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# Some Asymptotic Problems in Fully Nonlinear Elliptic Equations and Stochastic Control.

ROBERT JENSEN (*) - PIERRE LOUIS LIONS (**)

## Introduction.

In this paper we consider various penalization problems (or singular perturbation problems) where the penalty is on the dependence of solutions in certain directions. (Our meaning will be clear after examining the examples below.) The effect of such a penalization on the limit problem is to cause the limit solution to be independent of some of the original variables. In this way we obtain various limit problems in reduced dimensions.

We shall consider a few examples of our results to clarify our meaning. Let $\mathcal{O}$ be a bounded regular domain in $\mathbb{R}^n$ and let $\tilde{\mathcal{O}}$ be a bounded regular domain in $\mathbb{R}^m$. We denote $\mathcal{O}\times\tilde{\mathcal{O}}$ by $\mathcal{Q}$, i.e. $\mathcal{Q}\equiv\mathcal{O}\times\tilde{\mathcal{O}}$. In everything that follows, $x$ will denote a generic point in $\mathcal{O}$ and $y$ a generic point in $\tilde{\mathcal{O}}$.

EXAMPLE 1. $u^\varepsilon$ is a solution of:

$$(1)\qquad -\sum_{i,j} a_{ij}(x,y)\frac{\partial^2 u}{\partial x_i\,\partial x_j}+\sum_i b_i(x,y)\frac{\partial u^\varepsilon}{\partial x_i}+c(x,y)u^\varepsilon-\sum_i\frac{\partial^2 u^\varepsilon}{\partial y_i^2}+\frac{1}{\varepsilon}|D_y u^\varepsilon| = f(x,y)\quad \text{in } \mathcal{Q},$$

$$(2)\qquad u^\varepsilon = 0 \quad \text{on } \partial\mathcal{O}\times\bar{\tilde{\mathcal{O}}},$$

$$(3)\qquad \frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \bar{\mathcal{O}}\times\partial\tilde{\mathcal{O}},$$

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and where $n$ denotes the unit outward normal to $\partial\tilde{\mathcal{O}}$ and $D_y \equiv (\partial/\partial y_1, \ldots, \ldots, \partial/\partial y_m)$. We assume the coefficients $a_{ij}(x, y)$, $b_i(x, y)$ and $c(x, y)$ and the data $f(x, y)$ in (1) are smooth functions of $(x, y)$ and in addition we assume $c(x, y)$ is nonnegative and there is a positive constant $c$ such that

$$\sum_{i,j} a_{ij}(x, y)\xi_i\xi_j \geqslant c|\xi|^2 \quad \text{for all } (x, y) \text{ in } \mathcal{Q} \text{ and all } \xi \text{ in } \mathbb{R}^n. \tag{4}$$

*We prove that as $\varepsilon$ goes to zero $u^\varepsilon$ converges to the unique solution, $u(x)$, of the Hamilton-Jacobi-Bellman equation*

$$\begin{cases} \sup\limits_{y\in\tilde{\mathcal{O}}} \Big[ -\sum\limits_{i,j} a_{ij}(x, y) \dfrac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum\limits_i b_i(x, y) \dfrac{\partial u}{\partial x_i}(x) \\ \qquad\qquad\qquad + c(x, y)u(x) - f(x, y)\Big] = 0 \quad \text{in } \mathcal{O}; \\ u = 0 \quad \text{on } \partial\mathcal{O}. \end{cases} \tag{5}$$

As a consequence of this result we may approximate problem (5) (which is fully nonlinear in the sense that it is a second order equation in which the nonlinearity involves the second derivatives) by a simpler problem, namely (1)-(3), where the nonlinearity involves only the first derivatives. In this way we build a *simple approximation of the general Hamilton-Jacobi-Bellman equation* (HJB for short); we believe such an approximation could have useful numerical applications.

HJB equations occur as the optimality equations in the general problem of *optimal continuous control of stochastic differential equations* and are currently used in problems of management, economy and engineering. (See W. H. Fleming and R. Rishel [21] for an exposition of optimal stochastic control and HJB equations; for the most general results concerning the solution of (5) see L. C. Evans and P. L. Lions [19]; P. L. Lions [25], [26], [27], [35] and [32].)

Let us also point out that the asymptotic problem (1)-(3) can itself be interpreted in the light of optimal stochastic control and it is possible to give a *probabilistic proof of the convergence of $u^\varepsilon$ to $u$.*

EXAMPLE 2. $u^\varepsilon$ is a solution of (1), (2) and

$$u^\varepsilon = \psi \quad \text{on } \bar{\mathcal{O}} \times \partial\tilde{\mathcal{O}}, \tag{6}$$

where $\psi(x)$ is a smooth function independent of $y$ which vanishes on $\partial\mathcal{O} \times \partial\tilde{\mathcal{O}}$ and we make the same assumptions as in Example 1.

*We prove that as $\varepsilon$ goes to zero $u^\varepsilon$ converges to the unique solution, $u$, of the obstacle problem for an HJB equation, i.e.*

$$(7)\qquad \begin{cases} \max\left\{\sup\limits_{y\in\tilde{\mathcal{O}}}\left[-\sum\limits_{i,j} a_{ij}(x,y)\dfrac{\partial^2 u}{\partial x_i\,\partial x_j}(x)+\sum\limits_i b_i(x,y)\dfrac{\partial u}{\partial x_i}(x)+c(x,y)u(x)\right.\right. \\ \qquad\qquad\qquad\left.\left. -f(x,y)\right],\ u(x)-\psi(x)\right\}=0 \quad \text{in } \mathcal{O}, \\ u=0 \quad \text{on } \partial\mathcal{O}. \end{cases}$$

This result is, a priori, a bit surprising since a boundary condition like (6) becomes asymptotically the constraint condition

$$u(x)\leqslant\psi(x)\quad \text{in } \mathcal{O}.$$

However, in the light of optimal stochastic control this result can be easily understood. Indeed, we can again give a probabilistic proof of this result. ((7) corresponds to a problem in stochastic control where we combine optimal time problems and optimal continuous control.)

We also wish to point out that it is quite easy to conjecture a false result. Indeed, consider the following formal analysis. As $\varepsilon$ goes to zero the effect of the penalization should imply $u^\varepsilon$ converges to a function $u(x)$. Since $u^\varepsilon(x,y)=\psi(x)$ if $(x,y)\in\mathcal{O}\times\partial\tilde{\mathcal{O}}$ and since $u$ is independent of $y$ it is plausible to guess that $u(x)=\psi(x)$.

The above conjecture, although it appears reasonable, is false because from (1) we deduce that

$$-\sum_{i,j} a_{ij}(x,y)\frac{\partial^2 u^\varepsilon}{\partial x_i\,\partial x_j}(x,y)+\sum_i b_i(x,y)\frac{\partial u^\varepsilon}{\partial x_i}(x,y)+c(x,y)u^\varepsilon(x,y)$$
$$-\sum_i\frac{\partial^2 u^\varepsilon}{\partial y_i^2}(x,y)\leqslant f(x,y)\quad \text{in } \mathcal{O} \text{ for all } y \text{ in } \tilde{\mathcal{O}}.$$

So, by letting $\varepsilon$ go to zero we conclude

$$(8)\qquad -\sum_{i,j} a_{ij}(x,y)\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x)+\sum_i b_i(x,y)\frac{\partial u}{\partial x_i}(x)+c(x,y)u(x)\leqslant f(x,y)$$
$$\text{in } \mathcal{O} \text{ for all } y \text{ in } \tilde{\mathcal{O}}.$$

This inequality is not in general satisfied if $u=\psi$. Therefore, in general $u^\varepsilon$ cannot converge to $\psi$ but rather becomes asymptotically as near as possible to $\psi$, while taking into account the inequalities of (8). This higly imprecise argument leads to (7).

EXAMPLE 3. $u^\varepsilon$ is the solution of:

$$(9) \qquad -\sum_{i,j} a_{ij}(x, y) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j}(x, y) + \sum_i b_i(x, y) \frac{\partial u^\varepsilon}{\partial x_i}(x, y) + c(x, y) u^\varepsilon(x, y) \\ - \frac{\partial^2 u^\varepsilon}{\partial y^2}(x, y) + \beta\left(\frac{1}{\varepsilon} \frac{\partial u^\varepsilon}{\partial y}(x, y)\right) = f(x, y) \quad \text{in } \mathcal{Q},$$

plus boundary conditions (2) and (3) and where $\tilde{\mathcal{O}} = (0, 1)$. We make the same assumptions as in Example 1 and we assume $\beta$ is a strictly increasing function on $\mathbb{R}$ such that $\beta(0) = 0$.

*We prove that as $\varepsilon$ goes to zero $u^\varepsilon$ converges to the solution, $u$, of*

$$(10) \quad \begin{cases} -\sum_{i,j} a_{ij}(x, 0) \dfrac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_i b_i(x, 0) \dfrac{\partial u}{\partial x_i}(x) + c(x, 0) u(x) = f(x, 0) \text{ in } \mathcal{O} \\ u = 0 \quad \text{on } \partial\mathcal{O}. \end{cases}$$

If $\beta(t) = t$ this result is easily interpreted from the stochastic viewpoint. Indeed, in this case (9) becomes a linear second order elliptic equation and the associated diffusion process in the $y$ variable is a reflected diffusion process with a drift intensity of $1/\varepsilon$ and directed towards $y = 0$.

EXAMPLE 4. $u^\varepsilon$ is the solution of:

$$(11) \qquad -\sum_{i,j} a_{ij}(x, y) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j}(x, y) + \sum_i b_i(x, y) \frac{\partial u^\varepsilon}{\partial x_i}(x, y) + c(x, y) u^\varepsilon(x, y) \\ - \sum_i \frac{\partial^2 u^\varepsilon}{\partial y_i^2}(x, y) + \frac{1}{\varepsilon}\left(-\sum_i \frac{\partial^2 u^\varepsilon}{\partial y_i^2}(x, y)\right)^+ = f(x, y) \quad \text{in } \mathcal{Q}$$

plus the boundary conditions (2) and (3); we make the same assumptions as in Example 1.

*We prove that as $\varepsilon$ goes to zero $u^\varepsilon$ converges to the unique solution, $u$, of the HJB equation* (5).

As before, *this result can be interpreted in terms of optimal stochastic control* since (11) itself is a (particular) HJB equation corresponding to a control problem where the intensity of the Brownian motion in the $y$ variables is controlled and can take (« at each time and on each trajectory ») any value between 1 and $1 + 1/\varepsilon$. By (3) we impose Neumann boundary conditions which means that the Brownian motion is reflected at the boundary and so the asymptotic behavior of the solutions $u^\varepsilon$ of (11) is related to some *ergodic phenomena*.

EXAMPLE 5. $u^\varepsilon$ is the solution of:

$$(12) \qquad -\sum_{i} a_{ij}(x)\frac{\partial^2 u^\varepsilon}{\partial x_i\,\partial x_j} + \sum_{i} b_i(x,y)\frac{\partial u^\varepsilon}{\partial x_i}(x,y) + c(x,y)u^\varepsilon(x,y) + \beta\left(\frac{1}{\varepsilon}\left(-\sum_{i}\frac{\partial^2 u^\varepsilon}{\partial y_i^2}(x,y)\right)\right) = f(x,y) \quad \text{in } \mathscr{Q}$$

plus the boundary conditions (2) and (3); where

$$(13) \qquad 0 < \alpha_1 \leqslant \beta'(t) \leqslant \alpha_2 \quad \text{for all } t \text{ in } \mathbb{R},$$

and $\beta(0) = 0$. We also make the same assumptions as in Example 1.

*We prove that as $\varepsilon$ goes to zero $u^\varepsilon$ converges to the unique solution, $u$, of the following nonlinearly averaged equation* (NLAE *for short*)

$$(14) \qquad \begin{cases} \displaystyle\int_{\tilde{\mathcal{O}}} \gamma\left[f(x,y) + \sum_{i,j} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x) - \sum_{i} b_i(x,y)\frac{\partial u}{\partial x_i}(x) - c(x,y)\,u(x)\right] \cdot dy = 0 \quad \text{in } \mathcal{O}, \\ u = 0 \quad \text{on } \partial\mathcal{O}, \end{cases}$$

where $\gamma = \beta^{-1}$.

*We study the general class of these nonlocal problems and we prove that they are well posed under very general assumptions.* When $\beta$ is convex (14) turns out to be the HJB equation and in addition any HJB equation can be approximated by the NLAE appearing above. When $\beta$ is linear (14) reduces to a linear equation with averaged coefficients.

This kind of averaging phenomena appears to be similar to those known in Homogenization Theory (for example A. Bensoussan, J. L. Lions and G. Papanicolaou [3], A. Bensoussan [2] and E. de Giorgi and S. Spagnolo [14]). However, the nonlinear averaging we study is apparently new. This averaging principle is not restricted to second-order elliptic problems and has been used to obtain some new uniqueness results for Navier-Stokes equations (see T. Cafflish and P. L. Lions [8] and C. Foias and P. L. Lions [22]).

We do not yet fully understand this problem from the stochastic point of view and we hope to come back to this point in some future study. This phenomena seems to be a combination of Ergodic Theory and Stochastic Control (or Stochastic Differential Games when $\beta$ is not convex).

We shall not present any more examples although we consider many other problems and variants in this paper. The examples we have presented should give a good general idea of the nature of the problems we shall consider.

The methods we employ are purely analytical and rely heavily on the maximum principle. We do not give any detailed probabilistic proofs but we frequently attempt to explain why the results should hold based on probabilistic considerations (i.e. we sketch probabilistic proofs).

Finally we wish to point out that the problems considered here are vaguely reminiscent of some asymptotic problems arising in elasticity (see P. G. Ciarlet and P. Destuynder [11] and [12], P. G. Ciarlet [10] and P. G. Ciarlet and P. Rabier [13]) and to various singular perturbation problems occuring in deterministic optimal control, e.g. the simplification of large-scale systems, (see J. P. Chow and P. V. Kokotovic [9] and R. E. O'Malley [36]).

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## *A*) FIRST ORDER PENALIZATIONS

In this part of our paper we consider only penalizations of the first $y$ derivatives. These are problems of the sort which include Examples (1)-(3) of the Introduction.

We first introduce some notation which we shall keep throughout this paper (including Part $B$). As in the Introduction $\mathcal{O}$ and $\tilde{\mathcal{O}}$ will denote two bounded regular connected domains in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. We let $a_{ij}(x, y)$, $b_i(x, y)$, $c(x, y)$ and $f(x, y)$ for $1 \leqslant i, j \leqslant n$ be real valued functions on $\mathcal{Q} = \mathcal{O} \times \tilde{\mathcal{O}}$ which satisfy

$$
(15) \quad \begin{cases} a_{ij}(x, y) = a_{ji}(x, y) \text{ and } a_{ij}, b_i, c, f \text{ are in } C^{0,1}(\overline{\mathcal{Q}}) \text{ for } 1 \leqslant i, j \leqslant n, \\ \|D_x^\alpha p(\cdot, y)\|_{L^\infty(\mathcal{Q})} \leqslant C \text{ independent of } y \text{ for } |\alpha| \leqslant 2 \text{ and } p = a_{ij}, b_i, c, \\ \qquad \text{or } f, \ c(x, y) \geqslant 0 \text{ in } \mathcal{Q}\,. \end{cases}
$$

We also assume (4) from the Introduction (uniform ellipticity).

REMARK. In many of the results below we will not need all of the regularity of (15) but for the sake of simplicity we will not consider such generalizations here. Nor will we discuss generalizations to the case of degenerate operators, i.e. when (4) does not hold.

## I. – Penalization of the length of the gradient in $y$.

### 1. *Neumann boundary conditions.*

In this section we consider the problem of Example 1 given by (1)-(3). We first explain why (1)-(3) has a unique solution.

PROPOSITION I.1. *Under assumptions* (4) *and* (15) *there exists a unique solution* $u^\varepsilon \in C^{2,\alpha}(\bar{\mathscr{Q}})$ *for some* $\alpha$ *in* $(0,1)$ *of* (1)-(3).

PROOF. Let $u, v$ be two solutions of (1)-(3) and set $w = u - v$. Then $w$ satisfies the boundary conditions (2) and (3); furthermore

$$-\sum_{i,j} a_{ij}(x,y)\frac{\partial^2 w}{\partial x_i\,\partial x_j}(x,y) + \sum_i b_i(x,y)\frac{\partial w}{\partial x_i}(x,y) + c(x,y)w(x,y) - \sum_i \frac{\partial^2 w}{\partial y_i^2}(x,y) + B(x,y)\cdot D_y w(x,y) = 0$$

where $B \in L^\infty(\mathscr{Q})$ is defined by $1/\varepsilon(|D_y u| - |D_y v|) = B\cdot D_y w$. We see that uniqueness is, therefore, an easy consequence of standard uniqueness results for linear second order elliptic equations.

