cdot k\cdot} + \mu, \\
(bc)_{jk} &= Y_{\cdot jk\cdot} - Y_{\cdot j\cdot\cdot} - Y_{\cdot\cdot k\cdot} + \mu, \\
(abc)_{ijk} &= Y_{ijk\cdot} - Y_{ij\cdot\cdot} - Y_{i\cdot k\cdot} + Y_{i\cdots} - (bc)_{jk}, \\
p_m &= Y_{\cdots m} - \mu, \\
q_{ijkm} &= Y_{ijkm} - Y_{ijk\cdot} - Y_{\cdots m} + \mu.
\end{aligned}$$

It is easy to check that the sum of all the components defined above is identically equal to Y_{ijkm}. We have now

$$(1 + A + B + C + AB + AC + BC + ABC + P + ABCP)$$

quantities, in place of our original $ABCP$, but the following properties indicate the dependencies:

$$0 = \sum_i a_i = \sum_j b_j = \sum_k c_k = \sum_{ij} (ab)_{ij} = \sum_{ik} (ac)_{ik} = \sum_{jk} (bc)_{jk}$$
$$= \sum_{ijk} (abc)_{ijk} = \sum_m p_m = \sum_{ijk} q_{ijkm} = \sum_m q_{ijkm}.$$

These relations follow by definition of the parameters and *not* by assumption.

The quantities defined above can be given physical interpretation. We shall do this for representative cases:

μ is the "true" over-all conceptual response if all treatment combinations were applied to all experimental units.

a_i is the difference between the mean of the "true" responses if all treatments consisting of the ith level of $\mathcal{A}$ in combination with every level of $\mathcal{B}$ and every level of $\mathcal{C}$ were applied to all experimental units, and μ; we refer to a_i as the main effect or simply the effect of the ith level of $\mathcal{A}$. It should be noted that $a_i - a_{i'}$ is the difference between the responses due to level i of $\mathcal{A}$ and level i' of $\mathcal{A}$ averaged over all levels of other factors and all units.

$(ab)_{ij}$ is the difference between the effect of the jth level of $\mathcal{B}$ in combination with the ith level of $\mathcal{A}$ and the main effect of the jth level of $\mathcal{B}$. (The symmetry between $\mathcal{A}$ and $\mathcal{B}$ is obvious from the definition of $(ab)_{ij}$.) We call $(ab)_{ij}$ the interaction effect, or simply the interaction, of the ith level of $\mathcal{A}$ and the jth level of $\mathcal{B}$.

p_m measures the difference between the mean response from all combinations of levels of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ on unit m compared to μ. Thus the p_m measure the (average) variability of units with respect to the treatment combinations. Because of the direction of our interest, we refer to the p_m as the additive unit errors or simply as unit errors.

q_{ijkm} is similarly seen to represent the interaction of treatment combination (ijk) with unit m: and we refer to the q_{ijkm} as interactive unit errors or unit-treatment interactions.

A number of items deserve explicit mention even though some have been indicated in the literature, specifically by Yates [16, 17, 18] and others. (1) The definitions of effects and interactions are relative to a given scale of measurement of response. Transformations of the scale would lead to radically different effects and interactions associated with the treatment population. To speak of the main effect of level 2 of factor $\mathcal{A}$ as being large is meaningless unless a particular scale of response is implicit. Similarly the entire concept of interaction has meaning only relative to a given scale. Two factors can, with no contradiction, have negligible interactions on one scale and large interactions on another scale. (2) For a given scale of measurement of response, the definition (and interpretation) of, say, the effect of the ith level of $\mathcal{A}$ depends not only on all other levels of $\mathcal{A}$ included in the experimental population but also on all levels of all other factors as well as on the relevant population of experimental units. The generalization to other effects and interactions is immediate. It is of interest and importance to note that the difference of the effects, say $a_i - a_i'$, of two levels of $\mathcal{A}$ becomes independent of other levels of $\mathcal{A}$ under consideration but remains entirely dependent on the levels of the *other* factors and the population of experimental units. (3) If we have a scale of observation such that, for instance, all interactions $(ab)_{ij}$ are negligible, or $\mathcal{A}$ and $\mathcal{B}$ are additive, then the difference $a_i - a_i'$ becomes independent of which levels of $\mathcal{B}$ are included in the study. This points up the enormous simplification in the summarization of relevant information and in understanding of the situation which is effected when we can operate on a scale in which interactions may be neglected. (4) If the levels of factor are essentially identical in terms of their influence on response, then, on any scale, interactions with that factor will be negligible. Similarly, if experimental units are fairly homogeneous, then one would expect that, for most scales of observations, the variability of units would be largely described by the unit errors, p_m, and the unit-treatment interactions would be negligible. (5) The "reparametrization" of the population of "true responses" to effects, interactions, and unit errors focuses our attention on summary properties of the experimental situation. This has the advantages that (i) the analysis of variance mean squares are interpretable in terms of these parameters, which have a physical interpretation; (ii) knowledge of certain of the parameters is often essentially the information we desire from the experiment; (iii) by means of the decomposition given by the population model it is often simpler to appreciate and evaluate assumptions which may be implicit or explicit in a particular procedure or inference.

We turn our attention now to the development of a formal usable statistical model, for the actual experimental observations, in terms of the parameters defined above. We recall that our experiment could be regarded as giving us a random (within the restrictions of the design) sample, the $\{x_{i^*j^*k^*f}\}$, of size $\sum_{i^*j^*k^*} n_{i^*j^*k^*}$, from the set of $ABCP$ populations represented by the random variables $\{y_{ijkm}\}$, where $y_{ijkm} = Y_{ijkm} + \epsilon_{ijkm}$. To write an explicit model for the $x_{i^*j^*k^*f}$ it is useful and convenient to employ certain "dummy" random variables which we now proceed to define.

Let $\alpha_i^{i^*} = 1$ if selected level i^* of factor corresponds to level i in the population of levels of $\mathfrak{A}$;

$= 0$ otherwise.

Thus, if the 2nd *selected* level of factor $\mathfrak{A}$ corresponds to the 5th *population* level of factor $\mathfrak{A}$, then $\alpha_5^2 = 1$.

Similarly we define the sets $\{\beta_j^{j^*}\}$ and $\{\gamma_k^{k^*}\}$.

Because of the specification of random selection these quantities are random variables some of whose distributional properties are easily written down. For example; (1) The $\{\alpha_i^{i^*}\}$, $\{\beta_j^{j^*}\}$, $\{\gamma_k^{k^*}\}$ are groupwise statistically independent; (2) Pr $(\alpha_i^{i^*} = 1) = 1/A$; (3) $P(\alpha_i^{i^*}\alpha_{i'}^{i^*} = 0) = 1$, $i \neq i'$; (4) $P(\alpha_i^{i^*}\alpha_{i'}^{i^{*\prime}} = 1) = (1/A(A - 1))$, $i^* \neq i^{*\prime}$, $i \neq i'$; (5) $P(\beta_j^{j^*} = 0) = (B - 1)/B$; etc. We note that the α's, β's, and γ's are associated with the random selection of the factor levels to be tested.

We turn now to the specification of association of *selected* treatment combinations with experimental units. To this end we define

$\delta_m^{i^*j^*k^*f} = 1$ if the fth replicate of selected treatment combination $(i^*j^*k^*)$ is tested on unit m of the population of experimental units;
$= 0$, otherwise.

In view of the random selection of units for test and the randomization of treatment combinations to experimental units, it follows that the $\{\delta_m^{i^*j^*k^*f}\}$ are random variables with the following properties: (1) They are statistically independent of the α's, β's, and γ's defined above; (2) $P(\delta_m^{i^*j^*k^*f} = 1) = 1/P$; (3) $P(\delta_m^{i^*j^*k^*f}\delta_m^{i^{*\prime}j^{*\prime}k^{*\prime}f'} = 0) = 1$, $(i^*j^*k^*f) \neq (i^{*\prime}j^{*\prime}k^{*\prime}f')$; (4) $P(\delta_m^{i^*j^*k^*f}\delta_{m'}^{i^*j^*k^*f} = 0) = 1$, $m \neq m'$; (5) $P(\delta_m^{i^*j^*k^*f}\delta_{m'}^{i^{*\prime}j^{*\prime}k^{*\prime}f'} = 1) = (1/P(P - 1))$, $(i^*j^*k^*f) \neq (i^{*\prime}j^{*\prime}k^{*\prime}f')$, $m \neq m'$; etc.

It is now simple to write an explicit model for the observations $x_{i^*j^*k^*f}$, as follows:

$$x_{i^*j^*k^*f} = \mu + \sum_i \alpha_i^{i^*}a_i + \sum_j \beta_j^{j^*}b_j + \sum_k \gamma_k^{k^*}c_k + \sum_{ij} \alpha_i^{i^*}\beta_j^{j^*}(ab)_{ij}$$

$$+ \sum_{ik} \alpha_i^{i^*}\gamma_k^{k^*}(ac)_{ik} + \sum_{jk} \beta_j^{j^*}\gamma_k^{k^*}(bc)_{jk} + \sum_{ijk} \alpha_i^{i^*}\beta_j^{j^*}\gamma_k^{k^*}(abc)_{ijk}$$

$$+ \sum_m \delta_m^{i^*j^*k^*f}p_m + \sum_{ijkm} \alpha_i^{i^*}\beta_j^{j^*}\gamma_k^{k^*}\delta_m^{i^*j^*k^*f}(q_{ijkm} + \epsilon_{ijkm}).$$

The correspondence of terms between this and what is given in Section 7 will be apparent on inspection.

From the point of view of our development, the random variables in this model are the α's, β's, γ's, and δ's, which take on the values 0 and 1 with probabilities specified by the experimental design and procedure, and the ϵ's. All other quantities are regarded as fixed, unknown parameters defined on the array of "true" responses $\{Y_{ijkm}\}$.

This model, together with the properties of its components, contains all the implications of our procedures of random selection and allocation, as well as all assumptions we have made in the conceptual frame of reference for the analysis. It is in a sense "sufficient" for the general experimental situation and design, together with the additional assumptions which we made explicit in Section 6. This model can therefore be employed quite formally in any statistical manipulation or evaluations of the experiment, without reference to any other features.

The complexity of the model is only in its initial appearance. It is easy to handle in algebraic manipulations and, in particular, makes into an elementary algebraic operation the evaluation of expectations of various functions of the observations.

Toward the end of this section we shall illustrate how the statistical model is employed in evaluation of expectations of analysis of variance mean squares. Before doing this we give the explicit definitions of the components of variation, which have appeared in the ems in previous sections, in terms of the components of the population model. These are as follows:

$$\sigma_a^2 = \frac{1}{A-1}\sum_i a_i^2; \qquad \sigma_b^2 = \frac{1}{B-1}\sum_j b_j^2; \qquad \sigma_c^2 = \frac{1}{C-1}\sum_k c_k^2;$$

$$\sigma_{ab}^2 = \frac{1}{(A-1)(B-1)}\sum_{ij}(ab)_{ij}^2; \qquad \sigma_{ac}^2 = \frac{1}{(A-1)(C-1)}\sum_{ik}(ac)_{ik}^2;$$

$$\sigma_{bc}^2 = \frac{1}{(B-1)(C-1)}\sum_{jk}(bc)_{jk}^2; \qquad \sigma_{abc}^2 = \frac{1}{(A-1)(B-1)(C-1)}\sum_{ijk}(abc)_{ijk}^2;$$

$$\sigma_p^2 = \frac{1}{(P-1)}\sum_m p_m^2; \qquad \sigma_q^2 = \frac{1}{ABC(P-1)}\sum_{ijkm} q_{ijkm}^2; \qquad \sigma^2 = E(\epsilon_{ijkm}^2);$$

$$Q_{ap}^2 = \frac{1}{(A-1)(P-1)}\sum_{im} q_{i\cdot\cdot m}^2; \qquad Q_{bp}^2 = \frac{1}{(B-1)(P-1)}\sum_{jm} q_{\cdot j\cdot m}^2;$$

$$Q_{cp}^2 = \frac{1}{(C-1)(P-1)}\sum_{km} q_{\cdot\cdot km}^2;$$

$$Q_{abp}^2 = \frac{1}{(A-1)(B-1)(P-1)}\sum_{ijm}(q_{ij\cdot m} - q_{i\cdot\cdot m} - q_{\cdot j\cdot m})^2;$$

$$Q_{acp}^2 = \frac{1}{(A-1)(C-1)(P-1)}\sum_{ikm}(q_{i\cdot km} - q_{i\cdot\cdot m} - q_{\cdot\cdot km})^2;$$

$$Q_{bcp}^2 = \frac{1}{(B-1)(C-1)(P-1)} \sum_{ijk} (q_{\cdot jkm} - q_{\cdot j \cdot m} - q_{\cdot \cdot km})^2;$$

$$Q_{abcp}^2 = \frac{1}{(A-1)(B-1)(C-1)(P-1)}$$
$$\sum_{ijkm} (q_{ijkm} - q_{ij \cdot m} - q_{i \cdot km} + q_{i \cdot \cdot m} - q_{\cdot jkm} + q_{\cdot j \cdot m} + q_{\cdot \cdot km})^2.$$

With the exception of σ_q^2 the definition of these components is according to the scheme

$$\frac{\text{(sum of squares of quantities)}}{\text{(no. of quantities} - \text{no. of linear dependencies)}}.$$

While there is no doubt that the σ_a^2, σ_b^2, etc., reflect the variability of the populations $\{a_i\}$, $\{b_j\}$, etc., some further justification for the method of choice of divisors is in order. An important (and perhaps sufficient) justification is that such a method of definition simplifies the appearance of the ems and the variances of certain linear estimates. For further insight we might argue that the measure of dispersion wanted for, say, the $\{a_i\}$ is essentially that for the $\{Y_{i\ldots}\}$, a fundamental measure of the dispersion of which is one of Gini's mean differences, namely, the average of squares of differences between pairs from the population. For the case of the $\{Y_{i\ldots}\}$ this is

$$\begin{aligned} G_a &= \frac{1}{A(A-1)} \sum_{i \neq i'} (Y_{i\ldots} - Y_{i'\ldots})^2 \\ &= \frac{2}{A-1} \sum_i (Y_{i\ldots} - Y_{\ldots.})^2 \\ &= 2\sigma_a^2. \end{aligned}$$

(The factor 2 arises because each pair, in inverted order, appears twice.) The same argument applies to σ_b^2, σ_c^2, and σ_p^2. For the case of a measure of dispersion of, say, the $\{(ab)_{ij}\}$, we might argue that this should reflect the magnitude of interactions in the two-way array $\{Y_{ij\ldots}\}$, a fundamental measure of which is a mean square "double difference"

$$G_{ab} = \frac{1}{AB(A-1)(B-1)} \sum_{\substack{i \neq i' \\ j \neq j'}} [(Y_{ij\ldots} - Y_{ij'\ldots}) - (Y_{i'j\ldots} - Y_{i'j'\ldots})]^2.$$

Now the quantity in square brackets is identical with

$$(ab)_{ij} - (ab)_{ij'} - (ab)_{i'j} + (ab)_{i'j'},$$

and remembering that $\sum_i (ab)_{ij} = \sum_j (ab)_{ij} = 0$, and hence that

$$\sum_{ij} (ab)_{ij}^2 = -\sum_{\substack{i \\ j \neq j'}} (ab)_{ij}(ab)_{ij'} = -\sum_{\substack{i \neq i' \\ j}} (ab)_{ij}(ab)_{i'j} = \sum_{\substack{i \neq i' \\ j \neq j'}} (ab)_{ij}(ab)_{i'j'},$$

it is easy to find that

$$G_{ab} = 4\sigma_{ab}^2 .$$

(Again, the factor 4 arises because essentially the same quantity is permitted to appear four times.) The same argument applies to σ_{ac}^2, σ_{bc}^2, Q_{ap}^2, Q_{bp}^2, and Q_{cp}^2, and can be extended in the obvious way to σ_{abc}^2, Q_{abp}^2, Q_{acp}^2, Q_{bcp}^2, and Q_{abcp}.