To prove the existence of $u^\varepsilon$ we start by considering the solution, $\bar{u}$, in $C^2(\bar{\mathscr{Q}})$ of

$$-\sum_{i,j} a_{ij}(x,y)\frac{\partial^2 \bar{u}}{\partial x_i\,\partial x_j}(x,y) + \sum_i b_i(x,y)\frac{\partial \bar{u}}{\partial x_i}(x,y) + c(x,y)\bar{u}(x,y) - \sum_i \frac{\partial^2 \bar{u}}{\partial y_i^2}(x,y) = f(x,y) \quad \text{in } \mathscr{Q}, \tag{16}$$

plus boundary conditions (2) and (3).

Next we consider the solution, $u$, of the HJB equation (5) (see P. L. Lions [25] and L. C. Evans and P. L. Lions [19] for existence results). The function $u$ obviously satisfies the boundary conditions (2) and (3) when thought of as a function of both $x$ and $y$. Thus $\bar{u}$ is a supersolution and $u$ is a subsolution of (1). Then by the same argument used to prove uniqueness we see that

i) $u(x) \leqslant \bar{u}(x,y)$ for all $(x,y)$ in $\mathscr{Q}$;

ii) $u(x) \leqslant u^\varepsilon(x,y) \leqslant \bar{u}(x,y)$ for all $(x,y)$ in $\mathscr{Q}$.

The existence result is completed by applying a result due to H. Amann and M. G. Crandall [1]. □

REMARK I.1. Since we have proved $u(x) \leqslant u^\varepsilon(x, y) \leqslant \bar{u}(x, y)$ this provides an $L^\infty$ bound on $u^\varepsilon$ independent of $\varepsilon$. However, an examination of the proofs in P. L. Lions [25] and L. C. Evans and P. L. Lions [19] shows that we have in fact

$$\|D_x^\alpha u^\varepsilon(\cdot, y)\|_{L^\infty(\mathcal{O})} \leqslant C \quad \text{for } |\alpha| \leqslant 2 \text{ independent of } y \text{ and } \varepsilon .$$

We now state the main result of this section.

THEOREM I.1. *Under assumptions* (4) *and* (15), *as* $\varepsilon$ *goes to zero* $u^\varepsilon$ *converges to the solution,* $u$, *of the HSB equation* (5) *in* $L^p(\mathcal{Q})$ *for any* $p < \infty$ *and a.e.*

Before proceeding to the proof we make several remarks.

REMARK I.2. The proof we give uses the existence of $u$, the solution of (5). It is actually possible to prove Theorem I.1 in such a way that we also construct the solution, $u$, of (5). In this way one avoids introducing the penalized system used by L. C. Evans and A. Friedman [18]. The key step in this process still remains the derivation of a priori estimates as in [19] and [25] and this is independent of the chosen approximation scheme.

PROOF OF THEOREM I.1. We first note that if $\varepsilon < \varepsilon'$ then $u^\varepsilon$ is a subsolution of (1) for $\varepsilon'$. By the proof of Proposition I.1 we conclude

$$u(x) \leqslant u^\varepsilon(x, y) \leqslant u^{\varepsilon'}(x, y) \quad \text{in } \bar{\mathcal{Q}} \text{ if } \varepsilon < \varepsilon' .$$

Thus $u^\varepsilon$ decreases monotonically to some function $\underline{u}(x, y)$. By Remark I.1 since $D_x^\alpha u^\varepsilon$ are bounded independently of $y$ and $\varepsilon$ for $|\alpha| \leqslant 2$ we deduce from the equation

$$|D_y u^\varepsilon| \leqslant \varepsilon \sum_{i,j} \frac{\partial^2 u^\varepsilon}{\partial y_i^2} + C\varepsilon \quad \text{and} \quad D_y u^\varepsilon \to \bar{0} \quad \text{in } (\mathcal{D}'(\mathcal{O}))^m.$$

Therefore, there is a function $v(x) \in W^{2,\infty}(\mathcal{O})$ with $v = 0$ on $\partial\mathcal{O}$ and such that

$$\underline{u}(x, y) = v(x) \quad \text{a.e. in } \mathcal{O} .$$

Furthermore,

$$u^\varepsilon \to v \quad \text{in the weak star topology on } L^\infty(\mathcal{Q}) .$$

From our first inequality we conclude $v \geqslant u$ in $\bar{\mathcal{O}}$. On the other hand from (1) we deduce that

$$-\sum_{i,j} a_{ij}(x, y) \frac{\partial^2 v}{\partial x_i \partial x_j}(x) + \sum_i b_i(x, y) \frac{\partial v}{\partial x_i}(x) + c(x, y) v(x) \leqslant f(x, y) \quad \text{in } \mathcal{D}'(\mathcal{O})$$

for any $y$ in $\tilde{\mathcal{O}}$. This implies (since $v \in W^{2,\infty}(\mathcal{O})$) that

$$\sup_{y\in\tilde{\mathcal{O}}}\left[-\sum_{i,j} a_{ij}(x,y)\frac{\partial^2 v}{\partial x_i\,\partial x_j}(x)+\sum_i b_i(x,y)\frac{\partial v}{\partial x_i}(x)+c(x,y)v(x)-f(x,y)\right]\leqslant 0 \quad \text{a.e. in } \mathcal{O}.$$

By applying the general results of [27] (see also [26], [5] or the proof of uniqueness of solutions of the HJB equation given in [25]) we conclude that $v(x)\leqslant u(x)$ in $\bar{\mathcal{O}}$. This completes the proof of Theorem I.1. □

REMARK I.3. We give a formal proof of Theorem I.1 which we believe clarifies our result. Suppose $u^\varepsilon(x, y)$ converges to $v(x, y)$ in $C^2(\bar{\mathcal{Q}})$. Obviously $v$ is independent of $y$ so $v(x, y) = v(x)$ (because otherwise the penalization term in (1) would become unbounded). We claim that

$$\sup_{y\in\tilde{\mathcal{O}}}\left[-\sum_{i,j} a_{ji}(x,y)\frac{\partial u^\varepsilon}{\partial x_i\,\partial x_j}(x,y)+\sum_i b_i(x,y)\frac{\partial u^\varepsilon}{\partial x_i}(x,y)+c(x,y)u^\varepsilon(x,y)-\sum_i\frac{\partial^2 u^\varepsilon}{\partial y_i^2}(x,y)-f(x,y)\right]=0 \quad \text{in } \mathcal{O}.$$

If the equation above is true then clearly $v = \lim_{\varepsilon\to 0} u^\varepsilon$ also satisfies this equation, i.e. $v$ is the solution of (5).

In order to prove our claim fix $x_0$ in $\mathcal{O}$ and let $y_0 = y(x_0)$ be a maximum point of $u^\varepsilon(x, y)$. If $y_0 \in \tilde{\mathcal{O}}$ then $D_y u^\varepsilon(x_0, y_0) = 0$ while if $y_0 \in \partial\tilde{\mathcal{O}}$ then $D_y u^\varepsilon(x_0, y_0) = 0$ because of (3). We have therefore proved our claim and completed our formal proof.

Let us now consider a probabilistic interpretation of Theorem I.1. We must first describe the *stochastic control problem associated with* (5).

An admissible system, $\mathcal{A}$, consists of:

i) a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a normalized $n$-dimensional Brownian motion $W_t$;

ii) an adapted process $y(t, \omega)$ taking values in $\tilde{\mathcal{O}}$ (sometimes called the control);

iii) a family of solutions, $\xi_x(t)$ for $x \in \bar{\mathcal{O}}$, of the stochastic differential equations:

$$\left\{\begin{aligned} d\xi_x(t) &= \sigma\big(\xi_x(t), y(t)\big)\,dW_t - b\big(\xi_x(t), y(t)\big)\,dt\,,\\ \xi_x(0) &= x \qquad \text{for } x\in\bar{\mathcal{O}}\,,\end{aligned}\right. \tag{17}$$

where $(\sigma_{ij})$ is the positive definite symmetric square root of $(2a_{ij})$.

We set

$$u(x) = \inf_{\mathscr{A}} E\left[\int_0^{\tau_x} f(\xi_x(t), y(t)) \exp\left\{-\int_0^t c(\xi_x(s), y(s))\, ds\right\} dt\right] \tag{18}$$

where $\tau_x$ is the first exit time of the process $\xi_x(t)$ from $\bar{\mathcal{O}}$ (or $\mathcal{O}$). One then verifies that $u(x)$ *is the solution of* (5) (see [26], [35] or [27]).

*Next we consider the stochastic representation of* $u^\varepsilon(x, y)$. In addition to the Brownian motion $W_t$ let $\tilde{W}_t$ be a normalized $m$-dimensional Brownian motion independent of $W_t$. For each $(x, y)$ in $\bar{\mathscr{Q}}$ we consider the stochastic differential equations with reflections

$$\left\{\begin{array}{l} d\xi_{x,y}(t) = \sigma(\xi_{x,y}(t), y_{x'y}(t))\, dW_t - b(\xi_{x,y}(t), y_{x,y}(t))\, dt\,, \\ dy_{x,y}(t) = \sqrt{2}\, d\tilde{W}_t + q(t)\, dt - 1_{\partial\tilde{\mathcal{O}}}(y_{x,y}(t))\, n(y_{x,y}(t))\, dA_t\,, \\ y_{x,y}(t) \in \tilde{\mathcal{O}} \quad \text{for all } t,\ \xi_{x,y}(0) = x \quad \text{and} \quad \mathrm{J}_{x,y}(0) = y\,, \end{array}\right. \tag{17'}$$

where $A_t$ is some increasing adapted continuous process with $A_0 = 0$ and $q(t)$ is any adapted process having values in $B_{1/\varepsilon} = \{|q| \leqslant 1/\varepsilon\}$. (The last term in the second equation in (17′) corresponds to the reflection i.e. the Neumann condition given by (3).)

If we now let $\mathscr{A}'$ be the admissible system generated by (17′) and the other appropriate corrections we can show that

$$u^\varepsilon(x, y) = \inf_{\mathscr{A}} E\left[\int_0^{\tau_{x,y}} f(\xi_{x,y}(t), y_{x,y}(t)) \cdot \exp\left\{-\int_0^t c(\xi_{x,y}(s), y_{x,y}(s))\, ds\right\} dt\right], \tag{18'}$$

where $\tau_{x,y}$ is the first exit time of the process $\xi_{x,y}(t)$ from $\mathcal{O}$. Indeed, (18′) is just an application of Ito's formula (since we know there is a smooth solution of (1)-(3)) and related results may be found in [21].

It is now possible to understand (at least intuitively) the principle underlying Theorem I.1. As $\varepsilon$ goes to zero the class of admissible $q(t)$'s becomes larger and larger. Eventually it becomes « dense » in some sense in the space of all possible bounded adapted processes from $\mathscr{A}$. Since we can approximate any adapted continuous process $y(t)$ by some process $y_{x,y}(t)$ and since adapted continuous processes are enough for (18) *we conclude*

$$u^\varepsilon(x, y) \downarrow u(x) \quad \textit{as } \varepsilon \downarrow 0 \quad \textit{for all } (x, y) \textit{ in } \bar{\mathscr{Q}}.$$

(Since $u \in C(\bar{\mathscr{Q}})$ this proves by Dini's lemma that the convergence is actually uniform in $\bar{\mathscr{Q}}$.)

REMARK I.4. Theorem I.1 is totally independent of the boundary conditions in $x$, i.e. condition (2). Analogous results hold, for example, if (2) is replaced by a Neumann condition or even nonhomogeneous boundary conditions.

REMARK I.5. It would be very interesting to have an estimate on the rate of convergence of $u^\varepsilon$ to $u$ (for example in $C(\bar{\mathscr{Q}})$). We have been able to prove the following such estimate for the special case when $u$ is very smooth and the supremum in (5) is obtained at a unique point, $y$, which depends smoothly on $x$. The estimate is

$$\|u^\varepsilon - u\|_{L^\infty(\mathscr{Q})} \leqslant C\varepsilon \, .$$

We hope to come back to this point in some future study.

2. *Dirichlet boundary conditions.*

In this section we shall consider the problem given in Example 2. Thus for each $\varepsilon > 0$ we let $u^\varepsilon$ be the solution of (1), (2) and (6). We assume some additional regularity on the function, $\psi$, appearing in (6), namely,

$$\psi \in C^{2,\alpha}(\bar{\mathscr{Q}}) \quad \text{for some } \alpha > 0 \text{ and } \psi = 0 \text{ on } \partial\mathscr{O} \, . \tag{19}$$

By arguments similar to those used in the previous section it is possible to prove the existence of a unique solution, $u^\varepsilon$, of (1), (2) and (6). In addition we also establish the estimate (just as in the previous section)

$$\tilde{u}(x) \leqslant u^\varepsilon(x, y) \leqslant \bar{u}(x, y) \quad \text{for all } (x, y) \text{ in } \bar{\mathscr{Q}} \, ,$$

$$\|D_x^\alpha u^\varepsilon\|_{L^\infty(\mathscr{Q})} \leqslant C \quad \text{(independent of } \varepsilon) \text{ for } |\alpha| \leqslant 2 \, ,$$

where $\tilde{u}$ and $u$ are the respective solutions of (7) (see [35] for a solution of this problem) and

$$\tag{16'} \begin{cases} -\sum\limits_{i,j} a_{ij}(x, y) \dfrac{\partial^2 \bar{u}}{\partial x_i \, \partial x_j}(x, y) + \sum\limits_i b_i(x, y) \dfrac{\partial \bar{u}}{\partial x_i}(x, y) + c(x, y)\bar{u}(x, y) \\ \qquad\qquad\qquad\qquad - \sum\limits_i \dfrac{\partial^2 \bar{u}}{\partial y_i^2}(x, y) = f(x, y) \quad \text{in } \mathscr{Q} \\ \bar{u} \in C^2(\bar{\mathscr{Q}}) \quad \text{plus boundary conditions (2) and (6)} \, . \end{cases}$$

We now have the main result of this section.

THEOREM I.2. *Under assumptions* (4) *and* (15) *on the coefficients and under assumption* (19) $u^\varepsilon$ *converges to the solution,* $\tilde{u}$, *of the obstacle problem*

(7) *for the HJB equation in* $L^p(\mathcal{Q})$ *for any* $p < \infty$ *and a.e.*

PROOF OF THEOREM I.2. The first part of the proof of Theorem I.1 remains valid in this context. Thus we have

$$\begin{cases} u^\varepsilon(x, y) \downarrow \underline{u}(x, y) \geqslant \tilde{u}(x) & \text{in } \overline{\mathcal{Q}} \\ \underline{u}(x, y) = \psi(x) \quad \text{on } \overline{\mathcal{O}} \times \partial\tilde{\mathcal{O}} & \text{and} \quad \underline{u} \text{ is u.s.c.}, \\ \underline{u}(x, y) = v(x) \quad \text{a.e. in } \mathcal{Q},\ v \in W^{2,\infty}(\mathcal{O}),\ v = 0 & \text{on } \partial\mathcal{O}, \\ \sup\limits_{y \in \tilde{\mathcal{O}}} \left[ -\sum\limits_{i,j} a_{ij}(x, y) \dfrac{\partial^2 v}{\partial x_i \partial x_j}(x) + \sum\limits_i b_i(x, y) \dfrac{\partial v}{\partial x_i}(x) + c(x, y) v(x) - f(x, y) \right] \leqslant 0 & \\ \qquad\qquad\qquad \text{a.e. in } \mathcal{O}. \end{cases}$$

If we prove that $v(x) \leqslant \psi(x)$ then by use of the maximum principle we can establish $v(x) \leqslant \tilde{u}(x)$ in $\mathcal{O}$. Thus just as before $v(x) = \tilde{u}(x)$.

In order to show $v(x) \leqslant \psi(x)$ we use the next lemma.