The structure of the Q^2 quantities is, from their definition and that of q_{ijkm}, such that they reflect interactions of treatment factors with experimental units. For example

$$Q_{ap}^2 = \frac{1}{(A-1)(B-1)} \sum_{im} (Y_{i..m} - Y_{i...} - Y_{...m} + Y_{....})^2,$$

which reflects the interactions of levels of $\mathfrak{A}$ with experimental units. In view of the role which the unit treatment interaction components of variation play in the ems, it was felt that a distinctive notation for them would be worth while. So far as their formal definitions are concerned there are no distinctions between the σ^2's (except σ_q^2) and the Q^2's.

The essential reasons for the definition of σ_q^2 which was used are that σ_q^2 appears in the expectation of the residual mean square and that such a definition shortens some of the formulae. It is easily checked that

$$\sigma_q^2 = \frac{(A-1)}{A} Q_{ap}^2 + \frac{(B-1)}{B} Q_{bp}^2 + \frac{(C-1)}{C} Q_{cp}^2 + \frac{(A-1)(B-1)}{AB} Q_{abp}^2 + \frac{(A-1)(C-1)}{AC} Q_{acp}^2 + \frac{(B-1)(C-1)}{BC} Q_{bcp}^2 + \frac{(A-1)(B-1)(C-1)}{ABC} Q_{abcp}^2 .$$

We proceed now to show how to use the statistical model in deriving the expectation of the $\mathfrak{A}$ mean square, $A^* = 1/(a-1)A'$, for the case of proportional numbers (Section 9). This will illustrate the basis for the results given in previous sections.

We have

$$A' = \sum_{i^*j^*k^*} n_{i^*j^*k^*}(x_{i^*...} - x_{....})^2 = rVW \sum_{i^*} u_{i^*}(x_{i^*...} - x_{....})^2.$$

The statistical model can now be substituted into this expression, and determining the expectation becomes a purely algebraic operation when one uses freely the fact that the expectation of a sum is the sum of the expectations. Thus,

$$A' = rVW \sum_{i^*} u_{i^*}[a_{i^*}^* - a_{.}^* + (ab)_{i^*.}^* - (ab)_{..}^* + (ac)_{i^*.}^* - (ac)_{..}^* + (abc)_{i^*..}^* - (abc)_{...}^* + p_{i^*...}^* - p_{....}^* + q_{i^*...}^* - q_{....}^* + \epsilon_{i^*...}^* - \epsilon_{....}^*]^2,$$

where

$$a_{i^*}^* = \sum_i \alpha_i^{i^*} a_i ; \qquad a_{.}^* = \frac{1}{U} \sum_{i^*} u_{i^*} a_{i^*}^* ;$$

$$(ab)^*_{i^*\cdot} = \frac{1}{V}\sum_{j^*} v_{j^*}(ab)^*_{i^*j^*} = \frac{1}{V}\sum_{j^*ij} v_{j^*}\alpha_i^{i^*}\beta_j^{j^*}(ab)_{ij};$$

$$(ab)^*_{\cdot\cdot} = \frac{1}{UV}\sum_{i^*j^*} u_{i^*}v_{j^*}(ab)^*_{i^*j^*};$$

$$p^*_{i^*\cdots} = \frac{1}{ru_{i^*}VW}\sum_{\substack{j^*k^*f\\m}} \delta_m^{i^*j^*k^*f}p_m;$$

$$q^*_{i^*\cdots} = \frac{1}{ru_{i^*}VW}\sum_{\substack{j^*k^*f\\ijkm}} \alpha_i^{i^*}\beta_j^{j^*}\gamma_k^{k^*}\delta_m^{i^*j^*k^*f}q_{ijkm};$$

etc.

It is easy to check that unlike terms in the above expression are uncorrelated. For example,

$$E(a^*_{i^*}p^*_{i^*\cdots}) = E(\sum_i \alpha_i^{i^*}a_i)(\sum_{j^*k^*f}\delta_m^{i^*j^*k^*f}p_m) = [\sum_i a_iE(\alpha_i^{i^*})][\sum_{\substack{j^*k^*f\\m}} p_mE(\delta_m^{i^*j^*k^*f})],$$

since the α's and δ's are independent. But $E(\alpha_i^{i^*}) = 1/A$ for all i and i^*;

$$E(\delta_m^{i^*j^*k^*f}) = 1/P,$$

for all m, i^*, j^*, k^*, and f, and $\sum_i a_i = \sum_m p_m = 0$. Similarly, the expectation of all other cross-product terms may be shown to be zero.

(In the event that, for example, $\mathfrak{A}$ is fixed, i.e., $A = a$, one will in general not renumber the levels at random, so that in our notation i^* and i would be the *same* index in making the formal correspondence. As we mentioned elsewhere, for symmetric functions of the observations (in our sample involving all levels of $\mathfrak{A}$) no difficulty arises. In the section on linear estimation which involves non-symmetric functions we shall give an extended notation. For the present, if we use the convention that when $A = a$, i^* and i are the same index, then, for example, $\sum_i \alpha_i^3 a_i = a_3$, since $\alpha_i^i = 1$ with probability 1 and $\alpha_i^{i'} = 0$, $i \neq i'$ with probability 1, using our convention. Then $a^*_{i^*}$ would become a_i, a constant, and since $E(p^*_{i^*\cdots}) = 0$, the above result and its analogues remain true.)

Hence

$$\begin{aligned}E(A') = rVW \textstyle\sum_{i^*} u_{i^*}E\{[a^*_{i^*} - a^*_\cdot]^2 &+ [(ab)^*_{i^*\cdot} - (ab)^*]^2 + [(ac)^*_{i^*\cdot} - (ac)^*_{\cdot\cdot}]^2\\ &+ [(abc)^*_{i^*\cdot\cdot} - (abc)^*_{\cdot\cdot\cdot}]^2 + [p^*_{i^*\cdots} - p^*_{\cdots}]^2\\ &+ [q^*_{i^*\cdots} - q^*_{\cdots}]^2 + [\epsilon^*_{i^*\cdots} - \epsilon^*_{\cdots}]^2\}.\end{aligned}$$

Now,

$$\begin{aligned}\sum_{i^*} u_{i^*}E(a^*_{i^*} - a^*_\cdot)^2 &= E\left\{\sum_{i^*} u_{i^*}a^{*2}_{i^*} - \frac{1}{U}(\sum_{i^*} u_{i^*}a^*_{i^*})^2\right\}\\ &= E\left\{\sum_{i^*} u_{i^*}(\sum_i \alpha_i^{i^*}a_i)^2 - \frac{1}{U}(\sum_{i^*} u_{i^*}\sum_i \alpha_i^{i^*}a_i)^2\right\}\\ &= E\left\{\sum_{i^*j} u_{i^*}\alpha_i^{i^*}a_i^2 - \frac{1}{U}\sum_{i^*i} u_{i^*}^2\alpha_i^{i^*}a_i^2 - \frac{1}{U}\sum_{\substack{i^*\neq i^{*\prime}\\ i\neq i'}} u_{i^*}u_{i^{*\prime}}\alpha_i^{i^*}\alpha_{i'}^{i^{*\prime}}a_ia_{i'}\right\},\end{aligned}$$

where we have used the facts that

$$P(\alpha_i^{i^*}\alpha_{i'}^{i^*} = 0) = 1, \; i \neq i';$$
$$P(\alpha_i^{i^*}\alpha_i^{i^{*\prime}} = 0) = 1, \; i^* \neq i^{*\prime};$$
$$(\alpha_i^{i^*})^2 = \alpha_i^{i^*}.$$

Recalling now that

$$E(\alpha_i^{i^*}) = \frac{1}{A}, \qquad E(\alpha_i^{i^*}\alpha_{i'}^{i^{*\prime}}) = \frac{1}{A(A-1)},$$

and $\sum_{i \neq i'} a_i a_i' = -\sum_i a_i^2$, we obtain

$$\begin{aligned}
(\sum_i a_i^2)&\left(\frac{1}{A}U - \frac{1}{UA}\sum_i u_{i^*}^2 + \frac{1}{UA(A-1)}\sum_{i^* \neq i^{*\prime}} u_{i^*}u_{i^{*\prime}}\right)\\
&= \frac{U}{(A-1)}(\sum_i a_i^2)\left[\frac{(A-1)}{A} - \frac{(A-1)}{A}U^* + \frac{1}{U^2A}(U^2 - \sum_{i^*} u_{i^*}^2)\right]\\
&= U\sigma_a^2\frac{1}{A}(A - 1 - AU^* + U^* + 1 - U^*)\\
&= U(1 - U^*)\sigma_a^2.
\end{aligned}$$

Hence the coefficient of σ_a^2 in the expectation of $A^* = 1/(a-1)A'$, the mean square for $\mathcal{A}$, is

$$rUVW\frac{(1-U^*)}{(a-1)}$$

as given in Table 1. Note that if all $u_{i^*} = 1$, which would be the case if the total number of observations of each observed level i^* of $\mathcal{A}$ is the same, then $U = a$, $U^* = 1/a$, and the coefficient becomes rVW.

As another example we consider

$$\sum_{i^*} u_{i^*}E[p_{i^*\ldots}^* - p_{\ldots\ldots}^*]^2 = E\left\{\sum_{i^*} u_{i^*}\left[\frac{1}{ru_{i^*}VW}\sum_{\substack{j^*k^*f\\m}} \delta p_m - \frac{1}{rUVW}\sum_{\substack{i^*j^*k^*f\\m}} \delta p_m\right]^2\right\},$$

where δ denotes $\delta_m^{i^*j^*k^*f}$

$$= \frac{1}{r^2V^2W^2}E\left\{\sum_{i^*}\frac{1}{u_{i^*}}\left(\sum_{\substack{j^*k^*f\\m}} \delta p_m\right)^2 - \frac{1}{U}\left(\sum_{\substack{i^*j^*k^*f\\m}} \delta p_m\right)^2\right\}.$$

If the expressions in parentheses are expanded, it will be seen that a number of the terms will vanish because of relationships like

$$P(\delta_m^{i^*j^*k^*f}\delta_{m'}^{i^*j^*k^*f} = 0) = 1, \qquad m \neq m';$$

and

$$P(\delta_m^{i^*j^*k^*f}\delta_m^{i^{*\prime}j^{*\prime}k^{*\prime}f'} = 0) = 1, \qquad (i^*j^*k^*f) \neq (i^{*\prime}j^{*\prime}k^{*\prime}f').$$

If we also use the facts that

$$E(\delta_m^{i^*j^*k^*f}) = \frac{1}{P},$$

$$E(\delta^{i^*j^*k^*f}\,\delta_{m'}^{i^{*\prime}j^{*\prime}k^{*\prime}f'}) = \frac{1}{P(P-1)}, \qquad m \neq m',\ (i^*j^*k^*f) \neq (i^{*\prime}j^{*\prime}k^{*\prime}f'),$$

$$\sum_m p_m^2 = -\sum_{m \neq m'} p_m p_{m'},$$

and if we use (j^*k^*f) to denote $\sum_{j^*k^*f}(1)$, etc., then the expectation we seek to evaluate is

$$\frac{\sum_m p_m^2}{r^2V^2W^2}\left\{\sum_{i^*}\frac{1}{u_{i^*}}\left[\frac{1}{P}(j^*k^*f) - \frac{1}{P(P-1)}(j^*k^*f \neq f')\right.\right.$$

$$- \frac{1}{P(P-1)}(j^* \neq j^{*\prime}k^*ff') - \frac{1}{P(P-1)}(j^*k^* \neq k^{*\prime}ff')$$

$$\left. - \frac{1}{P(P-1)}(j^* \neq j^{*\prime}k^* \neq k^{*\prime}ff')\right] - \frac{1}{UP}\left[(i^*j^*k^*f) - \frac{1}{P-1}\right.$$

$$\cdot\{(i^*j^*k^*f \neq f') + (i^*j^* \neq j^{*\prime}k^*ff')$$

$$+ (i^*j^*k^* \neq k^*ff') + (i^*j^* \neq j^{*\prime}k^* \neq k^{*\prime}ff')$$

$$+ (i^* \neq i^{*\prime}j^*k^*ff') + (i^* \neq i^{*\prime}j^* \neq j^{*\prime}k^*ff')$$

$$\left.\left. + (i^* \neq i^{*\prime}j^*k^* \neq k^{*\prime}ff') + (i^* \neq i^{*\prime}j^* \neq j^{*\prime}k^* \neq k^{*\prime}ff')\}\right]\right\}.$$

It remains only to write down the various values of the sums and collect terms. Thus

$$(j^*k^*f) = \sum_{j^*k^*f}(1) = \sum_{j^*k^*} n_{i^*j^*k^*} = ru_{i^*}\sum_{j^*k^*} v_{j^*}w_{k^*} = ru_{i^*}VW;$$

$$(j^*k^*f \neq f') = \sum_{j^*k^*f \neq f'}(1) = \sum_{j^*k^*} n_{i^*j^*k^*}(n_{i^*j^*k^*} - 1)$$

$$= r^2u_{i^*}^2\sum_{j^*} v_{j^*}^2 \sum_{k^*} w_{k^*}^2 - ru_{i^*}VW$$

$$= r^2u_{i^*}^2V^*V^2W^*W^2 - ru_{i^*}VW;$$

$$(j^* \neq j^{*\prime}k^*ff') = \sum_{j^*k^*f}\ \sum_{\substack{j^{*\prime}k^*f' \\ j^* \neq j^{*\prime}}}(1)$$

$$= \sum_{\substack{j^* \neq j^{*\prime} \\ k^*}} n_{i^*j^*k^*}n_{i^*j^{*\prime}k^*} = r^2u_{i^*}^2\sum_{j^* \neq j^{*\prime}} v_{j^*}v_{j^{*\prime}}\sum_{k^*} w_{k^*}^2$$