LEMMA I.1. *Let* $\underline{u}$ *be a u.s.c. function on* $\overline{\mathcal{Q}}$ *and assume* $\underline{u} = v(x)$ *a.e. in* $\mathcal{Q}$ *where* $v \in c(\overline{\mathcal{O}})$ *then*

$$v(x) \leqslant \inf_{y \in \partial\tilde{\mathcal{O}}} \underline{u}(x, y) \quad \text{in } \mathcal{O}.$$

PROOF OF LEMMA I.1. Let $x \in \overline{\mathcal{O}}$ and let $y \in \partial\tilde{\mathcal{O}}$. We only need to prove that $v(x) \leqslant \underline{u}(x, y)$. Indeed, $v$ and $\underline{u}$ agree on a dense subset of $\mathcal{Q}$ and so there are sequences $x_i \to x$ and $y_i \to y$ such that $v(x_i) = \underline{u}(x_i, y_i)$. However, $v$ is continuous and $\underline{u}$ is u.s.c. so we conclude

$$v(x) = \lim_{i \leftarrow \infty} \underline{u}(x_i, y_i) \leqslant \underline{u}(x, y). \quad \square$$

This lemma in turn completes the proof of Theorem I.2. $\square$

REMARK I.6. Recall the false heuristic argument given in the introduction. If $u^\varepsilon(x, y) \to u(x)$ then one would expect $u$ to satisfy $u = \psi$ on $\overline{\mathcal{O}} \times \partial\tilde{\mathcal{O}}$ and thus $u \equiv \psi$. Obviously, this is not true since as we pointed out in the Introduction $\psi$ does not in general satisfy (7). This shows that there are boundary layers near $\mathcal{O} \times \partial\tilde{\mathcal{O}}$.

Again let us consider a *probabilistic interpretation of our results.* We keep the same notations as in the previous section and consider $(\xi_{x,y}, y_{x,y})$ solutions of

$$(17\bar{f}) \qquad \begin{cases} d\xi_{x,y}(t) = \sigma(\xi_{x,y}(t), y_{x,y}(t))\, dW_t - b(\xi_{x,y}(t), y_{x,y}(t))\, dt, \\ dy_{x,y}(t) = \sqrt{2}\, d\tilde{W}_t + q(t)\, dt, \\ \xi_{x,y}(0) = x \quad \text{and} \quad y_{x,y}(0) = y \quad \text{for } (x, y) \in \overline{\mathcal{Q}}, \end{cases}$$

where $q(t)$ is any adapted process taking values in $B_{1/\varepsilon}$. It is well known (see [21] for example) that $u^\varepsilon$ is given by

$$u^\varepsilon(x, y) = \inf_{\mathscr{A}'} E\Bigg[\int_0^{\tau_{x,y}\wedge\tilde{\tau}_{x,y}} f(\xi_{x,y}(t), y_{x,y}(t))\cdot\exp\left\{-\int_0^t c(\xi_{x,y}(s), y_{x,y}(s))\, ds\right\} dt + \mathscr{X}_{[0,\infty)}(\tilde{\tau}_{x,y} - \tau_{x,y})\, \psi(\xi_{x,y}(\tau_{x,y}\wedge\tilde{\tau}_{x,y}))\Bigg],$$

where $\tau_{x,y}$ and $\tilde{\tau}_{x,y}$ are respectively the first exit times of $\xi_{x,y}$ from $\bar{\mathcal{O}}$ and $y_{x,y}$ from $\bar{\tilde{\mathcal{O}}}$ and $\mathscr{X}_{[0,\infty)}$ is the characteristic function of the non-negative half line.

It can be shown that it is possible to approximate any process $y(t)$ with values in $\bar{\mathcal{O}}$ by a process of the form $y_{x,y}(t)$ with larger and larger drift controls $q(t)$ (at least if $y \notin \partial\tilde{\mathcal{O}}$ for in this case $y_{x,y}(t)$ will exit from $\bar{\tilde{\mathcal{O}}}$ instantaneously). In addition, if $\theta$ is any stopping time we can use $q(t)$ to insure that $\tilde{\tau}_{x,y}$ approximates $\theta$.

*The combination of these two facts shows that if $(x, y)$ is in $\bar{\mathcal{Q}}$ and $y \notin \partial\tilde{\mathcal{O}}$ then $u^\varepsilon(x, y)\downarrow\tilde{u}(x)$.* In particular, from Dini's lemma this implies that *the convergence is uniform on compact subsets of $\bar{\mathcal{O}}\times\tilde{\mathcal{O}}$.*

REMARK I.7. If we replace (6) by

$$u^\varepsilon(x, y) = \psi(x, y) \quad \text{on } \bar{\mathcal{O}}\times\partial\tilde{\mathcal{O}} \tag{6'}$$

where $\psi$ is a function on $\bar{\mathcal{O}}\times\partial\tilde{\mathcal{O}}$ which is assumed to be of class $C^{2,\alpha}$ for some $\alpha > 0$ and $\psi(x, y) = 0$ on $\partial\mathcal{O}\times\partial\tilde{\mathcal{O}}$ then Theorem I.2 (and its proof) is valid with $\psi(x)$ replaced by $\inf \psi(x, y)$ in (7).

### 3. *Variants and related problems.*

Periodic boundary conditions on $y$. Let $\tilde{\mathcal{O}} = (0, l_1)\times\ldots\times(0, l_m)$ for positive constants $l_1, \ldots, l_m$. Consider the solution, $u^\varepsilon$, of (1), (2) plus periodicity in $y$. That is, we replace (3) in Example 1 by

$$\begin{cases} D_y^\alpha u^\varepsilon(x, y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_m) = D_y^\alpha u^\varepsilon(x, y_1, \ldots, y_{i-1}, l_i, y_{i+1}, \ldots, y_m) \\ \text{for } 1\leqslant i\leqslant m,\ y_j\in(0, l_j) \text{ and } |\alpha|\leqslant 1\,. \end{cases} \tag{20}$$

We can again show *the existence of a unique $u^\varepsilon$ and can prove that $u^\varepsilon$ converges to the solution, $u$, of* (5) *a.e.*

Noncylindrical domains. Let $\mathcal{Q} \subset \mathbb{R}^n \times \mathbb{R}^m$ be a bounded connected domain with smooth boundary. Consider the solution, $u^\varepsilon$, of (1) plus the boundary condition

$$u^\varepsilon(x, y) = \psi(x, y) \quad \text{on } \partial\mathcal{Q} . \tag{21}$$

Define $\mathcal{O}$ by

$$\mathcal{O} \equiv \{x \in \mathbb{R}^n \mid \text{there is a } y \in \mathbb{R}^m \text{ such that } (x, y) \in \mathcal{Q}\} .$$

Assume additionally that $\mathcal{O}$ is a domain with a smooth boundary. Under some natural regularity assumptions on the coefficients of the operator in (1), on $\psi$ and on $\mathcal{Q}$ it is possible to prove that *$u^\varepsilon(x, y)$ converges to the solution, $\tilde{u}(x)$, of the following equation*

$$\begin{cases} \max\left[\sup\limits_{y \in \bar{\mathcal{O}}_x}\left\{-\sum\limits_{i,j} a_{ij}(x, y)\dfrac{\partial^2 \tilde{u}}{\partial x_i\, \partial x_j}(x) + \sum\limits_i b_i(x, y)\dfrac{\partial \tilde{u}}{\partial x_i}(x)\right.\right. \\ \qquad\qquad \left.\left. + c(x, y)\tilde{u}(x) - f(x, y)\right\}, \tilde{u}(x) - \tilde{\psi}(x)\right] = 0 \quad \text{a.e. in } \mathcal{O} \\ \tilde{u}(x) = \tilde{\psi}(x) \quad \text{on } \partial\mathcal{O} \end{cases}$$

where $\mathcal{O}_x = \{y \in \mathbb{R}^m | (x, y) \in \mathcal{Q}\}$ and $\tilde{\psi}(x) = \inf\limits_{y \in \bar{\mathcal{O}}_x} \psi(x, y)$. We shall not develop these types of results any further at this time.

Stochastic differential games. Consider now the case of three bounded regular domains $\mathcal{O}$, $\tilde{\mathcal{O}}$ and $\hat{\mathcal{O}}$ in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively. We use $x$, $y$ and $z$ to denote generic points in these domains. In place of the problem in Example 1 given by (1)-(3) consider the problem of finding a solution, $u^{\varepsilon,\delta}$, of:

$$\begin{cases} -\sum\limits_{i,j} a_{ij}(x)\dfrac{\partial^2 u^{\varepsilon,\delta}}{\partial x_i\, \partial x_j}(x, y, z) + \sum\limits_i b_i(x, y, z)\dfrac{\partial u^{\varepsilon,\delta}}{\partial x_i}(x, y, z) \\ \qquad + c(x, y, z)u^{\varepsilon,\delta}(x, y, z) - \sum\limits_i \dfrac{\partial^2 u^{\varepsilon,\delta}}{\partial y_i^2}(x, y, z) - \sum\limits_l \dfrac{\partial^2 u^{\varepsilon,\delta}}{\partial z_l^2}(x, y, z) \\ \qquad + \dfrac{1}{\varepsilon}|D_y u^{\varepsilon,\delta}(x, y, z)| - \dfrac{1}{\delta}|D_z u^{\varepsilon,\delta}(x, y, z)| = f(x, y, z) \\ \qquad\qquad\qquad\qquad \text{in } \mathcal{Q} = \mathcal{O} \times \tilde{\mathcal{O}} \times \hat{\mathcal{O}} , \\ u^{\varepsilon,\delta} \in C^2(\bar{\mathcal{Q}}) , \quad u^{\varepsilon,\delta} = 0 \quad \text{on } \partial\mathcal{O} \times \tilde{\mathcal{O}} \times \hat{\mathcal{O}} , \\ \dfrac{\partial u^{\varepsilon,\delta}}{\partial n} = 0 \quad \text{on } \bar{\mathcal{O}} \times \partial\tilde{\mathcal{O}} \times \bar{\hat{\mathcal{O}}} , \end{cases} \tag{22}$$

and where $a_{ij}$, $b_i$, $c$ and $f$ are all smooth functions on $\bar{\mathcal{Q}}$ and $(a_{ij})$ is uniformly elliptic.

It is easy to show *existence and uniqueness of a solution*, $u^{\varepsilon,\delta}$, *of* (22) *and it can be shown that if one takes the iterated limit* $\lim_{\delta\to 0}\lim_{\varepsilon\to 0} u^{\varepsilon,\delta}$ *the iterated limit exists and is a function*, $u(x)$, *independent of* $y$ *and* $z$ *and* $u(x)$ *is a solution of*

$$(23)\quad \begin{cases} -\sum_{i,j} a_{ij}(x)\dfrac{\partial^2 u}{\partial x_i\,\partial x_j}(x) \\ \quad + \inf_{z\in\hat{\mathcal{O}}}\sup_{y\in\tilde{\mathcal{O}}}\left\{\sum_i b_i(x,y,z)\dfrac{\partial u}{\partial x_i}(x) + c(x,y,z)u(x) - f(x,y,z)\right\} = 0 \quad \text{in } \mathcal{O}, \\ u\in C^2(\overline{\mathcal{O}}) \quad \text{and} \quad u=0 \quad \text{on } \partial\mathcal{O}. \end{cases}$$

From the stochastic point of view one has:

$$u^{\varepsilon,\delta}(x,y,z) = \sup_{\mathscr{B}''}\inf_{\mathscr{A}''} E\left[\int_0^\tau f(\xi_{x,y,z}(t), y_{x,y,z}(t), \zeta_{x,y,z}(t)) \cdot\exp\left\{-\int_0^t c(\xi_{x,y,z}(s), y_{x,y,z}(s), \zeta_{x,y,z}(s))\,ds\right\}dt\right]$$

where $\mathscr{A}''$ and $\mathscr{B}''$ are the appropriate admissible systems (analogous to (17′) and (18′)) and $\tau$ is the first exit time from $\overline{\mathcal{O}}$ of the process $\xi_{x,y,z}(t)$. The stochastic differential equations for $\xi_{x,y,z}$, $y_{x,y,z}$ and $\zeta_{x,y,z}$ are

$$\begin{aligned} d\xi_{x,y,z}(t) &= \sigma(\xi_{x,y,z}(t))\,dW_t - b(\xi_{x,y,z}(t), y_{x,y,z}(t), \zeta_{x,y,z}(t))\,dt \\ dy_{x,y,z}(t) &= \sqrt{2}\,d\tilde{W}_t + q(t,\omega)\,dt - 1_{\partial\tilde{\mathcal{O}}}(y_{x,y,z}(t))\,n(y_{x,y,z}(t))\,dA_t \\ d\zeta_{x,y,z}(t) &= \sqrt{2}\,d\hat{W}_t + r(t,\omega)\,dt - 1_{\partial\hat{\mathcal{O}}}(\zeta_{x,y,z}(t))\,n(\zeta_{x,y,z}(t))\,dB_t \\ \xi_{x,y,z}(0) &= x,\ y_{x,y,z}(0) = y,\ \zeta_{x,y,z}(0) = z \end{aligned}$$

where $W_t$, $\tilde{W}_t$ and $\hat{W}_t$ are three normalized independent Brownian motions on $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively, $q(t,\omega)$ and $r(t,\omega)$ are adapted processes taking values in $B_{1/\varepsilon}$ and $B_{1/\delta}$ respectively and $A_t$ and $B_t$ are continuous decreasing adapted processes such that $A_0 = 0$ and $B_0 = 0$ and for all $(x, y, z, t)$ in $\overline{\mathscr{Q}}\times[0,\infty)$ we have $(y_{x,y,z}(t), \zeta_{x,y,z}(t)) \in \overline{\tilde{\mathcal{O}}}\times\overline{\hat{\mathcal{O}}}$.

We have that $u(x) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} u^{\varepsilon,\delta}(x,y,z)$ exists and

$$u(x) = \sup_{\mathscr{A}}\inf_{\mathscr{B}} E\left[\int_0^{\tau_x} f(\xi_x(t), y(t), \zeta(t))\exp\left\{-\int_0^t c(\xi_x(s), y(s), \zeta(s))\,ds\right\}dt\right]$$

where $\mathscr{A}$ and $\mathscr{B}$ are the appropriate admissible systems (analogous to (17) and (18)), $\tau_x$ is the first exit time from $\bar{\mathcal{O}}$ of $\xi_x(t)$

$$d\xi_x(z) = \sigma(\xi_x(t))\, dW_t - b(\xi_x(t), y(t), \zeta(t))\, dt, \ \xi_x(0) = x$$

and $y(t, \omega)$ and $\zeta(t, \omega)$ are adapted processes taking values in $\bar{\bar{\mathcal{O}}}$ and $\bar{\hat{\mathcal{O}}}$ respectively.

REMARK I.8. If the matrix $a_{ij}(x)$ is replaced by a matrix $a_{ij}(x, y, z)$ (i.e. with dependence on $y$ and $z$) we can still prove that $\lim_{\delta\to 0} \lim_{\varepsilon\to 0} u^{\varepsilon,\delta}(x, y, z)$ exists but we no longer know that the limit solves (in any strong sense) the analogue of (23) to this case.

4. *The Cauchy problem.*

All the problems we have so far considered have natural extensions to time dependent problems. We shall only give one example (the time dependent analogue of Example 1) and we assume the functions $a_{ij}$, $b_i$, $c$ and $f$ are time independent to simplify our notation.

Let $\mathcal{Q}' \equiv \mathcal{Q}\times(0, T)$ for some $T > 0$ and let $u^\varepsilon$ be the solution of:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(x, y, t) - \sum\limits_{i,j} a_{ij}(x, y)\dfrac{\partial^2 u^\varepsilon}{\partial x_i\, \partial x_j}(x, y, t) + \sum\limits_i b_i(x, y)\dfrac{\partial u^\varepsilon}{\partial x_i}(x, y, t) \\ \qquad + c(x, y)u^\varepsilon(x, y, t) - \sum\limits_i \dfrac{\partial^2 u^\varepsilon}{\partial y_i^2}(x, y, t) + \dfrac{1}{\varepsilon}|D_y u^\varepsilon(x, y, t)| = f(x, y) \quad \text{in } \mathcal{Q}', \\ u^\varepsilon \in C^{2,1}(\mathcal{Q}') \cap C(\overline{\mathcal{Q}'}) \\ u^\varepsilon = 0 \quad \text{on } \partial\mathcal{O}\times\bar{\tilde{\mathcal{O}}}\times[0, T], \qquad \dfrac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \bar{\mathcal{O}}\times\partial\tilde{\mathcal{O}}\times[0, T], \\ u^\varepsilon(x, y, 0) = u_0(x, y) \quad \text{for } (x, y) \quad \text{in } \bar{\mathcal{Q}}, \end{cases}$$

where $u_0$ is a given function in $C(\bar{\mathcal{Q}})$ satisfying:

$$u_0 = 0 \quad \text{on } \partial\mathcal{O}\times\bar{\tilde{\mathcal{O}}} \quad \text{and} \quad D_x^2 u_0 \in L^\infty(\mathcal{Q}).$$

Then as $\varepsilon$ goes to zero, $u^\varepsilon(x, y, t)$ converges a.e. to a function, $u(x, t)$, in $W^{2,1,\infty}(\mathcal{Q}')$ which is the unique solution of:

$$\frac{\partial u}{\partial t}(x, t) + \sup_{y\in\tilde{\mathcal{O}}}\Big[-\sum_{i,j} a_{ij}(x, y)\frac{\partial^2 u}{\partial x_i\, \partial x_j}(x, t) + \sum_i b_i(x, y)\frac{\partial u}{\partial x_i}(x, t)$$
$$+ c(x, y)u(x, t) - f(x, y)\Big] = 0 \qquad \text{in } \mathcal{O}(0, T),$$

$$u = 0 \quad \text{on } \partial\mathcal{O}\times[0, T], \qquad u(x, 0) = \inf_{y\in\tilde{\mathcal{O}}} u_0(x, y) \quad \text{for } x \text{ in } \mathcal{O}.$$

This problem also has an optimal stochastic control interpretation but we shall not consider it here. Notice that we have obtained the Cauchy problem for HJB equations but the initial data has changed and thus there is a boundary layer near $t = 0$. Cauchy problems for HJB equations are considered in N. V. Krylov [23], M. V. Safonov [38] and P. L. Lions [34] and [35].