$$= r^2u_{i^*}^2(V^2 - V^2V^*)W^2W^*;$$

$$(i^*j^*k^*f) = rUVW;$$

$$(i^*j^*k^*f \neq f') = r^2U^2U^*V^2V^*W^2W^* - rUVW;$$

$(i^*j^* \neq j^{*\prime}k^*ff') = r^2U^2U^*(V^2 - V^2V^*)W^2W^*;$

etc. Thus the coefficient of $\sum_m p_m^2$ is

$$\frac{1}{r^2V^2W^2}\frac{1}{P(P-1)}\left\{\sum_{i^*}\frac{1}{u_{i^*}}\,[ru_{i^*}VW(P-1) - (r^2u_{i^*}^2V^*V^2W^*W^2 - ru_{i^*}VW)\right.$$
$$- r^2u_{i^*}^2(V^2 - V^2V^*)W^2W^* - r^2u_{i^*}^2V^2V^*(W^2 - W^2W^*)$$
$$- r^2u_{i^*}^2V^2W^2(1 - V^*)(1 - W^*)]$$
$$- \frac{1}{U}[rUVW(P-1) - (r^2U^2U^*V^2V^*W^2W^* - rUVW)$$
$$- r^2U^2U^*V^2(1 - V^*)W^2W^* - r^2U^2U^*V^2V^*W^2(1 - W^*)$$
$$- r^2U^2U^*W^2(1 - V^*)(1 - W^*) - r^2U^2(1 - U^*)V^2V^*W^2W^*$$
$$- r^2U^2(1 - U^*)V^2(1 - V^*)W^2W^* - r^2U^2(1 - U^*)V^2V^*W^2(1 - W^*)$$
$$\left.- r^2U^2V^2W^2(1 - U^*)(1 - V^*)(1 - W^*)]\right\}$$

$$= \frac{1}{rVW}\frac{1}{P(P-1)}\{\sum_{i^*}[(P-1) + 1 - ru_{i^*}VW\{V^*W^* + (1 - V^*)W^*$$
$$+ V^*(1 - W^*) + (1 - V^*)(1 - W^*)\}]$$
$$- [(P-1) + 1 - rUVW\{U^*V^*W^* + U^*(1 - V^*)W^* + U^*V^*(1 - W^*)$$
$$+ U^*(1 - V^*)(1 - W^*) + (1 - U^*)V^*W^* + (1 - U^*)(1 - V^*)W^*$$
$$+ (1 - U^*)V^*(1 - W^*) + (1 - U^*)(1 - V^*)(1 - W^*)\}]\}$$

$$= \frac{1}{rVW}\frac{1}{P(P-1)}(aP - P) = \frac{(a-1)}{rVW(P-1)}.$$

Thus the coefficient of σ_p^2 in the expectation of A* is

$$rVW\frac{(a-1)}{rVW}\frac{1}{(a-1)} = 1,$$

as given in Table 1.

In a similar way one can complete the evaluation of $E(A')$, proceeding from component to component; and of course the other mean squares may be handled in the same fashion. In view of the symmetry of factors one can write down, at once, $E(B^*)$ and $E(C^*)$ from the results for $E(A^*)$, and likewise for I_{AB}^*, I_{AC}^*, and I_{BC}^*. A check on results is that the expectation of the total sum of squares should equal the sum of the expectations.

The complexity of the formulae and also of the algebra is considerably simplified when the number of observations per cell is a constant, say r.

While the operations with the statistical model may appear tedious, this is more apparent than real, in that with some familiarity with the technique a good deal of the writing can be decreased through short-cut notation and simplification by inspection. Furthermore the operations are quite elementary and mechanical; and in addition to the symmetries we have mentioned, there are others, such as the symmetry with respect to σ_{ab}^2 and σ_{ac}^2 in $E(A^*)$.

13. A more symmetric form for ems; extension of results. The general formulae for ems may be put in a more symmetric form which is simpler in appearance and which makes very simple the extension of the results to four or more factors. The modified form of the results involves certain linear combination of the defined components of variation in terms of which the expectations of mean squares have the *appearance* corresponding to an "all factors random, number of units infinite" situation. This general pattern for ems, involving appropriate and definite rules for forming the linear combinations of components of variation, has been obtained by one or both of the present authors for more complex designs and situations than we have studied in this paper; ramifications will be discussed in later communications.

We shall consider, for definiteness, the results of Table 1 on ems for the case of proportional numbers. These results are given in Table 5 in terms of the following notation:

$$\Sigma_a = \sigma_a^2 - \frac{1}{B}\sigma_{ab}^2 - \frac{1}{C}\sigma_{ac}^2 - \frac{1}{P}Q_{ap}^2 + \frac{1}{BC}\sigma_{abc}^2 + \frac{1}{BP}Q_{abp}^2 + \frac{1}{CP}Q_{acp}^2 - \frac{1}{BCP}Q_{abcp}^2 .$$

Σ_b and Σ_c are defined analogously.

$$\Sigma_{ab} = \sigma_{ab}^2 - \frac{1}{C}\sigma_{abc}^2 - \frac{1}{P}Q_{abp}^2 + \frac{1}{CP}Q_{abcp}^2 .$$

Σ_{ac} and Σ_{bc} are defined analogously.

$$\Sigma_{abc} = \sigma_{abc}^2 - \frac{1}{P}Q_{abcp}^2 .$$

$$\Sigma_p = \sigma_p^2 - \frac{1}{A}Q_{ap}^2 - \frac{1}{B}Q_{bp}^2 - \frac{1}{C}Q_{cp}^2 + \frac{1}{AB}Q_{abp}^2 + \frac{1}{AC}Q_{acp}^2 + \frac{1}{BC}Q_{bcp}^2 - \frac{1}{ABC}Q_{abcp}^2 .$$

$$\Sigma_{ap} = Q_{ap}^2 - \frac{1}{B}Q_{abp}^2 - \frac{1}{C}Q_{acp}^2 + \frac{1}{BC}Q_{abcp}^2 .$$

Σ_{bp} and Σ_{cp} are defined analogously.

$$\Sigma_{abp} = Q_{abp}^2 - \frac{1}{C}Q_{abcp}^2 .$$

Σ_{acp} and Σ_{bcp} are defined analogously.

$$\Sigma_{abcp} = Q_{abcp}^2 .$$

$$\begin{aligned}\Sigma_0 &= \sigma^2 + \Sigma_{abcp} + \Sigma_{bcp} + \Sigma_{acp} + \Sigma_{abp} + \Sigma_{cp} + \Sigma_{bp} + \Sigma_{ap} + \Sigma_p \\ &= \sigma^2 + \sigma_p^2 + \sigma_q^2.\end{aligned}$$

TABLE 5

Symmetric form for the results of Table 1

Mean squares	Expected mean squares
A^*	$rUVW \frac{(1 - U^*)}{(a - 1)} (\Sigma_a + V^*\Sigma_{ab} + W^*\Sigma_{ac} + V^*W^*\Sigma_{abc}) + \Sigma_0$
B^*	$rUVW \frac{(1 - V^*)}{(b - 1)} (\Sigma_b + U^*\Sigma_{ab} + W^*\Sigma_{bc} + U^*W^*\Sigma_{abc}) + \Sigma_0$
C^*	$rUVW \frac{(1 - W^*)}{(c - 1)} (\Sigma_c + U^*\Sigma_{ac} + V^*\Sigma_{bc} + U^*V^*\Sigma_{abc}) + \Sigma_0$
I^*_{AB}	$rUVW \frac{(1 - U^*)(1 - V^*)}{(a - 1)(b - 1)} (\Sigma_{ab} + W^*\Sigma_{abc}) + \Sigma_0$
I^*_{AC}	$rUVW \frac{(1 - U^*)(1 - W^*)}{(a - 1)(c - 1)} (\Sigma_{ac} + V^*\Sigma_{abc}) + \Sigma_0$
I^*_{BC}	$rUVW \frac{(1 - V^*)(1 - W^*)}{(b - 1)(c - 1)} (\Sigma_{bc} + U^*\Sigma_{abc}) + \Sigma_0$
I^*_{ABC}	$rUVW \frac{(1 - U^*)(1 - V^*)(1 - W^*)}{(a - 1)(b - 1)(c - 1)} \Sigma_{abc} + + \Sigma_0$
R^*	Σ_0

An inverse relationship giving the σ^2 and Q^2 quantities explicitly in terms of the Σ's is easily written down.