## II. – Penalization of the derivative in $y$.

### 1. *Neumann boundary conditions.*

We shall now specify the domain $\tilde{\mathcal{O}}$ but we shall allow more general penalty functions as in Example 3. Assume $\tilde{\mathcal{O}} = (0, 1)$ and that (4) and (5) hold. Let $u^\varepsilon$ be the solution of the differential equation (9) with boundary conditions (2) and (3). To simplify our exposition we shall assume

$$\beta \in C^1(\mathbb{R}), \quad \limsup_{|t|\to\infty} \frac{|\beta(t)|}{|t|} < \infty . \tag{24}$$

(If $\beta$ is convex or if $(a_{ij})$ does not depend on $y$ then (24) is not necessary.)

We shall now show why (9), (2), (3) has a unique solution, $u^\varepsilon$. In view of [1] it is enough to find two functions $\overline{u}$ and $\underline{u}$ soch that $\overline{u}, \underline{u} \in W^{2,\infty}(\mathcal{Q})$, $\overline{u} \geqslant \underline{u}$ and

$$\begin{cases} -\sum_{i,j} a_{ij}(x, y) \dfrac{\partial^2 \overline{u}}{\partial x_i \partial x_j}(x, y) + \sum_i b_i(x, y) \dfrac{\partial \overline{u}}{\partial x_i}(x, y) + c(x, y)\overline{u}(x, y) \\ \qquad\qquad -\dfrac{\partial^2 \overline{u}}{\partial y^2}(x, y) + \beta\left(\dfrac{1}{\varepsilon}\dfrac{\partial \overline{u}}{\partial y}(x, y)\right) \geqslant f(x, y) \quad \text{in } \mathcal{Q} \\ -\sum_{i,j} a_{ij}(x, y) \dfrac{\partial^2 \underline{u}}{\partial x_i \partial x_j}(x, y) + \sum_i b_i(x, y) \dfrac{\partial \underline{u}}{\partial x_i} + c(x, y)\underline{u}(x, y) \\ \qquad\qquad -\dfrac{\partial^2 \underline{u}}{\partial y^2}(x, y) + \beta\left(\dfrac{1}{\varepsilon}\dfrac{\partial \underline{u}}{\partial y}(x, y)\right) \leqslant f(x, y) \quad \text{in } \mathcal{Q} \\ \overline{u} \text{ and } \underline{u} \text{ satisfy the boundary conditions (2) and (3)}. \end{cases}$$

Finding such functions is accomplished by letting $\underline{u}(x, y) = \underline{u}(x)$ be the solution of

$$\begin{cases} \sup_{y\in\tilde{\mathcal{O}}} \left\{-\sum_{i,j} a_{ij}(x, y) \dfrac{\partial^2 \underline{u}}{\partial x_i \partial x_j}(x) + \sum_i b_i(x, y) \dfrac{\partial \underline{u}}{\partial x_i}(x) + c(x, y)\underline{u}(x) - f(x, y)\right\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } \mathcal{O} \\ \underline{u} \in W^{2,\infty}(\mathcal{O}) \quad \text{and} \quad \underline{u} = 0 \quad \text{on } \partial\mathcal{O} \end{cases}$$

and $\overline{u}(x, y) = \overline{u}(x)$ is taken as the solution of

$$\begin{cases} \inf\limits_{y\in\bar{\mathcal{O}}} \left\{ -\sum\limits_{i,j} a_{ij}(x, y) \dfrac{\partial^2 \overline{u}}{\partial x_i\, \partial x_j}(x) + \sum\limits_i b_i(x, y) \dfrac{\partial \overline{u}}{\partial x_i}(x) + c(x, y)\overline{u}(x) - f(x, y) \right\} = 0 \\ \hfill \text{a.e. in } \mathcal{O} \\ \overline{u} \in W^{2,\infty}(\mathcal{O}) \quad \text{and} \quad \overline{u} = 0 \quad \text{on } \partial\mathcal{O}\,. \end{cases}$$

We now state the main result.

THEOREM II.1. *Under assumptions* (4), (15), (24) *and assuming* $c(x, y) > 0$, $\beta'(0) > 0$, $t\beta(t) > 0$ *if* $t \neq 0$ *and* $\liminf\limits_{|t|\to\infty} |\beta(t)| = \infty$ *then as* $\varepsilon$ *goes to zero the solution,* $u^\varepsilon$, *of* (9), (2), (3) *converges uniformly on* $\overline{\mathcal{Q}}$ *to* $u(x)$ *the solution of the linear problem* (10).

REMARK II.1. As in the preceding sections the boundary condition in $x$ is of no real importance to the validity of our result and we could equally well treat other boundary conditions on $\partial\mathcal{O}$.

We shall first give an analytical proof and then sketch a probabilistic proof for the special case when $\beta(t)$ is linear.

PROOF OF THEOREM II.1. Let $u(x)$ be the solution of (10). There is a constant $C_0$ such that

$$\Bigg| -\sum_{i,j} \big(a_{ij}(x, y) - a_{ij}(x, 0)\big) \frac{\partial^2 u}{\partial x_i\, \partial x_j}(x) + \sum_i \big(b_i(x, y) - b_i(x, 0)\big) \frac{\partial u}{\partial x_i}(x) \\ + \big(c(x, y) - c(x, 0)\big)\, u(x) - \big(f(x, y) - f(x, 0)\big) \Bigg| \leqslant C_0 y$$

for all $(x, y) \in \overline{\mathcal{Q}}$.

Because of the assumptions made on $\beta$ there is a constant, $C_1$, such that

$$\beta(C_1 y) \geqslant C_0 y \quad \text{for all } y \text{ in } [0, 1]\,.$$

Therefore, if we define $\overline{u}$ by $\overline{u}^\varepsilon(x, y) \equiv u(x) + \frac{1}{2}\varepsilon C_1 y^2 + \varepsilon(C_1/\alpha)$ where $\alpha > 0$ is chosen so that $c(x, y) \geqslant \alpha > 0$ on $\overline{\mathcal{Q}}$ then

$$\Bigg[ -\sum_{i,j} a_{ij}(x, y) \frac{\partial^2 \overline{u}^\varepsilon}{\partial x_i\, \partial x_j}(x, y) + \sum_i b_i(x, y) \frac{\partial \overline{u}^\varepsilon}{\partial x_i}(x, y) \\ + c(x, y)\overline{u}^\varepsilon(x, y) - \frac{\partial^2 \overline{u}^\varepsilon}{\partial y^2}(x, y) + \beta\left(\frac{1}{\varepsilon}\frac{\partial \overline{u}^\varepsilon}{\partial y}(x)\right) - f(x, y) \Bigg] \\ \geqslant -C_0 y + c(x, y)\left[\frac{1}{2}\varepsilon C_1 y^2 + \varepsilon \frac{C_1}{\alpha}\right] - \varepsilon C_1 + \beta(C_1 y) \\ \geqslant \frac{1}{2}\varepsilon C_1 c(x, y) y^2 \quad \text{in } \overline{\mathcal{Q}}\,.$$

Using this inequality and the maximum principle we deduce that

$$u^\varepsilon(x, y) \leqslant \bar{u}^\varepsilon(x, y) \quad \text{in } \bar{\mathscr{Q}}.$$

In the same way we deduce that if $\underline{u}_\varepsilon$ is defined by $\underline{u}(x, y) \equiv u(x) - \frac{1}{2}\varepsilon C_1 y^2 - \varepsilon(C_1/\alpha)$ then

$$u^\varepsilon(x, y) \geqslant \underline{u}_\varepsilon(x, y) \quad \text{in } \bar{\mathscr{Q}}.$$

As our final conclusion we obtain

$$\|u^\varepsilon - u\|_{L^\infty(\mathscr{Q})} \leqslant C\varepsilon$$

for some constant, $C$, independent of $\varepsilon$. □

COROLLARY II.1. *Under the assumptions of Theorem II.1 there is a constant, $C$, independent of $\varepsilon$ such that*

$$\|u^\varepsilon - u\|_{L^\infty(\mathscr{Q})} \leqslant C\varepsilon .$$

We now consider a probabilistic proof of Theorem II.1 in the special case where $\beta(t) = t$. For this case we have

$$u^\varepsilon(x, y) = E\Big[\int_0^{\tau_x^\varepsilon} f(\xi_{x,y}^\varepsilon(t), y_y^\varepsilon(t)) \exp\Big\{-\int_0^t c(\xi_{x,y}^\varepsilon(s), y_y^\varepsilon(s))\, ds\Big\}\, dt\Big]$$

where $\tau_x^\varepsilon$ is the exit time of $\xi_{x,y}^\varepsilon$ from $\bar{\mathcal{O}}$ and $\xi_{x,y}^\varepsilon$ is the solution of

$$\begin{aligned} d\xi_{x,y}^\varepsilon &= \sigma(\xi_{x,y}^\varepsilon, y_y^\varepsilon)\ dW_t - b(\xi_{x,y}^\varepsilon, y_y^\varepsilon)\, dt \\ \xi_{x,y}^\varepsilon(0) &= x \end{aligned}$$

and $y_y^\varepsilon$ is the solution of

$$\begin{aligned} dy_y^\varepsilon &= \sqrt{2}\, d\tilde{W}_t - \frac{1}{\varepsilon}\, dt + [1_{\{0\}}(y_y^\varepsilon) - 1_{\{1\}}(y_y^\varepsilon)]\, dA_t, \\ y_y^\varepsilon(0) &= y \end{aligned}$$

where $\tilde{W}_t$ is a one-dimensional Brownian motion independent of the $n$-dimensional Brownian motion, $W_t$, and $A_t$ is some continuous increasing process such that $A_0 = 0$.

We claim that

$$\sup_{y\in[0,1]} E\left[\int_0^\infty |y_y^\varepsilon(t)|^2 e^{-\lambda t}\, dt\right] \to 0 \qquad \text{as } \varepsilon \text{ goes to zero},$$

for every $\beta > 0$. Indeed, observe that if we define $v^\varepsilon(y)$ by

$$v^\varepsilon(y) \equiv E\left[\int_0^\infty |y_y^\varepsilon(t)|^2 e^{-\lambda t}\, dt\right]$$

then $v^\varepsilon$ is the solution of the o.d.e.

$$\begin{aligned} &-(v^\varepsilon)''(y) + \frac{1}{\varepsilon}(v^\varepsilon)'(y) + \lambda v^\varepsilon(y) = y^2 \quad \text{in } (0,1) \\ &(v^\varepsilon)'(0) = (v^\varepsilon)'(1) = 0 . \end{aligned}$$

The claim now follows by constructing an explicit solution for $v^\varepsilon$.

Now, denote by $\xi_x(t)$ the solution of

$$\begin{aligned} &d\xi = \sigma(\xi, 0)\, dW_t - b(\xi, 0)\, dt \\ &\xi(0) = x . \end{aligned}$$

It is clear that

$$u(x) = E\left[\int_0^{\tau_x} f(\xi_x(t), 0) \exp\left\{-\int_0^t c(\xi_x(s), 0)\, ds\right\} dt\right]$$

for $x \in \overline{\mathcal{O}}$, where $\tau_x$ is the first exit time of $\xi_x(t)$ from $\overline{\mathcal{O}}$.

To prove Theorem II.1 it is enough to show

$$E[|\xi^\varepsilon_{x,y}(t) - \xi_x(t)|^2] \to 0 \qquad \text{as } \varepsilon \text{ goes to zero},$$

for all $t \geqslant 0$ and that the exit times converge. The exit times will converge if the first statement is true since we have assumed $\sigma$ is nondegenerate. So we consider $E[|\xi^\varepsilon_{x,y}(t) - \xi_x(t)|^2]$ and we have

$$E[|\xi^\varepsilon_{x,y}(t) - \xi_x(t)|^2] \leqslant CE\left[\int_0^t |\xi^\varepsilon_{x,y}(s) - \xi_x(s)|^2\, ds\right] + CE\left[\int_0^t |y_y^\varepsilon(s)|^2\, ds\right],$$

for some constant $C$. The conclusion we desire follows from an application of Gronwall's inequality.

REMARK II.2. It is possible to generalize the preceding results to some extent. Consider the problem:

find $u^\varepsilon$ in $C^2(\mathcal{Q}) \cap C(\overline{\mathcal{Q}})$ such that

$$(9') \qquad -\sum_{i,j} a_{ij}(x, y) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j}(x, y) + \sum_i b_i(x, y) \frac{\partial u^\varepsilon}{\partial x_i}(x, y) + c(x, y) u^\varepsilon(x, y) - \sum_i \frac{\partial^2 u^\varepsilon}{\partial y_i^2}(x, y) + \beta\left(\frac{1}{\varepsilon} \frac{\partial u^\varepsilon}{\partial y_1}(x, y)\right) = f(x, y) \quad \text{in } \mathcal{Q}$$

plus boundary conditions (2) and (3).

We assume additionally that there exists a unique point $y_0$ on $\partial\tilde{\mathcal{O}}$ such that $-n(y_0) = (1, 0 \ldots, 0)$. This assumption is given precisely by

$$\begin{cases} \text{there exists a function, } \varphi, \text{ in } W^{2,\infty}(\overline{\tilde{\mathcal{O}}}) \text{ such that} \\ \dfrac{\partial \varphi}{\partial y_1} \geqslant y_1, \ \dfrac{\partial \varphi}{\partial n} > 0 \quad \text{on } \partial\tilde{\mathcal{O}} \setminus \{y_0\}, \ \varphi(y_0) = y_{01} \\ \tilde{\mathcal{O}} \subset \{y | y_1 > y_{01}\} \quad \text{and} \quad y_0 \in \partial\tilde{\mathcal{O}}. \end{cases}$$

Under this assumption our previous results and proofs (appropriately modified) remain valid. An open problem is the case where there are many points at which $-n(x) = (1, 0, \ldots, 0)$.

## 2. *Periodic boundary conditions.*

Let $\mathcal{O} = (0, 1)$ and let (4) and (15) hold (regularity assumptions on the coefficients and uniform ellipticity of the operator). Consider a solution, $u^\varepsilon$, of (9) satisfying the boundary conditions (2) and

$$(25) \qquad u^\varepsilon(x, 0) = u^\varepsilon(x, 1) \quad \text{and} \quad \frac{\partial u^\varepsilon}{\partial y}(x, 0) = \frac{\partial u^\varepsilon}{\partial y}(x, 1) \quad \text{on } \overline{\mathcal{O}}.$$

We assume that $\beta$ is an increasing function such that

$$(26) \qquad \beta \in C^1(\mathbb{R}), \quad 0 < c_0 \leqslant \beta'(t) \leqslant c_1 \quad \text{for all } t \text{ in } \mathbb{R} \text{ and } \beta(0) = 0.$$

We shall use $\gamma(t)$ to denote the inverse of $\beta(t)$, i.e. $\gamma = \beta^{-1}$.

By assumption (26) there is no problem in finding solutions of (9), (2) and (25). In order to simplify our presentation we assume $a_{ij}(x, y) = a_{ij}(x)$, $b_i(x, y) = b_i(x)$ and $c(x, y) = c(x)$.