The form of the results given in Table 5 not only makes entirely clear the pattern for extension to more than three factors but also indicates what are, in general, the estimable quantities in the analysis of variance. It will be evident that an unbiased estimate, based on the analysis of variance mean squares, always exists for each $\sum$ quantity in Table 5. It is of interest that the Σ quantities depend only on the population sizes and not on the sample sizes.

To make explicit the pattern of extensions to more than three factors, we give $E(I^*_{AB})$ when we have four factors $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$. The notation and definitions implicit should be clear. We use X as analogous to U, V, W, and X^* as analogous to U^*, V^*, W^*, with definitions of components of variation as before. Then

$$E(I^*_{AB}) = \Sigma_0 + rUVWX \frac{(1 - U^*)(1 - V^*)}{(a - 1)(c - 1)} \cdot (W^*X^*\Sigma_{abcd} + W^*\Sigma_{abc} + X^*\Sigma_{abd} + \Sigma_{ab}),$$

where

$$\Sigma_{abcd} = Q^2_{abcd} - \frac{1}{P} Q^2_{abcdp},$$

$$\Sigma_{abc} = \sigma_{abc}^2 - \frac{1}{D}\sigma_{abcd}^2 - \frac{1}{P}Q_{abcp}^2 + \frac{1}{DP}Q_{abcdp}^2,$$

$$\Sigma_{ab} = \sigma_{ab}^2 - \frac{1}{C}\sigma_{abc}^2 - \frac{1}{D}\sigma_{abd}^2 - \frac{1}{P}Q_{abp}^2 + \frac{1}{CD}\sigma_{abcd}^2 + \frac{1}{CP}Q_{abcp}^2 + \frac{1}{DP}Q_{abdp}^2 - \frac{1}{CDP}Q_{abcdp}^2,$$

$$\Sigma_0 = \Sigma_p + \Sigma_{ap} + \Sigma_{bp} + \cdots + \Sigma_{abcdp} + \sigma^2,$$

etc.

Further discussion on the extension of results for expected mean squares to other designs, on a more formal (operational) statement of definition of the Σ quantities, and on the formal general reciprocal definition of the σ^2 and Q^2 quantities in terms of the Σ's is deferred to a later publication.

14. Estimation of effects, interactions and errors. In many factorial experiments one of the objectives of the experiment will be the estimation of contrasts, such as $\sum_i k_i a_i$, with $\sum_i k_i = 0$, and in particular differences such as $a_i' - a_i'$; also the uncertainty (usually as measured by variance) of such estimates needs to be estimated. The essential objective of this section is to illustrate briefly the use of the statistical model in such "linear estimation" problems.

To simplify the exposition we shall deal with the case of two factors, which is equivalent, formally, to putting $C = c = 1$ for the situation we developed earlier. We can now drop the subscripts k and k^* and all interactions involving $\mathcal{C}$. The population model becomes

$$y_{ijm} = \mu + a_i + b_j + (ab)_{ij} + p_m + q_{ijm} + \epsilon_{ijm}.$$

The statistical model becomes

$$x_{i^*j^*f} = \mu + \sum_i \alpha_i^{i^*} a_i + \sum_j \beta_j^{j^*} b_j + \sum_{ij} \alpha_i^{i^*}\beta_j^{j^*}(ab)_{ij} + \sum_m \delta_m^{i^*j^*f} p_m + \sum_{ijm} \alpha_i^{i^*}\beta_j^{j^*}\delta^{i^*j^*f}(q_{ijm} + \epsilon_{ijm}).$$

We recall that our experiment involved the random selection of a levels from A of factor $\mathcal{A}$ and b levels from B of $\mathcal{B}$, where $a \leqq A$, $b \leqq B$, and the random allocation of the ab selected treatment combinations to randomly selected experimental units (from a population of size P), so that each *selected* treatment (i^*j^*) appeared $n_{i^*j^*}$ times, $n_{i^*j^*} \geqq 1$.

For the case of $A > a$, $B > b$ the association of $n_{i^*j^*}$ values with population treatment combinations (ij) is a random one. For the case of $\mathcal{A}$ and $\mathcal{B}$ fixed factors one of two situations might exist, namely when i^* and i (and j^* and j) are taken as the same index or when the range of $i^*(j^*)$ is a random permutation of the range of $i(j)$. In the first case we can speak of having n_{ij} observations for treatment combination ij; in the second case we have $(\sum_{i^*j^*} \alpha_i^{i^*}\beta_j^{j^*} n_{i^*j^*})$ observations, a random variable having average value $1/ab\sum_{i^*j^*} n_{i^*j^*}$, for treatment (ij). To bypass this difficulty, we shall consider in this paper the case of equal numbers, i.e., $n_{i^*j^*} = r \geqq 1$, all i^*j^*.

Under this last condition,

$$x_{i^*\cdot\cdot} = \mu + a^*_{i^*} + b^*_{\cdot} + (ab)^*_{i^*\cdot} + p^*_{i^*\cdot\cdot} + q^*_{i^*\cdot\cdot} + \epsilon^*_{i^*\cdot\cdot}$$

where

$$a^*_{i^*} = \sum_i \alpha_i^{i^*} a_i, \qquad b^*_{\cdot} = \frac{1}{b} \sum_{j^*j} \beta_j^{j^*} b_j;$$

$$(ab)^*_{i^*\cdot} = \frac{1}{b} \sum_{j^*ij} \alpha_i^{i^*} \beta_j^{j^*} (ab)_{ij}; \qquad p^*_{i^*\cdot\cdot} = \frac{1}{rb} \sum_{j^*fm} \delta_m^{i^*j^*f} p_m;$$

etc.

With no further restrictions an $a \leqq A$, $b \leqq B$, let us put

$$x^{i\cdot\cdot} = \frac{\sum_{i^*} \alpha_i^{i^*} x_{i^*\cdot\cdot}}{\sum_{i^*} \alpha_i^{i^*}}$$

when the right-hand side is *determinate*. This quantity will be *indeterminate* whenever population level i of $\mathfrak{A}$ is *not* included among those actually selected, for then both numerator and denominator above will be zero. Then, when $x^{i\cdot\cdot}$ exists, the denominator above is 1 and

$$x^{i\cdot\cdot} = \mu + a^i + \sum_{i^*} \alpha_i^{i^*} [b^*_{\cdot} + (ab)^*_{i^*\cdot} + p^*_{i^*\cdot\cdot} + q^*_{i^*\cdot\cdot} + \epsilon^*_{i^*\cdot\cdot}]$$

$$= \mu + a_i + \frac{1}{b} \sum_{j^*j} \beta_j^{j^*} b_j + \frac{1}{b} \sum_{j^*j} \beta_j^{j^*} (ab)_{ij} + \sum_{i^*} \alpha_i^{i^*} [p^*_{i^*\cdot\cdot} + q^*_{i^*} + \epsilon^*_{i^*\cdot\cdot}].$$

It should be noted that this statistical model for $x^{i\cdot\cdot}$ is *conditional* on level i of $\mathfrak{A}$, having been one of the selected a levels of $\mathfrak{A}$; hence, in this expression, we take $P(\alpha_i^{i^*} = 1) = 1/a$, which is the conditional probability that selected level i^* corresponds to population level i, given that i is selected.