THEOREM II.2. *Under assumptions* (4), (15) *and* (26) *and if the coefficients* $a_{ij}$, $b_i$ *and* $c$ *do not depend on* $y$ *then there is a unique solution,* $u^\varepsilon$, *to* (9), (2) *and* (25). *Furthermore, as* $\varepsilon$ *goes to zero the solutions* $u^\varepsilon$ *converge weakly in* $H^2(\mathcal{Q})$ *to* $u \in C^2(\overline{\mathcal{O}})$, *the solution of*

$$-\sum_{i,j} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_i b_i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x) = \bar{f}(x) \quad \text{in } \mathcal{O}$$

$$u = 0 \quad \text{on } \partial\mathcal{O}$$

*where* $\bar{f}(x)$ *is given by*

$$\int_0^1 \gamma(f(x, y) - \bar{f}(x))\, dy = 0 \quad \text{for all } x \text{ in } \mathcal{O}\,. \tag{27}$$

REMARK II.3. The same type of result holds if the coefficients $a_{ij}$, $b_i$, $c$ do depend on $y$. In this case the limit as $\varepsilon$ goes to zero is the solution of

$$\tag{28} \begin{cases} \displaystyle\int_0^1 \gamma\Big(f(x, y) + \sum_{i,j} a_{ij}(x, y) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \\ \displaystyle\qquad\qquad - \sum_{i,j} b_i(x, y) \frac{\partial u}{\partial x_i}(x) - c(x, y)u(x)\Big)\, dy = 0 \quad \text{in } \mathcal{O} \\ u = 0 \qquad \text{on } \partial\mathcal{O}\,. \end{cases}$$

If $\beta$ is convex then $u^\varepsilon$ converges to a solution of (28) (but with $a_{ij}(x)$ replaced by $a_{ij}(x, y)$). These results will be developed further in Part $B$) on nonlinear averaging phenomena.

REMARK II.4. Because of (26) we see that $\gamma = \beta^{-1}$ also satisfies the same assumption. This implies the existence for each $x \in \overline{\mathcal{O}}$ of $t = t(x)$ such that

$$\int_0^1 \gamma(f(x, y) - t)\, dy = 0\,.$$

By the implicit function theorem $t$ is a $C^1$ function of $x$ and we have

$$\int_0^1 \gamma'(f(x, y) - t) \frac{\partial f}{\partial x_i}(x, y)\, dy = \left\{\int_0^1 \gamma'(f(x, y) - t)\, dy\right\} \frac{\partial t}{\partial x_i}(x)\,.$$

This shows that there is a unique function, $\bar{f}$, of (27) and that $\bar{f}$ is $C^1$ on $\overline{\mathcal{O}}$.

Thus, there is a *unique* solution, $u(x)$, of the linear problem given in Theorem II.2.

PROOF OF THEOREM II.2. We first show that $u^\varepsilon$ is bounded in $H^2(\mathcal{Q})$. Multiply (9) by $(1/\varepsilon)(\partial u^\varepsilon/\partial y)$ and using (25) we obtain

$$\int_{\mathcal{O}}\left(\int_0^1 \beta\left(\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial y}\right)\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial y}\,dy\right)dx \leqslant \|f\|_{L^2}\left\|\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial y}\right\|_{L^2} + C\|D_x u^\varepsilon\|_{L^2}\left\|\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial y}\right\|_{L^2} + \sum_{i,j}\int a_{ij}(x)\frac{\partial^2 u^\varepsilon}{\partial x_i\,\partial x_j}\left(\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial y}\right)dx\,dy\,.$$

In view of (26) this implies

$$c_0\left\|\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial y}\right\|_{L^2}^2 \leqslant \{\|f\|_{L^2} + C\|D_x u^\varepsilon\|_{L^2}\}\left\|\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial y}\right\|_{L^2} - \sum_{i,j}\int_{\mathcal{O}}\frac{\partial}{\partial x_j}(a_{ij}(x))\frac{\partial u^\varepsilon}{\partial x_i}\left(\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial y}\right)dx\,dy - \frac{1}{2\varepsilon}\sum_{i,j}\int_{\mathcal{O}}\frac{\partial}{\partial y}\left(a_{ij}(x)\frac{\partial u^\varepsilon}{\partial x_i}\frac{\partial u^\varepsilon}{\partial x_j}\right)dx\,dy$$

and using the periodicity condition, (25) again, we conclude

$$c_0\left\|\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial y}\right\|_{L^2} \leqslant \|f\|_{L^2} + C\|D_x u^\varepsilon\|_{L^2}\,.$$

Using well-known regularity results for linear equations gives us

$$\|u^\varepsilon\|_{H^2(\mathcal{Q})} \leqslant C + C\|u^\varepsilon\|_{H^1(\mathcal{Q})} \leqslant C + C\|u^\varepsilon\|^{\frac{1}{2}}_{H^2(\mathcal{Q})}\|u^\varepsilon\|^{\frac{1}{2}}_{L^2(\mathcal{Q})}\,.$$

As we have seen before (in the preceding section) it is easy to obtain $L^\infty$ estimates for $u^\varepsilon$ and so this proves our claim.

We may now extract a subsequence, still denoted by $u^\varepsilon$, which converges weakly in $H^2(\mathcal{Q})$ and strongly in $H^1(\mathcal{Q})$ to some function $u$ which will depend only on $x$ since $D_y u^\varepsilon \xrightarrow[L^2]{} 0$. In addition $u$ satisfies $u = 0$ on $\partial\mathcal{O}$ so we only need to show (in view of Remark II.7) that

$$-\sum_{i,j} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x) + \sum_i b_i(x)\frac{\partial u}{\partial x_i}(x) + c(x)u(x) + \beta(\varphi) - f(x,y) = 0 \quad \text{a.e. in } \mathcal{Q}$$

for some $\varphi$ in $L^2(\mathcal{Q})$ such that

$$\int_0^1 \varphi(x, y)\, dy = 0 \qquad \text{a.e. in } \mathcal{O}\,.$$

Since $(1/\varepsilon)(\partial u^\varepsilon/\partial y)$ is bounded in $L^2$ we have

$$\varphi^\varepsilon = \frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial y} \to \varphi \qquad \text{weakly in } L^2$$

and because of (25)

$$\int_0^1 \varphi(x, y)\, dy = 0 \qquad \text{a.e. in } \mathcal{O}\,.$$

Since $\beta$ is a maximal monotone operator we will be done (by some general results due to H. Brézis [6]) if we show

$$\operatorname{ilm}\sup_{\varepsilon\to 0}\int_{\mathcal{Q}} \beta(\varphi^\varepsilon)\varphi^\varepsilon \leqslant \int_{\mathcal{Q}} \psi\varphi$$

where

$$\psi(x, y) = f(x, y) + \sum_{i,j} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\, \partial x^j}(x) - \sum_i b_i(x)\frac{\partial u}{\partial x_i}(x) - c(x)u(x)\,.$$

Using (25) we find

$$\int_{\mathcal{Q}} \beta(\varphi^\varepsilon)\varphi^\varepsilon = \int_{\mathcal{Q}}\left\{f(x, y)\varphi^\varepsilon(x, y) - \sum_i b_i(x)\frac{\partial u^\varepsilon}{\partial x_i}(x, y)\varphi^\varepsilon(x, y) - c(x)u^\varepsilon(x, y)\varphi^\varepsilon(x, y)\right\} - \sum_{i,j}\int_{\mathcal{Q}}\left(\frac{\partial}{\partial x_j}a_{ij}(x)\right)\frac{\partial u^\varepsilon}{\partial x_i}\varphi^\varepsilon$$

and since $u^\varepsilon \to u$ in $H^1$ while $p^\varepsilon \rightharpoonup p$ weakly in $L^2$ we conclude

$$\lim_{\varepsilon\to 0}\int_{\mathcal{Q}} \beta(\varphi^\varepsilon)\varphi^\varepsilon = \int_{\mathcal{Q}} \psi\varphi\,. \qquad \square$$

REMARK II.5. The proof above shows that we actually have:

$$\begin{cases} u^\varepsilon \text{ converges strongly in } H^2(\mathcal{Q}) \text{ to } u(x)\,, \\ \dfrac{1}{\varepsilon}\dfrac{\partial u^\varepsilon}{\partial y} \text{ converges strongly in } L^2(\mathcal{Q}) \text{ to } \gamma(f(x, y) - \bar{f}(x)). \end{cases}$$

In conclusion we state the following result.

PROPOSITION II.1. *Under the assumptions of Theorem II.2, let $u^\varepsilon$ be the unique solution of*

$$(9'') \qquad -\sum_{i,j} a_{ij}(x)\frac{\partial^2 u^\varepsilon}{\partial x_i\,\partial x_j}(x,y)+\sum_i b_i(x)\frac{\partial u^\varepsilon}{\partial x_i}(x,y)+c(x)u^\varepsilon(x,y) - \frac{\partial^2 u^\varepsilon}{\partial y^2}(x,y)+\frac{1}{\varepsilon}\beta\left(\frac{\partial u^\varepsilon}{\partial y}\right)=f(x,y) \quad \text{in } \mathcal{Q}$$

*together with boundary conditions* (2) *and* (25). *Then, as $\varepsilon$ goes to zero, $u^\varepsilon$ converges weakly in $H^2(\mathcal{Q})$ to $u$ which is the solution of*

$$\begin{cases} -\sum_{i,j} a_{ij}(x)\dfrac{\partial^2 u}{\partial x_i\,\partial x_j}(x)+\sum_i b_i(x)\dfrac{\partial u}{\partial x_i}(x)+c(x)u(x)=\displaystyle\int_0^1 f(x,y)\,dy & \text{in } \mathcal{O}\,, \\ u\in C^2(\bar{\mathcal{O}})\,, \quad u=0 \quad \text{on } \partial\mathcal{O}\,. \end{cases}$$

The proof of Proposition II.1 is very similar to the proof of Theorem II.2 and we will only sketch it. One first obtains the $H^2$ bounds on the solutions in the same way as in the proof of Theorem II.2. If we set

$$\bar{u}^\varepsilon(x)=\int_0^1 u^\varepsilon(x,y)\,dy$$

we have

$$-\sum_{i,j} a_{ij}(x)\frac{\partial^2 \bar{u}^\varepsilon}{\partial x_i\,\partial x_j}+\sum_i b_i(x)\frac{\partial \bar{u}^\varepsilon}{\partial x_i}+c(x)\bar{u}(x)=\int_0^1 f(x,y)\,dy + \int_0^1\left|\frac{1}{\varepsilon}\beta\left(\frac{\partial u^\varepsilon}{\partial y}\right)-\frac{1}{\varepsilon}\beta'(0)\frac{\partial u^\varepsilon}{\partial y}\right|dy$$

where the last term on the right is bounded by

$$C\int_0^1 \frac{1}{\varepsilon}\left|\frac{\partial u^\varepsilon}{\partial y}\right|\gamma\left(\left|\frac{\partial u^\varepsilon}{\partial y}\right|\right)dy\,,$$

where $\gamma(t)$ is bounded, $\gamma(t)\to 0$ as $t\searrow 0$ and thus this term converges to zero in $L^1(\mathcal{O})$.

From here the conclusion of the proof is straightforward. □

## III. – One-sided penalization of the derivative in $y$.

1. *Neumann boundary conditions.*

Let $\tilde{\mathcal{O}} = (0,1)$ and let (4) and (15) hold (regularity assumptions on the coefficients and uniform ellipticity). We again consider a solution, $u^\varepsilon$, of (9), (2) and (3). However, we now make the following assumption on $\beta$. We assume

$$(29) \qquad \beta \in C(\mathbb{R}), \quad \beta \text{ is convex}, \quad \beta(t) = 0 \text{ if } t \leqslant 0 \text{ and } \beta(t) > 0 \text{ if } t > 0 .$$

A typical example is $\beta(t) = t^+$.

Our first result is on the solvability of (9), (2) and (3) under the assumption (29) on $\beta$ and gives an a priori bound on the solution, $u^\varepsilon$.

PROPOSITION III.1. *Under assumptions* (4), (15) *and* (29) *there exists a unique solution, $u^\varepsilon$, of* (9), (2) *and* (3). *Furthermore*

$$\begin{cases} \|D_x^2 u^\varepsilon\|_{L^\infty(\mathcal{Q})} \leqslant \text{const}; & \left\|\dfrac{\partial^2 u^\varepsilon}{\partial y^2}\right\|_{L^\infty(\mathcal{Q})} \leqslant \text{const}; \\ \|u^\varepsilon\|_{W^{1,\infty}(\mathcal{Q})} \leqslant \text{const}. & \end{cases}$$

*In particular*

$$\|u^\varepsilon\|_{W^{2,p}(\mathcal{Q})} \leqslant \text{const} \qquad \textit{for any } p < \infty .$$

PROPOSITION III.2. *Since the uniqueness of a solution to* (9), (2) *and* (3) *is a straightforward consequence of the maximum principle we shall prove only* the existence of a solution. In order to prove the existence of a solution we apply the results of [28] (and its proof) from which we conclude that we need only to exhibit a subsolution $\underline{u}$ satisfying

$$\begin{cases} -\sum\limits_{i,j} a_{ij}(x,y)\dfrac{\partial^2 \underline{u}}{\partial x_i\,\partial x_j}(x,y) + \sum\limits_i b_i(x,y)\dfrac{\partial \underline{u}}{\partial x_i}(x,y) + c(x,y)\,\underline{u}(x,y) \\ \qquad\qquad -\dfrac{\partial^2 \underline{u}}{\partial y^2}(x,y) + \beta\!\left(\dfrac{1}{\varepsilon}\dfrac{\partial \underline{u}}{\partial y}(x,y)\right) \leqslant f(x,y) \qquad \text{a.e. in } \mathcal{Q}, \\ \underline{u} \in W^{2,\infty}(\mathcal{Q}) \quad \text{and} \quad \underline{u} \text{ satisfies (2) and (3)}. \end{cases}$$

If we denote by $\underline{u}(x)$ the solution of the HJB equation

$$\begin{cases} \sup\limits_{y\in(0,1)}\Big\{-\sum\limits_{i,j} a_{ij}(x,y)\dfrac{\partial^2\underline{u}}{\partial x_i\,\partial x_j}(x)+\sum\limits_i b_i(x,y)\dfrac{\partial\underline{u}}{\partial x_i}(x)+c(x,y)\,\underline{u}(x)-f(x,y)\Big\}=0 \\ \hfill \text{a.e. in } \mathscr{Q} \\ \underline{u}\in W^{2,\infty}(\mathscr{O}),\ \underline{u}=0 \quad \text{on } \partial\mathscr{O} \end{cases}$$

then it is clear that $\underline{u}$ is just the subsolution we need since $\beta(0)=0$.

We proceed to the a priori estimate on the solution. Since $\beta$ is convex we may apply the arguments in [25] and [19] from which we obtain

$$\|D_x^\alpha u^\varepsilon\|_{L^\infty(\mathscr{Q})}\leqslant\text{const}\qquad\text{for } |\alpha|\leqslant 2\ .$$

In particular this implies that

$$\left|-\frac{\partial^2u^\varepsilon}{\partial y^2}+\beta\left(\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial y}\right)\right|\leqslant\text{const}\qquad\text{in } \mathscr{Q}\ .$$

Let $(x_0,y_0)$ be a point at which $\partial u^\varepsilon/\partial y$ attains its maximum. If $(\partial u^\varepsilon/\partial y)(x_0,y_0)\leqslant 0$ then $\beta\ \big((1/\varepsilon)(\partial u^\varepsilon/\partial y)\big)=0$ in $\mathscr{Q}$ so

$$\left\|\frac{\partial^2u^\varepsilon}{\partial y^2}\right\|_{L^\infty(\mathscr{Q})}\leqslant\text{const}\ .$$

On the other hand, if $(\partial u^\varepsilon/\partial y)(x_0,y_0)>0$ then $(x_0,y_0)\in\mathscr{Q}$ and since $(\partial^2u^\varepsilon/\partial y^2)(x_0,y_0)=0$ this implies

$$\beta\left(\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial y}(x_0,y_0)\right)\leqslant\text{const}$$

and in all cases

$$0\leqslant\beta\left(\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial y}\right)\leqslant\text{const}\qquad\text{in } \mathscr{Q}\ .$$

This concludes our proof. □

We now state our main result for this section.

THEOREM III.1. *Under assumptions* (4), (15) *and* (29) *the solution,* $u^\varepsilon$ *of* (9), (2) *and* (3) *converges weakly in* $W^{2,p}(\mathscr{Q})$ *as* $\varepsilon$ *goes to zero to the maximum* *solution, $u$, of the HJB equation*

$$
(31)\quad \begin{cases} \max\left\{\left(-\sum_{i,j} a_{ij}(x,y)\dfrac{\partial^2 u}{\partial x_i\,\partial x_j}(x,y)+\sum_i b_i(x,y)\dfrac{\partial u}{\partial x_i}(x,y)\right.\right. \\ \qquad\left.\left.+\,c(x,y)u(x,y)-\dfrac{\partial^2 u}{\partial y^2}(x,y)-f(x,y)\right),\ \dfrac{\partial u}{\partial y}\right\}=0 \quad a.e.\ in\ \mathscr{Q} \\ u=0 \quad \text{on } \partial\mathcal{O}\times\bar{\tilde{\mathcal{O}}} \quad \text{and} \quad \dfrac{\partial u}{\partial y}(x,0)=\dfrac{\partial u}{\partial y}(x,1)=0 \quad \text{in } \tilde{\mathcal{O}}. \end{cases}
$$

*In addition, $u\in W^{2,p}(\mathscr{Q})$ for $p<\infty$, $D_x^2 u\in L^\infty(\mathscr{Q})$ and $\partial^2 u/\partial y^2\in L^\infty(\mathscr{Q})$.*

REMARK III.1. If the boundary condition (3) is replaced by some Dirichlet condition, like

$$
(6')\qquad u^\varepsilon=0 \quad \text{on } \bar{\mathcal{O}}\times\partial\tilde{\mathcal{O}}
$$

then we do not know what the limit of $u^\varepsilon$ is. We conjecture that $u^\varepsilon$ converges to $u$ where $u$ is a solution of (31) but with boundary conditions replaced by

$$
\begin{cases} u=0 \quad \text{on } \partial\tilde{\mathcal{O}}, \quad u(x,1)=0 \quad \text{in } \bar{\mathcal{O}}, \\ \max\left(u(x,0),\ \dfrac{\partial u}{\partial y}(x,0)\right)=0 \quad \text{in } \bar{\mathcal{O}}. \end{cases}
$$

There would be a boundary layer near $\bar{\mathcal{O}}\times\{0\}$.

We proceed to the proof of Theorem III.1.

PROOF OF THEOREM III.1. First we show that if a subsequence $u^{\varepsilon_n}$ converges in $C^1$ and weakly in $W^{2,p}(\mathscr{Q})$ (for $p<\infty$) to a function $u$ then $u$ is a solution of (31). Indeed, because of the penalization term we have

$$
\frac{\partial u}{\partial y}\leqslant 0 \quad \text{in } \mathscr{Q}.
$$

Next, if $\omega_\alpha=\{(x,y)\in\mathscr{Q}\,|\,\partial u/\partial y\leqslant-\alpha<0\}$ then for any $\alpha>0$ there is an $n_\alpha$ such that when $n\geqslant n_\alpha$

$$
\frac{\partial u^{\varepsilon_n}}{\partial y}\leqslant 0 \quad \text{on } \overline{\omega_\alpha}.
$$

Therefore we have on $\overline{\omega_\alpha}$

$$
-\sum_{i,j} a_{ij}(x,y)\frac{\partial^2 u^{\varepsilon_n}}{\partial x_i\,\partial x_j}(x,y)+\sum_i b_i(x,y)\frac{\partial u^{\varepsilon_n}}{\partial x_i}(x,y)+c(x,y)u^{\varepsilon_n}(x,y) -\frac{\partial^2 u^{\varepsilon_n}}{\partial y^2}(x,y)=f(x,y)
$$

and thus, in the limit

$$-\sum_{i,j} a_{ij}(x,y)\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x,y)+\sum_i b_i(x,y)\frac{\partial u}{\partial x_i}(x,y)+c(x,y)u(x,y)-\frac{\partial^2 u}{\partial y^2}(x,y)=f(x,y)\qquad\text{on } \omega_\alpha .$$

Since we obviously have

$$-\sum_{i,j} a_{ij}(x,y)\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x,y)+\sum_i b_i(x,y)\frac{\partial u}{\partial x_i}(x,y)+c(x,y)u(x,y)-\frac{\partial^2 u}{\partial y^2}(x,y)-f(x,y)\leqslant 0\qquad\text{a.e. in } \mathcal{Q}$$

these facts imply that $u$ is a solution of (31).

We now prove that any such limit, $u$, is a maximal solution of (31). This of course proves that the sequence $u^\varepsilon$ itself converges to the maximal solution of (31). Indeed, let $v$ be any solution of (31). Then $v$ is a subsolution of (9), (2) and (3) since $\beta(t)=0$ if $t\leqslant 0$. It follows that $v\leqslant u^\varepsilon$ for any $\varepsilon>0$ and this proves our claim.

It is also possible to give a probabilistic proof of the preceding result because all of the differential equations have an interpretation in terms of optimal stochastic control. To simplify the presentation we shall present briefly and formally the optimal stochastic control problem associated with (31).

Denote by $\tilde\sigma$ the following $(n+1)\times(n+1)$-matrix

$$\tilde\sigma(x,y)=\begin{pmatrix}\sigma(x,y) & \vdots & 0\\ \cdots\cdots & & \cdots\cdots\\ 0 & \vdots & \sqrt{2}\end{pmatrix},$$

and let $\tilde b(x,y)$ denote the vector in $\mathbb{R}^n\times\mathbb{R}$

$$\tilde b(x,y)=\begin{pmatrix}b(x,y)\\ 0\end{pmatrix}.$$

Let $e_y$ be the vector in $\mathbb{R}^n\times\mathbb{R}$:

$$e_y=\begin{pmatrix}0\\1\end{pmatrix}.$$

Consider the solution, $\xi_{x,y}(t)$, of the controlled stochastic process,

$$d\xi_{x,y}(t)=\Theta(t,\omega)\tilde\sigma(\xi_{x,y}(t))\,d\tilde W_t-\Theta^2(t,\omega)\tilde b(\xi_{x,y}(t))\,dt-\big(1-\Theta^2(t,\omega)\big)e_y\,dt+1_{\{0\}}(\xi_{x,y}(t))\,e_y\,dA_t-1_{\{1\}}(\xi_{x,y}(t))\,e_y\,dA_t$$

where $A_t$ is some continuous nondecreasing process such that $A_0 = 0$ and where $\Theta(t, \omega)$, the control, is any progressively measurable process taking values in $[0, 1]$. Then

$$u(x, y) = \inf_{\Theta} E\left[\int_0^{\tau_{x,y}} \Theta^2(t, \omega) f(\xi_{x,y}(t)) \exp\left\{-\int_0^t \Theta^2(t, \omega) c(\xi_{x,y}(s))\, ds\right\} dt\right]$$

where $\tau_{x,y}$ is the first exit time from $\bar{\mathcal{O}}$ of the process $\xi^1_{x,y}(t)$ where $\xi^1_{x,y}(t)$ denotes the projection of $\xi_{x,y}(t)$ onto $\mathbb{R}^n$.

This control problem corresponds to the situation where at each time and on each trajectory one may choose between the diffusion process (governed by the linear second order differential operator) and the pure deterministic process in the positive $y$-direction. This is a control problem with a degenerate diffusion process. □

2. *Periodic boundary conditions.*

We continue to assume that $\tilde{\mathcal{O}} = (0, 1)$ and that (4) and (15) hold. We still consider the solution, $u^\varepsilon$, of (9) and (2) but we replace (3) by (25) and we still assume $\beta$ satisfies (29).

In the same way as in Proposition III.1 we prove the existence and uniqueness of a solution, $u^\varepsilon$, of (9), (2) and (25). It is again possible to prove the following a priori estimates:

$$\|D_x^\alpha u^\varepsilon\|_{L^\infty(\mathcal{Q})} \leqslant \text{const} \qquad (\text{for } |\alpha| \leqslant 2)$$

and

$$\left\|\frac{\partial^2 u^\varepsilon}{\partial y^2}\right\|_{L^\infty(\mathcal{Q})} \leqslant \text{const}; \qquad \|u^\varepsilon\|_{W^{1,\infty}(\mathcal{Q})} \leqslant \text{const}\,.$$

The next theorem is our main result for this section.

THEOREM III.2. *Under assumptions* (4), (15) *and* (29) *the solution,* $u^\varepsilon$, *of* (9), (2) *and* (25) *converges weakly in* $W^{2,p}(\mathcal{Q})$ (*for* $p < \infty$) *as* $\varepsilon$ *goes to zero to the solution,* $u$, *of the HJB equation*

$$\left|\begin{array}{l} \sup\limits_{y\in(0,1)} \left\{-\sum\limits_{i,j} a_{ij}(x, y) \dfrac{\partial^2 u}{\partial x_i\, \partial x_j}(x) + \sum\limits_{i} b_i(x, y) \dfrac{\partial u}{\partial x_i}(x) \right. \\ \qquad\qquad\qquad\qquad \left. + c(x, y) u(x) - f(x, y)\right\} = 0 \qquad \text{a.e. in } \mathcal{O}\,, \\ u \in W^{2,\infty}(\mathcal{O})\,, \qquad u = 0 \qquad \text{on } \mathcal{O}\,. \end{array}\right.$$

REMARK III.2. It would be very interesting to have estimates on the rate of convergence of $u^\varepsilon$ to $u$. We were able to obtain such estimates only in very special cases (similar to those described in Remark I.5).

REMARK III.3. It is possible to give multidimensional variants of the above problem. For this case we take $\tilde{\mathcal{O}} = (0, L_1)\times\ldots\times(0, L_m)$ and use periodic boundary conditions in $y_i$ $(1\leqslant i\leqslant m)$ and we replace the penalization term $\beta\,\big((1/\varepsilon)(\partial u^\varepsilon/\partial y)\big)$ by $\sum_{i=1}^{m}\beta\big(1/\varepsilon)(\partial u^\varepsilon/\partial y_i)\big)$ (for example).

PROOF OF THEOREM III.2. We need only prove that if $u^\varepsilon$ converges to $\tilde{u}$ in the sup norm on $C^1$ and weakly in $W^{2,p}$, $p<\infty$, then $\tilde{u}\leqslant u$ where $u$ is the proposed limit for $u^\varepsilon$.

If $u^\varepsilon$ converges to $\tilde{u}$ then since $\beta$ is non-negative it is clear that

$$-\sum_{i,j} a_{ij}(x,y)\frac{\partial^2\tilde{u}}{\partial x_i\,\partial x_j}(x,y)+\sum_i b_i(x,y)\frac{\partial\tilde{u}}{\partial x_i}(x,y)$$
$$+\,c(x,y)\tilde{u}(x,y)-\frac{\partial^2\tilde{u}}{\partial y^2}(x,y)\leqslant f(x,y)\qquad\text{a.e. in } \mathcal{Q}\,.$$

In view of the penalization term it is also clear that $\tilde{u}(x,y)=\tilde{u}(x)$ and thus $\tilde{u}\in W^{2,\infty}(\mathcal{O})$ and $\tilde{u}=0$ on $\partial\mathcal{O}$. Therefore

$$\sup_{y\in(0,1)}\left\{-\sum_{i,j}a_{ij}(x,y)\frac{\partial^2\tilde{u}}{\partial x_i\,\partial x_j}(x)+\sum_i b_i(x,y)\frac{\partial\tilde{u}}{\partial x_i}(x)+c(x,y)\tilde{u}(x)-f(x,y)\right\}\leqslant 0$$
$$\text{a.e. in } \mathcal{O}$$

and using the maximum principle as in [25] we conclude that $\tilde{u}\leqslant u$ in $\mathcal{O}$. □

This asymptotic problem can also be understood in terms of optimal stochastic control since (9), (2) and (25) is an HJB equation where only the drift in the $y$-direction is controlled. Using this interpretation it is easy to obtain the limit result; we shall not give more details.

We conclude this section with a comment on the problem (9), (2) and (25). This problem appears to be some kind of regularized, continuous version of the approximation shceme introduced in [18]. In [18] the problem,

$$\max_{1\leqslant i\leqslant m}\left\{-\sum_{k,l}a^i_{kl}(x)\,\frac{\partial^2 u}{\partial x_k\,\partial x_l}(x)+\sum_k b^i(x)\frac{\partial u}{\partial x_k}(x)+c^i(x)u(x)-f^i(x)\right\}=0$$

is approximated by a system written as follows

$$\begin{cases} -\sum_{k,l} a^i_{kl}(x)\dfrac{\partial^2 u^\varepsilon_i}{\partial x_k\,\partial x_l}(x) + \sum_k b^i_k(x)\dfrac{\partial u^\varepsilon_i}{\partial x_k}(x) + c^i(x)u^\varepsilon_i(x) \\ \qquad\qquad + \beta\left(\dfrac{u^\varepsilon_i - u^\varepsilon_{i+1}}{\varepsilon}(x)\right) = f^i(x) \quad \text{in } \mathcal{O}, \\ u^\varepsilon_i = 0 \text{ on } \partial\mathcal{O} \text{ (by definition } u^\varepsilon_i = u^\varepsilon_{m+1}). \end{cases}$$

If we consider $i$ as giving a discretization of $(0,1)$ the continuous version of the problem above is

$$\begin{cases} -\sum_{k,l} a_{kl}(x,y)\dfrac{\partial^2 u^\varepsilon}{\partial x_k\,\partial x_l}(x,y) + \sum_{k,l} b(x,y)\dfrac{\partial u^\varepsilon}{\partial x_k}(x,y) + c(x,y)u^\varepsilon(x,y) \\ \qquad\qquad + \beta\left(\dfrac{1}{\varepsilon}\dfrac{\partial u^\varepsilon}{\partial y}(x,y)\right) = f(x,y) \quad \text{in } \mathcal{Q}, \\ u^\varepsilon = 0 \quad \text{on } \partial\mathcal{O}\times\overline{\mathcal{O}}, \\ u^\varepsilon \text{ periodic in } y. \end{cases}$$

This is nearly the problem (9), (2) and (25); the only difference is that in (9) we have an additional smoothing term, the « viscosity term » $\partial^2 u^\varepsilon/\partial y^2$, which in some sense smoothes the problem. The probabilistic interpretation of the problem above is given in A. Bensoussan and P. L. Lions [4].

## *B*) SECOND ORDER PENALIZATIONS

In this part of our paper we consider penalizations involving the second derivatives in $y$. The notations are the same as in Part $A$) and we assume that the regularity assumption (5) and the ellipticity condition (4) hold. By $\Delta_y$ we mean the Laplacian in $y$, i.e. $\Delta_y = \sum_j \partial^2/\partial y_j^2$.

In Section I we show that the problem

$$(32) \qquad -\sum_{i,j} a_{ij}(x,y)\frac{\partial^2 u^\varepsilon}{\partial x_i\,\partial x_j}(x,y) + \sum_i b_i(x,y)\frac{\partial u^\varepsilon}{\partial x_i}(x,y) + c(x,y)u^\varepsilon(x,y) + \beta\left(-\frac{\Delta_y u^\varepsilon}{\varepsilon}(x,y)\right) = f(x,y) \quad \text{a.e. in } \mathcal{Q}$$

with boundary conditions (2) and (3) is well posed. In Section II we consider the asymptotic problem, i.e. $\varepsilon$ goes to zero. We assume that $\beta$ essentially satisfies (29) in this section. In Section III we again consider the asymptotic problem but in this case we assume that $\beta$ satisfies (26). This case gives what we call nonlinear averaged equations (NLAE); we study these problems in Section IV.

REMARK. A reference to a theorem from the previous part of this paper will be denoted by « Theorem A ... ». For example Theorem II.2 of Part $A$) would be written as « Theorem A.II.2 ».

## I. – Some fully nonlinear elliptic equations.

We consider the following problem: find $u$ a solution of

$$(32') \qquad -\sum_{i,j} a_{ij}(x, y)\frac{\partial^2 u}{\partial x_i\, \partial x_j}(x, y) + \sum_i b_i(x, y)\frac{\partial u}{\partial x_i}(x, y) + c(x, y)u(x, y) + \beta\big(-\Delta_y u(x, y)\big) = f(x, y) \quad \text{a.e. in } \mathcal{Q}$$

plus the boundary conditions (2) and (3). We assume that $\beta$ is an increasing continuous function such that

$$(33) \qquad \beta(0) = 0\,, \qquad \alpha(t-s)^2 \leqslant (t-s)\big(\beta(t)-\beta(s)\big) \leqslant c(t-s)^2 \quad \text{for all } t \text{ and } s\,,$$

where $\alpha$ and $c$ are positive constants.

This problem is clearly highly nonlinear and very few existence results are known for such problems (an exception is the class of HJB equations) (see [15], [16] and [20]).