In the last expression, all terms after the first two on the right-hand side have expectation zero, whatever the relation of B to b. For example

$$E[\sum_{j^*j} \beta_j^{j^*} (ab)_{ij}] = \frac{1}{B} \sum_{j^*} [\sum_j (ab_{ij}] = 0;$$

$$E[\sum_{i^*} \alpha_i^{i^*} p^*_{i^*\cdot\cdot}] = E\left[\frac{1}{rb} \sum_{i^*} \alpha_i^{i^*} \sum_{j^*fm} \delta_m^{i^*j^*f} p_m\right] = \frac{1}{rbaP} \sum_{i^*j^*f} (\sum_m p_m) = 0;$$

$$E[\sum_{i^*} \alpha_i^{i^*} q^*_{i^*\cdot\cdot}] = E\left[\frac{1}{rb} \sum_{i^*} \alpha_i^{i^*} \sum_{j^*f} \sum_{ijm} \alpha_i^{i^*} \beta_j^{j^*} \delta_m^{i^*j^*f} q_{ijm}\right]$$

$$= E\left[\frac{1}{rb} \sum_{i^*j^*f} \sum_{jm} \alpha_i^{i^*} \beta_j^{j^*} \delta_m^{i^*j^*f} q_{ijm}\right]$$

$$= \frac{1}{rb} \frac{1}{a} \frac{1}{B} \frac{1}{P} \sum_{i^*j^*f} \sum_j (\sum_m q_{ijm}) = 0;$$

etc.

Thus $x^{i\cdot\cdot}$ is an unbiased estimate of

$$\mu + a_i = Y_{i\cdot\cdot},$$

which is the conceptual over-all mean "true" response from all (population) treatment combinations involving level i of $\mathcal{A}$ on all (population) experimental units. Hence, an unbiased estimate of the difference of the main effects of levels i and i' of $\mathcal{A}$, $a_i - a_{i'}$, is given by $(x^{i\cdot\cdot} - x^{i'\cdot\cdot})$, when both of these quantities are determinate, independent of whether either $\mathcal{A}$ or $\mathcal{B}$ is fixed or not, and independent of whether interactions of factors with each other or with units are negligible or not.

It may be appropriate here to emphasize that the difference $(a_i - a_{i'})$ is *independent* of the *other* levels of $\mathcal{A}$ under study but is *very much dependent*, in general, on what population of levels of $\mathcal{B}$ and of experimental units is under consideration. (Note that the preceding sentence refers to population parameters and *not* to sample estimates.)

In considering uncertainties involved in the estimation of $(a_i - a_{i'})$ by $(x^{i\cdot\cdot} - x^{i'\cdot\cdot})$ it is clear from the model for $x^{i\cdot\cdot}$ that the estimate will be affected by the interactions of levels i and i' of $\mathcal{A}$ with levels of $\mathcal{B}$ only if $\mathcal{B}$ is not fixed, for if $\mathcal{B}$ is a fixed factor, then $\sum_{j^*j} \beta_j^{j^*}(ab)_{ij} = \sum_j (ab)_{ij} = 0$, independent of i. On the other hand if $\mathcal{B}$ is *not* a fixed factor, then the term

$$\frac{1}{b}\sum_{j^*j} \beta_j^{j^*}[(ab)_{ij} - (ab)_{i'j}]$$

does not vanish from $(x^{i\cdot\cdot} - x^{i'\cdot\cdot})$.

If factor $\mathcal{B}$ is fixed and, further, unit treatment interactions are negligible, i.e., all $q_{ijm} = 0$, then

$$x^{i\cdot\cdot} - x^{i'\cdot\cdot} = a_i - a_{i'} + \textstyle\sum_{i^*} (\alpha_i^{i^*} - \alpha_{i'}^{i^*})(p_{i^*\cdot\cdot}^* + \epsilon_{i^*\cdot\cdot}^*).$$

The variance of this estimate is

$$E[\sum_{i^*} (\alpha_i^{i^*} - \alpha_{i'}^{i^*})(p_{i^*\cdot\cdot}^* + \epsilon_{i^*\cdot\cdot}^*)]^2 = \frac{2\sigma^2}{rb} + E\left[\sum_{i^*} (\alpha_i^{i^*} - \alpha_{i'}^{i^*})\left(\frac{1}{rb}\sum_{j^*fm} \delta_m^{i^*j^*f} p_m\right)\right]^2$$

$$= \frac{2\sigma^2}{rb} + \frac{1}{r^2b^2} E\,[(\sum_{i^*j^*fm} \alpha_i^{i^*} \delta_m^{i^*j^*f} p_m^2 + \sum_{\substack{i^*j^*f\neq f' \\ m\neq m}} \alpha_i^{i^*} \delta_m^{i^*j^*f} \delta_{m'}^{i^*j^*f'} p_m p_{m'}$$

$$+ \sum_{\substack{i^*j^*\neq j^*ff' \\ m\neq m'}} \alpha_i^{i^*} \delta_m^{i^*j^*f} \delta_m^{i^*j^{*\prime}f'} p_m p_{m'}) + (\text{similar terms with } i' \text{ for } i)$$

$$- 2(\sum_{i^*\neq i^{*\prime}} \sum_{\substack{j^*ff' \\ m\neq m'}} \alpha_i^{i^*} \alpha_{i'}^{i^{*\prime}} \delta_m^{i^*j^*f} \delta_{m'}^{i^{*\prime}j^*f'} p_m p_{m'}$$

$$+ \sum_{\substack{i^*\neq i^{*\prime} \\ j^*\neq j^{*\prime}}} \sum_{\substack{ff' \\ m\neq m'}} \alpha_i^{i^*} \alpha_{i'}^{i^{*\prime}} \delta_m^{i^*j^*f} \delta_{m'}^{i^{*\prime}j^{*\prime}f'} p_m p_{m'})]$$

$$= \frac{2\sigma^2}{rb} + \frac{2}{r^2b^2}\left[\left(\frac{rb}{P} - \frac{r(r-1)b}{P(P-1)} - \frac{r^2b(b-1)}{P(P-1)}\right)\right.$$

$$+ \left(\frac{r^2b}{P(P-1)} + \frac{r^2b(b-1)}{P(P-1)}\right)\Bigg] \sum_m p_m^2$$

$$= \frac{2}{rb}(\sigma^2 + \sigma_p^2).$$

Hence under the conditions that (i) unit-treatment interactions are zero and (ii) $\mathfrak{B}$ is a fixed factor (i.e., $B = b$), the variance of the estimate of the difference of the main effects of two levels of $\mathfrak{A}$ is estimated unbiasedly by $2R^*/rb$, where R^*, is the "residual mean square" in the analysis of variance.

We consider next the variance of the estimate $(x^{i\cdot\cdot} - x^{i'\cdot\cdot})$ without the above restrictions. Then

$$\text{var}\,(x^{i\cdot\cdot} - x^{i'\cdot\cdot}) = \frac{2}{rb}(\sigma^2 + \sigma_p^2)$$

$$+ E\left[\frac{1}{b}\sum_{j^*j} \beta_j^{j^*}((ab)_{ij} - (ab)_{i'j}) + \sum_{i^*}(\alpha_i^{i^*} - \alpha_{i'}^{i^*})q_{i^*\cdot\cdot}^*\right]^2$$

$$= \frac{2}{rb}(\sigma^2 + \sigma_p^2) + \frac{1}{b^2}E\,\{\sum_{j^*j}\beta_j^{j^*}[(ab)_{ij} - (ab)_{i'j}]^2$$

$$+ \sum_{\substack{j^*\neq j^{*\prime}\\ j\neq j'}} \beta_j^{j^*}\beta_{j'}^{j^{*\prime}}[(ab)_{ij} - (ab)_{i'j}][(ab)_{ij'} - (ab)_{i'j'}]\}$$

$$+ \frac{1}{r^2b^2}E[\sum_{i^*}\alpha_i^{i^*}\sum_{\substack{j^*f\\ jm}}\beta_j^{j^*}\delta_m^{i^*j^*f}q_{ijm} - \sum_{i^*}\alpha_{i'}^{i^*}\sum_{\substack{j^*f\\ jm}}\beta_j^{j^*}\delta_m^{i^*j^*f}q_{i'jm}]^2$$

$$= \frac{2}{rb}(\sigma^2 + \sigma_p^2) + \frac{1}{b^2}\left\{\frac{b}{B}\sum_j[(ab)_{ij}^2 + (ab)_{i'j}^2 - 2(ab)_{ij}(ab)_{i'j}]\right.$$

$$\left. - \frac{b(b-1)}{B(B-1)}\sum_j[(ab)_{ij}^2 + (ab)_{i'j}^2 - 2(ab)_{ij}(ab)_{i'j}]\right\}$$

$$+ \frac{1}{r^2b^2}\left\{\left[\frac{rb}{BP} - \frac{r(r-1)b}{P(P-1)B}\right]\sum_{jm} q_{ijm}^2 - \frac{r^2b(b-1)}{P(P-1)B(B-1)}\sum_{\substack{j\neq j'\\ m}} q_{ijm}q_{ij'm}\right.$$

$$+ \left[\frac{rb}{PB} - \frac{r(r-1)b}{P(P-1)B}\right]\sum_{jm} q_{i'jm}^2 - \frac{r^2b(b-1)}{P(P-1)B(B-1)}\sum_{\substack{j\neq j'\\ m}} q_{i'jm}q_{i'j'm}$$

$$\left. + \frac{2r^2b}{P(P-1)B}\sum_{jm} q_{ijm}q_{i'jm} + \frac{2r^2b(b-1)}{P(P-1)B(B-1)}\sum_{\substack{j\neq j'\\ m}} q_{ijm}q_{i'j'm}\right\}$$

$$= \frac{2}{rb}(\sigma^2 + \sigma_p^2) + \frac{1}{b}\,\frac{(B-b)}{B(B-1)}\sum_j[(ab)_{ij} - (ab)_{i'j}]^2$$

$$+ \frac{1}{rb}\,\frac{1}{BP(P-1)}\,[(P-r)\sum_{jm}(q_{ijm}^2 + q_{i'jm}^2) + 2r\sum_{jm} q_{ijm}q_{i'jm}]$$

$$- \frac{(b-1)}{bP(P-1)B(B-1)} \sum_{\substack{j \neq j' \\ m}} (q_{ijm}q_{ij'm} + q_{i'jm}q_{i'j'm} - 2q_{ijm}q_{i'j'm})$$

$$= \frac{2}{rb}(\sigma^2 + \sigma_p^2) + \frac{(B-b)}{bB}\frac{1}{(B-1)}\sum_j [(ab)_{ij} - (ab)_{i'j}]^2$$

$$+ \frac{1}{rb}\frac{1}{B(P-1)}\sum_{jm}(q_{ijm}^2 + q_{i'jm}^2) - \frac{1}{b}\frac{1}{PB(P-1)}\sum_{jm}(q_{ijm} - q_{i'jm})^2$$

$$- \frac{(b-1)}{bP(P-1)B(B-1)}\sum_{\substack{j \neq j' \\ m}}(q_{ijm} - q_{i'jm})(q_{ij'm} - q_{i'j'm}).$$

Before considering the estimation of this variance we shall obtain a useful related quantity, namely, the average variance of estimates such as $(x^{i\cdot\cdot} - x^{i'\cdot\cdot})$

$$\frac{1}{A(A-1)}\sum_{i \neq i'} \operatorname{Var}(x^{i\cdot\cdot} - x^{i'\cdot\cdot})$$

$$= \frac{2}{rb}(\sigma^2 + \sigma_p^2 + \sigma_q^2) + \frac{2}{b}\left(\frac{B-b}{B}\left[\sigma_{ab}^2 - \frac{1}{P}Q_{abp}^2\right] - \frac{2}{P}Q_{ap}^2\right)$$

$$= \frac{2}{rb}[E(A^*) - rb\sigma_a^2].$$

This displays explicitly (what is after all obvious) the relationship of the average variance of estimates of differences and the analysis of variance ems. Clearly this average variance can be estimated unbiasedly only if conditions are such than an "unbiased error term" exists for σ_a^2, which will be true only when unit treatment interactions are negligible or/and the population of experimental units is very large. In general, the estimate of this average variance based on the error term for σ_a^2 given in Table 2 will be positively biased, i.e., the variance will tend to be overestimated. With regard to the component σ_{ab}^2 we note that its importance in this formula for the average variance of estimates of differences of main effects of levels of $\mathcal{B}$ is determined by the relationship of B (the population size of levels of factor $\mathcal{B}$ to b (the sample size for levels of $\mathcal{B}$) and *not* by any considerations concerning A and a.

If the average variance given above was felt not to be adequate as an estimate of the variance of a specific difference $(x^{i\cdot\cdot} - x^{i'\cdot\cdot})$ one could carry out what would amount to an analysis of variance involving only the observations of relevance, namely, those that go into $(x^{i\cdot\cdot} - x^{i'\cdot\cdot})$. Thus if we extend our previous notation to

$$x^{ij*f} = \frac{\sum_{i*} \alpha_i^{i*} x_{i*j*f}}{\sum_{i*} \alpha_i^{i*}}, \qquad \text{and} \qquad x^{ij*\cdot} = \frac{1}{r}\sum_i x^{ij*f},$$

then the sums of squares of this "partial" analysis of variance would be

$$\frac{2}{rb}(x^{i\cdot\cdot} - x^{i'\cdot\cdot})^2; \qquad \frac{r}{2}\sum_{j*}(x^{ij*\cdot} + x^{i'j*\cdot} - x^{i\cdot\cdot} - x^{i'\cdot\cdot})^2;$$

$$\frac{r}{2}\sum_{j*}(x^{ij*\cdot}-x^{i'j*\cdot}-x^{i\cdot\cdot}+x^{i'\cdot\cdot})^2;\ \sum_{j*f}[(x^{ij*f}-x^{ij*\cdot})^2+(x^{i'j*f}-x^{i'j*\cdot})]^2;$$

i.e., sums of squares for level i versus level i' of $\mathcal{A}$; sum of squares for levels of $\mathcal{B}$ averaged over levels i and i' of $\mathcal{A}$; sums of squares for interactions of levels of $\mathcal{B}$ with levels i and i' of $\mathcal{A}$; and residual. Clearly this partial analysis of variance will bear the same relation to the variance of $(x^{i\cdot\cdot} - x^{i'\cdot\cdot})$ as the complete analysis bears to the average variance of differences.

When interactions with experimental units are negligible, the residual mean square from the partial analysis of variance will have the same expectation as that for the complete one.

When $B > b$ and the interactions of levels i and i' of $\mathcal{A}$ with levels of $\mathcal{B}$ may be considerably different from the interactions of other levels of $\mathcal{A}$ with levels of $\mathcal{B}$, it may be worth while carrying out the partial analysis of variance to obtain estimates of the variance of the specific difference.

The preceding discussion can be applied symmetrically to factor $\mathcal{B}$ and be extended to a three or more factor situation. Similarly the statistical model can be employed formally to answer questions involving the estimation of specific interactions, or differences of such, and to find variances of such estimates.

So far as experimental unit variability is concerned, randomization is fully effective in providing unbiased linear estimates and in giving unbiased estimates of the component of variation corresponding to the additive unit errors; but, in general, randomization does not lead to the unbiased estimation of the contribution to variances of linear estimates due to the interactive unit error. It is, however, probably true that in many situations this latter bias will not be important.

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