THEOREM I.1. *Assume* (4), (15) *and* (33). *Then*

i) *there exists a solution, $u$, of* (32′), (2) *and* (3) *in* $H^2(\mathcal{Q}) \cap L^\infty(\mathcal{Q})$.

ii) *If in addition we assume*

(34) there exist constants $C_0$, $C > 0$ and $|\beta(t) - C_0 t| \leqslant C$ for all $t$ in $\mathbb{R}$ *then there exists a unique solution, $u$, of* (32′), (2) *and* (3) *in* $W^{2,p}(\mathcal{Q})$ (*for* $p < \infty$).

iii) *If we assume that $\beta$ is convex then there exists a unique solution, $u$, of* (32′), (2) *and* (3) *in* $W^{2,p}(\mathcal{Q})$ (*for* $p < \infty$) *which also satisfies*

$$D_x^2 u \in L^\infty(\mathcal{Q})\,, \qquad \Delta_y u \in L^\infty(\mathcal{Q})\,.$$

iv) *If $a_{ij}$, $b_i$ and $c$ do not depend on $y$ then there exists a unique solution $u$ of* (32′), (2) *and* (3) *in* $H^2(\mathcal{Q}) \cap L^\infty(\mathcal{Q})$.

REMARK I.1. If $\beta$ is convex it is possible to prove that $u \in W^{2,\infty}_{\mathrm{loc}}(\mathcal{Q})$ and if in addition $\beta$ is smooth and the coefficients are smooth it is possible to prove that $u$ is smooth by a result due to L. C. Evans [17].

The existence in $H^2(\mathcal{Q})$ only requires that $a_{ij} \in C(\overline{\mathcal{Q}})$, $b_i \in L^\infty(\mathcal{Q})$, $c \in L^\infty_+(\mathcal{Q})$ and $f \in L^2(\mathcal{Q})$. Indeed, one only needs to adapt the proof below using the results and methods of P. L. Lions [33]. Using a method somewhat similar to the one used in H. Brezis and L. C. Evans [7] it is possible to show that if $a_{ij}, b_i, c \in W^{1,\infty}(\mathcal{Q})$ and $f \in H^1(\mathcal{Q})$ then there is a solution of (32′), (2) and (3) in $H^3(\mathcal{Q})$.

REMARK I.2. We do not know if the $H^2$ solutions of (32′), (2) and (3) are unique.

Before proceeding to the proof of Theorem I.1 we point out that ii) is an obvious adaptation of a result of L. C. Evans [15] and we do not give a proof. In the case where $\beta$ is convex, iii), (32′) is an HJB equation and an easy modification of the arguments in [19] and [25] yield the results claimed. Therefore, we actually only present proofs for i) and iv) of Theorem I.1.

PROOF OF THEOREM I.1. To simplify our presentation we assume that $\beta \in C^1(\mathbb{R})$. As recalled above ii) is essentially contained in [15] and we use ii) to prove i). Indeed, there is clearly a sequence $\beta_k$ of penalty terms satisfying (34) and which converge uniformly on compact sets to $\beta$. We can assume w.l.o.g. that $\beta_k$ satisfy (33) uniformly in $k$.

Therefore there exists a sequence of unique solutions, $u_k$, of (32′), (2) and (3) with $\beta$ replaced by $\beta_k$; the solutions are in $W^{2,p}(\mathcal{Q})$ (for $p < \infty$).

We claim that there exists $\nu > 0$ and for every $\varepsilon > 0$ a constant $C_\varepsilon$ such that for all $u \in H^2(\mathcal{Q})$ which satisfy (2) and (3) we have

$$\left\|\left(-\sum_{i,j} a_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j} + \sum_i b_i \frac{\partial u}{\partial x_i} + cu + \beta(-\Delta_y u)\right)(-\Delta_y u)\right\|^2_{L^2(\mathcal{Q})} \geqslant \nu\{\|D_y^2 u\|^2_{L^2(\mathcal{Q})} + \|D_x D_y u\|^2_{L^2(\mathcal{Q})}\} - \varepsilon\|u\|^2_{H^2(\mathcal{Q})} - C_\varepsilon\|u\|^2_{H^1(\mathcal{Q})}. \tag{35}$$

Assume (35) for the moment and note that this gives

$$\|D_y^2 u_k\|_{L^2(\mathcal{Q})} + \|D_y D_x u_k\|_{L^2(\mathcal{Q})} \leqslant \varepsilon\|u_k\|_{H^2(\mathcal{Q})} + C_\varepsilon\|u_k\|_{H^1(\mathcal{Q})} \qquad \text{for every } \varepsilon > 0\,.$$

Using (32′) this implies that

$$\|u_k\|_{H^2(\mathcal{Q})} \leqslant C + C\|u_k\|_{H^1(\mathcal{Q})} \quad \text{for some } C \text{ independent of } k\,.$$

It is standard to show by appropriate supersolutions that $\|u_k\|_{L^\infty(\mathcal{Q})} \leqslant \text{const.}$

Therefore we conclude

$$\|u_k\|_{H^2(\mathcal{Q})} \leqslant \text{const} \qquad \text{and} \qquad \|u_k\|_{L^\infty(\mathcal{Q})} \leqslant \text{const} .$$

These are clearly the types of estimates that we desire.

We shall now prove (35). It is enough to prove (35) for smooth functions satisfying (2) and (3); we really need only to prove

$$\int_{\mathcal{Q}} \left(-\sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j}\right)(-\Delta_y u) \geqslant \nu \|D_x D_y u\|^2_{L_2(\mathcal{Q})} - \varepsilon \|u\|^2_{H^2(\mathcal{Q})} - C_\varepsilon \|u\|^2_{H^1(\mathcal{Q})} . \tag{35'}$$

Recall that $a_{ij}$ are Lipschitz continuous. Thus

$$\int_{\mathcal{Q}} \left(-\sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j}\right)(-\Delta_y u) = \sum_{i,j} \int_{\mathcal{Q}} a_{ij} \frac{\partial u}{\partial x_i}\left(-\Delta_y\left(\frac{\partial u}{\partial x_j}\right)\right) + \sum_{i,j} \int_{\mathcal{Q}} \frac{\partial}{\partial x_j}(a_{ij}) \frac{\partial u}{\partial x_i}(-\Delta_y u)$$

$$\begin{aligned}\int_{\mathcal{Q}} \left(-\sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j}\right)(-\Delta_y u) &= \sum_{j,i,k} \int_{\mathcal{Q}} a_{ij} \frac{\partial^2 u}{\partial x_i\, \partial y_k} \frac{\partial^2 u}{\partial x_j\, dy_k} + \sum_{i,j,k} \int \frac{\partial}{\partial y_k}(a_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_j\, \partial y_k} \\ &\quad + \sum_{i,j} \int_{\mathcal{Q}} \frac{\partial}{\partial x_j}(a_{ij}) \frac{\partial u}{\partial x_i}(-\Delta_y u) \geqslant \nu \|D_x D_y u\|^2_{L^2(\mathcal{Q})} - \varepsilon \|\Delta_y u\|^2_{L^2(\mathcal{Q})} - C \|D_x u\|^2_{L^2(\mathcal{Q})} ,\end{aligned}$$

and this is enough to prove our claim.

By our previous arguments $u_k$ are uniformly bounded in $H^2(\mathcal{Q}) \cap L^\infty(\mathcal{Q})$ and we may extract a subsequence (which for simplicity we also write as $u_k$) such that $u_k$ converges weakly in $H^2(\mathcal{Q})$ and strongly in $H^1(\mathcal{Q})$ and whose limit, $u$, is in $H^2(\mathcal{Q}) \cap L^\infty(\mathcal{Q})$ and satisfies (2) and (3).

We shall now show that $u$ satisfies the appropriate equation. To do this it is sufficient to show that

$$\limsup_{k \to \infty} \int_{\mathcal{Q}} (-\Delta_y u_k)(\beta_k(-\Delta_y u_k)) \leqslant \int_{\mathcal{Q}} (-\Delta_y u)\,\psi$$

where

$$\psi \equiv f + \sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j} - \sum_i b_i \frac{\partial u}{\partial x_i} - cu .$$

(Since $\beta_k$ converges to $\beta$ in the sense of graphs we may apply H. Brezis [6] and using the inequality above the desired result is obtained.)

Now we know

$$\int_{\mathscr{Q}}(-\Delta_y u_k))\beta_k(-\Delta_y u_k)) = \int_{\mathscr{Q}}(-\Delta_y u_k)f + \sum_{i,j}\int_{\mathscr{Q}}(-\Delta_y u_k)a_{ij}\frac{\partial^2 u_k}{\partial x_i\,\partial x_j} - \sum_i \int_{\mathscr{Q}}(-\Delta_y u_k)b_i\frac{\partial u_k}{\partial x_i} - \int_{\mathscr{Q}}(-\Delta_y u_k)cu_k$$

and since we may assume $u_k \to u$ strongly in $H^1(\mathscr{Q})$ the inequality that we need to prove is

$$\limsup_{k\to\infty}\sum_{i,j}\int_{\mathscr{Q}}(-\Delta_y u_k)a_{ij}\frac{\partial^2 u_k}{\partial x_i\,\partial x_j} \leqslant \sum_{i,j}\int_{\mathscr{Q}}(-\Delta_y u)a_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j}\,.$$

However, as we have seen in an earlier calculation

$$\sum_{i,j}\int_{\mathscr{Q}}(-\Delta_y u_k)a_{ij}\frac{\partial^2 u_k}{\partial x_i\,\partial x_j} = -\sum_{i,j,l}\int_{\mathscr{Q}}a_{ij}\frac{\partial^2 u_k}{\partial y_l\,\partial x_i}\,\frac{\partial^2 u_k}{\partial y_l\,\partial x_j} - \sum_{i,j,l}\int_{\mathscr{Q}}\frac{\partial a_{ij}}{\partial y_l}\,\frac{\partial u_k}{\partial x_j}\,\frac{\partial^2 u_k}{\partial y_l\,\partial x_i} - \sum_{i,j}\int_{\mathscr{Q}}\frac{\partial a_{ij}}{\partial x_i}\,\frac{\partial u_k}{\partial x_j}\,(-\Delta_y u_k)\,,$$

and the last two terms on the right of the equals sign converge to the same terms but with $u_k$ replaced by $u$. Since the first term on the right of the equals sign is convex in $D^2 u_k$ the weak convergence of $u_k$ to $u$ in $H^2(\mathscr{Q})$ implies that in the limit the first term on the right of the equals sign may be replaced by the same term with $u$ in place of $u_k$ if the equals sign is changed to the inequality « $\leqslant$ ». This proves part i) of our theorem.

We now consider part iv) of Theorem I.1. We note that there is a $\lambda_0 > 0$ such that: for some $\nu > 0$

$$\int_{\mathscr{Q}}\Big(-\sum_{i,j}a_{ij}\frac{\partial^2(u-v)}{\partial x_i\,\partial x_j} + \sum_i b_i\frac{\partial(u-v)}{\partial x_i} + c(u-v) + \lambda_0(u-v) + \beta(-\Delta_y u) - \beta(-\Delta_y v)\Big)(-\Delta_y(u-v)) \geqslant \nu\{\|D_y^2(u-v)\|^2_{L^2(\mathscr{Q})} + \|D_x D_y(u-v)\|^2_{L^2(\mathscr{Q})} + \|D_y(u-v)\|^2_{L^2(\mathscr{Q})}\}\,.$$

Indeed, the same argument as presented earlier in this proof shows that the quantity on the left of the equals sign is larger than

$$\nu_0\{\|D_y^2(u-v)\|^2_{L^2(\mathscr{Q})} + \|D_x D_y(u-v)\|^2_{L^2(\mathscr{Q})}\} - C\|D_y(u-v)\|_{L^2(\mathscr{Q})}\|D_x D_y(u-v)\|_{L^2(\mathscr{Q})} + \lambda_0\|D_y(u-v)\|^2_{L^2(\mathscr{Q})}.$$

By choosing $\nu = \nu_0/2$ and $\lambda_0$ sufficiently large we obtain the desired inequality.

We claim that the inequality just established guarantees that the solution, $u$, already constructed is unique for any $f \in L^\infty(\mathcal{Q})$ if $c(x) \geq \lambda_0$.

Indeed, by writing $c(x)$ as $c_0(x) + \lambda_0$ we may apply our previous inequality to obtain $D_y(u - v) = 0$, where $u$ and $v$ are solutions. Thus $(u - v)$ depends only on $x$ and by setting $w = u - v$ we find that

$$\begin{cases} -\sum_{i,j} a_{ij}(x) \dfrac{\partial^2 w}{\partial x_i\, \partial x_j}(x) + \sum_i b_i(x) \dfrac{\partial w}{\partial x_i}(x) + c_0(x) w(x) + \lambda_0 w(x) = 0 & \text{in } \mathcal{O}\,, \\ w = 0 \quad \text{on } \partial\mathcal{O}\,. \end{cases}$$

By uniqueness for linear problems we conclude $w \equiv 0$ in $\mathcal{O}$ and this establishes our claim.

Next we prove that if $u$ and $v$ are two solutions with different inhomogeneous terms $f$ and $g$ respectively then

$$\|u - v\|_{L^\infty(\mathcal{Q})} \leq C_{\lambda_0} \|f - g\|_{L^\infty(\mathcal{Q})}$$

where $C_{\lambda_0} < 1/\beta_0$.

It is clearly sufficient to prove this for the approximating solutions $u_n$ and $v_n$. In fact we prove

$$|(u_n - v_n)(x, y)| \leq w(x) \|f - g\|_{L^\infty(\mathcal{Q})} \quad \text{in } \overline{\mathcal{Q}}$$

where $w$ is the solution of

$$\begin{cases} -\sum_{i,j} a_{ij}(x) \dfrac{\partial^2 w}{\partial x_i\, \partial x_j}(x) + \sum_i b_i(x) \dfrac{\partial w}{\partial x_i}(x) + c(x) w(x) + \lambda_0 w(x) = 1 & \text{in } \mathcal{O} \\ w \in C^2(\overline{\mathcal{O}})\,, \quad w = 0 \quad \text{on } \partial\mathcal{O}\,. \end{cases}$$

This claim is an easy consequence of the maximum principle (use the version in [5]). By the strong maximum principle for linear equations we have

$$\|w\|_{L^\infty(\mathcal{Q})} < \frac{1}{\lambda_0}\,.$$

Denoting $\|w\|_{L^\infty(\mathcal{Q})}$ by $C_{\lambda_0}$ we have the desired result.

We now can finish our proof of the uniqueness of solutions of (32′), (2) and (3) when the coefficients do not depend on $y$. Indeed, if $u$ and $v$ are two solutions in $H^2(\mathcal{Q}) \cap L^\infty(\mathcal{Q})$ then they are also solutions of

$$\begin{aligned} &-\sum_{i,j} a_{ij}(x) \frac{\partial^2 u}{\partial x_i\, \partial x_j}(x, y) + \sum_i b_i(x) \frac{\partial u}{\partial x_i}(x, y) \\ &\qquad + c(x) u(x, y) + \lambda_0 u(x, y) + \beta(-\Delta_y u(x, y)) = f(x) + \lambda_0 u(x, y) \quad \text{in } \mathcal{Q} \end{aligned}$$

(and similarly for $v$). By our previous work we find

$$\|u - v\|_{L^\infty(\mathcal{Q})} \leqslant C_{\lambda_0} \lambda_0 \|u - v\|_{L^\infty(\mathcal{Q})}$$

and since $C_{\lambda_0} < 1/\beta_0$ we conclude $u = v$ in $\mathcal{Q}$. □

## II. – One-sided penalization.

In this section we consider the following equation with boundary condition (2) and (3). Consider the solution, $u^\varepsilon$, of

$$(36) \qquad -\sum_{i,j} a_{ij}(x, y) \frac{\partial^2 u^\varepsilon}{\partial x_i \, \partial x_j}(x, y) + \sum_i b_i(x, y) \frac{\partial u^\varepsilon}{\partial x_i}(x, y) \\ + c(x, y) u^\varepsilon(x, y) - \Delta_y u^\varepsilon(x, y) + \beta\left(-\frac{1}{\varepsilon} \Delta_y u^\varepsilon\right) = f(x, y) \quad \text{in } \mathcal{Q}$$

plus boundary conditions (2) and (3) and where we still assume (4) and (15) and $\beta$ satisfies

$$(37) \qquad \beta(t) = 0 \quad \text{if } t \leqslant 0; \quad \beta(t) > 0 \quad \text{if } t > 0\,, \quad \beta(t) \to \infty \quad \text{as } t \to \infty; \\ \beta \text{ is convex and Lipschitz on } \mathbb{R}\,.$$

In view of Theorem I.1 we know that there is a unique solution, $u^\varepsilon$, of (36), (2) and (3); $u^\varepsilon \in W^{2,p}(\mathcal{Q})$ for $p < \infty$. In addition, by the methods of estimation employed in [19] and [25] we have

$$\|D_x^\alpha u^\varepsilon\|_{L^\infty(\mathcal{Q})} \leqslant \text{constant} \qquad (\text{independent of } \varepsilon) \text{ for } |\alpha| \leqslant 2\,.$$

This implies (from (36)) that

$$\left\| -\Delta_y u^\varepsilon + \beta\left(-\frac{1}{\varepsilon} \Delta_y u^\varepsilon\right)\right\|_{L^\infty(\mathcal{O})} \leqslant \text{constant} \qquad (\text{independent of } \varepsilon)$$

and hence

$$\|-\Delta_y u^\varepsilon\|_{L^\infty(\mathcal{Q})} \leqslant \text{constant}; \qquad \left\|\beta\left(-\frac{1}{\varepsilon} \Delta_y u^\varepsilon\right)\right\|_{L^\infty(\mathcal{Q})} \leqslant \text{constant}\,.$$

THEOREM II.1. *Under assumptions* (4), (15) *and* (36) *the solution,* $u^\varepsilon$, *of* (36), (2) *and* (3) *satisfies*

$$\|D_x^\alpha u^\varepsilon\|_{L^\infty(\mathscr{Q})} \leqslant C\,, \quad \text{for } |\alpha| \leqslant 2\,; \quad \|\Delta_y u^\varepsilon\|_{L^\infty(\mathscr{Q})} \leqslant C\,; \tag{38}$$

$$\left\|\beta\left(-\frac{1}{\varepsilon}\,\Delta_y u^\varepsilon\right)\right\|_{L^\infty(\mathscr{Q})} \leqslant C \quad \textit{for some } C \textit{ independent of } \varepsilon\,.$$

*In particular,* $u^\varepsilon$ *is uniformly bounded in* $W^{2,p}(\mathscr{Q})$.

*As* $\varepsilon$ *goes to zero,* $u^\varepsilon$ *converges weakly in* $W^{2,p}(\mathscr{Q})$ *to the solution,* $u$, *of the HSB equation,* (5).

We shall now give our analytic proof of Theorem II.1; then we give the optimal stochastic control interpretation (note that since $\beta$ is convex (36) is an HJB equation).

PROOF OF THEOREM II.1. Let $u^\varepsilon$ converge weakly in $W^{2,p}(\mathscr{Q})$ to some function $u(x, y)$ satisfying (2) and (3) and

$$D_x^\alpha u \in L^\infty(\mathscr{Q}) \quad \text{for } |\alpha| \leqslant 2\,; \quad \Delta_y u \in L^\infty(\mathscr{Q})\,.$$

By (38) we also know that $-\Delta_y u \leqslant 0$ in $\mathscr{Q}$; by integrating this inequality over $\mathscr{Q}$ we get

$$0 \geqslant \int_{\mathscr{Q}} (-\,\Delta_y u) = \int_{\mathscr{O}} \left(\int_{\mathscr{O}} -\,\Delta_y u\right) = \int_{\mathscr{O}} \left(\int_{\partial\tilde{\mathscr{O}}} -\frac{\partial u}{\partial n}\right) = 0\,.$$

The inequality is strict unless $-\Delta u_y = 0$ a.e. in $\mathscr{Q}$ and then by (3) and uniqueness for linear equations we conclude that $u$ is independent of $y$.

Since $\beta$ is non-negative we find that $u$ satisfies

$$-\sum_{i,j} a_{ij}(x, y)\,\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x) + \sum_i b_i(x, y)\,\frac{\partial u}{\partial x_i}(x) + c(x, y)\,u(x) \leqslant f(x, y) \qquad \text{in } \mathscr{O}$$

and this implies $u \leqslant \tilde{u}$ in $\bar{\mathscr{O}}$ where $\tilde{u}$ is the solution of the HJB equation (5) with boundary condition (2). On the other hand since $\tilde{u}$ is a subsolution of (36), (2) and (3) we deduce (by the maximum principle of Bony, [5]) that

$$\tilde{u} \leqslant u^\varepsilon \quad \text{in } \bar{\mathscr{Q}}\,.$$

Thus $u = \tilde{u}$. □

We conclude this section with an *optimal stochastic control* interpretation of this result. To simplify matters we assume $\beta(t) = t^+$.

For any progressively measurable process $q(t, \omega)$ taking values between 0 and $1/\varepsilon$ we consider $(\xi, \eta) = (\xi_{x,y}, \eta_{x,y})$ solutions of the stochastic differential equations

$$d\xi(t) = \sigma(\xi, \eta)\, dW_t - b(\xi, \eta)\, dt\,, \qquad \xi(0) = x$$

$$d\eta(t) = \sqrt{2}\,(1 + q(t, \omega))\, d\tilde{W}_t - 1_{\partial\bar{\mathcal{O}}}(\eta(t))\, n(\eta(t))\, dA_t$$

where $A_t$ is some continuous adapted increasing process such that $A_0 = 0$ and $(W_t, \tilde{W}_t)$ is some normalized Brownian motion.

It is possible to prove that $u^\varepsilon$ is given by

$$u^\varepsilon(x, y) = \inf_{0 \leqslant q \leqslant 1/\varepsilon} E\left[\int_0^{\tau_{x,y}} f(\xi(t), \eta(t)) \exp\left\{-\int_0^t c(\xi(s), \eta(s))\, ds\right\} dt\right]$$

where, as before, $\tau_{x,y}$ is the exit time from $\bar{\mathcal{O}}$ of $\xi(t)$. Our result shows that as $\varepsilon$ goes to zero, *all possible processes* $q(t)$ *given above «approximate»* (*in some sense*) *all possible progressively measurable processes taking values in* $\bar{\bar{\mathcal{O}}}$.

It is worth noting that the result above gives a method to approximate any (continuous time) stochastic control problem where essentially only the intensity of the Brownian motion is controlled.

## III. – Nonlinear averaging principle.

We now turn our attention to the problem (32), (2) and (3) where we still assume (4) and (15) hold and where we assume that $\beta$ satisfies (26).

We denote $\beta^{-1}$ by $\gamma$; in view of Theorem I.1 we know there is a solution $u^\varepsilon$ of (32), (2) and (3) in $H^2(\mathcal{Q}) \cap L^\infty(\mathcal{Q})$. In addition, we know that this solution is unique if any of the following conditions are satisfied:

i) $\beta$ is convex;

ii) (34);

iii) the coefficients $a_{ij}$, $b_i$, $c$ do not depend on $y$.

Our main result of this section is:

THEOREM III.1. *Assume* (4), (15) *and* (26) *hold. If the coefficients* $(a_{ij})$ *do not depend on* $y$ *then any solution*, $u^\varepsilon$, *of* (32), (2) *and* (3) *is bounded in* $H^2(\mathcal{Q})$ (*independent of* $\varepsilon$). *Furthermore, as* $\varepsilon$ *goes to zero* $u^\varepsilon$ *converges weakly in* $H^2(\mathcal{Q})$ *to the solution*, $u$, *of* (14) *and* (2).

REMARK III.1. If $\beta$ is convex the solution of (14) and (2) is in $W^{2,\infty}(\mathcal{O})$.

REMARK III.2. If $\beta(t) = C_0 t$ then (34) is clearly satisfied and our result yields the particular case (well known when $\mathcal{O} = \emptyset$):

$$u^\varepsilon(x, y) \to u(x)$$

$$-\sum_{i,j} \bar{a}_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_i \bar{b}_i(x) \frac{\partial u}{\partial x_i}(x) + \bar{c}(x) u(x) = \bar{f}(x) \quad \text{in } \tilde{\mathcal{O}}$$

where

$$\bar{a}_{ij}(x) = \frac{1}{\text{measure } (\tilde{\mathcal{O}})} \int_{\tilde{\mathcal{O}}} a_{ij}(x, \cdot)$$

and similarly for $\tilde{b}_i(x)$, $\bar{c}(x)$ and $\bar{f}(x)$.

When $\beta$ is not linear then (14) is a nonlinearly averaged equation. There are a number of examples for higher order problems which indicate that the nonlinear averaging phenomena is not a property of just second order operators nor just elliptic operators (see [8] and [22] for related results in the context of Navier-Stokes equations).

REMARK III.3. From the stochastic point of view the case when $\beta$ is linear is a straightforward ergodic problem. When $\beta$ is nonlinear, say for example $\beta$ is convex, then Theorem III.1 is a combination of optimal stochastic control and ergodic theory.

We shall only prove the a priori estimates and establish the weak convergence of a subsequence of solutions, $u^\varepsilon$, of (32), (2) and (3) to a solution, $u$, of (14) and (2). In the next section we prove uniqueness results for solutions of (14) and (2) and in conjunction with the results we are about to establish this will yield a complete proof of Theorem III.1.

PROOF OF THEOREM III.1.

i) Proof of a priori estimates. We shall first prove that $u^\varepsilon$ is uniformly bounded in $L^\infty(\mathcal{Q})$; in view of the proof of existence of solutions for (32), (2) and (3) it will be sufficient to prove that there exist sub- and supersolutions independent of $\beta$ and in $L^\infty(\mathcal{Q})$.

Let $\bar{u}$ (resp. $u$) be the solution $\bar{u}(x, y) = \bar{u}(x)$ where $\bar{u}$ is the solution of

$$\begin{cases} \inf\limits_{y \in \tilde{\mathcal{O}}} \Big\{ -\sum_{i,j} a_{ij}(x, y) \dfrac{\partial^2 \bar{u}}{\partial x_i \partial x_j}(x) + \sum_i b_i(x, y) \dfrac{\partial \bar{u}}{\partial x_i}(x) \\ \qquad\qquad + c(x, y)\bar{u}(x) - \|f\|_{L^\infty(\mathcal{Q})} \Big\} = 0 \quad \text{in } \mathcal{O} \\ \bar{u} \in W^{2,\infty}(\mathcal{O}), \quad \bar{u} = 0 \quad \text{on } \partial\mathcal{O} \end{cases}$$

(resp.

$$\begin{cases} \sup_{v\in\tilde{\mathcal{O}}}\Big\{-\sum_{i,j} a_{ij}(x,y)\dfrac{\partial^2 \underline{u}}{\partial x_i\,\partial x_j}(x) + \sum_i b_i(x,y)\dfrac{\partial \underline{u}}{\partial x_i}(x) \\ \qquad\qquad + c(x,y)\,\underline{u}(x) + \|f\|_{L^\infty(\mathcal{D})}\Big\} = 0 \quad \text{in } \mathcal{O} \\ \underline{u}\in W^{2,\infty}(\mathcal{O})\,, \quad \underline{u} = 0 \quad \text{on } \partial\mathcal{O}\,. \end{cases}$$

It is clear that $\bar{u}$ is a supersolution and $\underline{u}$ is a subsolution and both are independent of $\beta$. From the proof of Theorem I.1 it follows that $\underline{u}\leqslant u^\varepsilon\leqslant\bar{u}$ in $\bar{\mathcal{D}}$.

If $a_{ij}$ does not depend on $y$ we proceed as follows. Multiply (32) by $(-\Delta_y u^\varepsilon)$ and integrate to obtain

$$\frac{C_0}{\varepsilon}\|\Delta_y u^\varepsilon\|^2_{L^2(\mathcal{D})} + \int_{\mathcal{D}}\Big(-\sum_{i,j} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_j}\Big)(-\Delta_y u^\varepsilon) \leqslant \big(C + C\|u^\varepsilon\|_{L^2(\mathcal{D})} + C\|D_x u^\varepsilon\|_{L^2(\mathcal{D})}\big)\|\Delta_y u^\varepsilon\|_{L^2(\mathcal{D})}$$

where $C$ denotes various constants independent of $\varepsilon$ and $u^\varepsilon$.

Integrating by parts twice, as in the proof of Theorem I.1, we deduce: there is $\nu > 0$ (independent of $\varepsilon$) such that

$$\frac{C_0}{\varepsilon}\|\Delta_y u^\varepsilon\|^2_{L^2(\mathcal{D})} + \nu\|D_x D_y u^\varepsilon\|^2_{L^2(\mathcal{D})} \leqslant \big(C + C\|u^\varepsilon\|_{L^2(\mathcal{D})} + C\|D_x u^\varepsilon\|_{L^2(\mathcal{D})}\big)\|\Delta_y u^\varepsilon\|_{L^2(\mathcal{D})}\,.$$

In particular this shows that

$$\frac{1}{\varepsilon}\|\Delta_y u^\varepsilon\|_{L^2(\mathcal{D})} \leqslant C + C\|u^\varepsilon\|_{H^1(\mathcal{D})}$$

and

$$\|D_x D_y u^\varepsilon\|_{L^2(\mathcal{D})} \leqslant C + C\|u^\varepsilon\|_{H^1(\mathcal{D})}\,.$$

Using these inequalities and equation (32) we deduce

$$\|D_x^2 u^\varepsilon\|_{L^2(\mathcal{D})} \leqslant C + C\|u^\varepsilon\|_{H^1(\mathcal{D})}\,.$$

In particular

$$\|u^\varepsilon\|_{H^2(\mathcal{D})} \leqslant C + C\|u^\varepsilon\|_{H^1(\mathcal{D})}$$

and we conclude as we did before.

ii) Proof of convergence. As we have seen, $\|(1/\varepsilon)\,\Delta_y u^\varepsilon\|_{L^2(\mathscr{Q})}$ is uniformly bounded in $\varepsilon$. We shall now show that if $u^\varepsilon$ converges weakly to some function $u(x, y) \in H^2(\mathscr{Q})$ then $u(x, y) = u(x)$ and $u$ satisfies (14) and (2). Indeed, $u$ clearly satisfies (2) and (3) and $\Delta_y u^\varepsilon = 0$ in $\mathscr{Q}$. Therefore $u(x, y) = u(x)$ and $u(x)$ satisfies (2).

To prove (14) holds for $u$ we need to prove that if $(-1/\varepsilon)\,\Delta_y u^\varepsilon$ converges in $L^2(\mathscr{Q})$ to $\varphi$ in $L^2(\mathscr{Q})$ then

$$\limsup_{\varepsilon} \int_{\mathscr{Q}} \left(-\frac{1}{\varepsilon}\,\Delta_y u^\varepsilon\right) \beta\left(\frac{1}{\varepsilon}\,\Delta_y u^\varepsilon\right) \leqslant \int_{\mathscr{Q}} \varphi\psi$$

where

$$\psi(x, y) = f(x, y) + \sum_{i,j} a_{ij}(x, y) \frac{\partial^2 u}{\partial x_i\,\partial x_j}(x) - \sum_i b_i(x, y) \frac{\partial u}{\partial x_i}(x) - c(x, y) u(x)\,.$$

However

$$\int_{\mathscr{Q}} \left(-\frac{1}{\varepsilon}\,\Delta_y u^\varepsilon\right) \beta\left(\frac{1}{\varepsilon}\,\Delta_y u^\varepsilon\right) = \int_{\mathscr{Q}} \left(-\frac{1}{\varepsilon}\,\Delta_y u^\varepsilon\right)\left(f + \sum_{i,j} a_{ij} \frac{\partial^2 u^\varepsilon}{\partial x_i\,\partial x_j} - \sum_i b_i \frac{\partial u^\varepsilon}{\partial x_i} - c u^\varepsilon\right),$$

and since $u^\varepsilon$ converges strongly in $H^1(\mathscr{Q})$ to $u$ we need only to prove

$$\limsup_{\varepsilon} \int_{\mathscr{Q}} \left(-\frac{1}{\varepsilon}\,\Delta_y u^\varepsilon\right)\left(\sum_{i,j} a_{ij} \frac{\partial u^\varepsilon}{\partial x_i\,\partial x_j}\right) \leqslant \int_{\mathscr{Q}} \varphi \left(\sum_{i,j} a_{ij} \frac{\partial^2 u^\varepsilon}{\partial x_i\,\partial x_j}\right).$$

By a calculation similar to the one in the proof of Theorem I.1 we have

$$\int_{\mathscr{Q}} \left(-\frac{1}{\varepsilon}\,\Delta_y u^\varepsilon\right)\left(\sum_{i,j} a_{ij} \frac{\partial^2 u^\varepsilon}{\partial x_i\,\partial x_j}\right) \leqslant \sum_{i,j} \int_{\mathscr{Q}} \frac{\partial a_{ij}}{\partial x_i} \frac{\partial u^\varepsilon}{\partial x_j}\left(-\frac{1}{\varepsilon}\,\Delta_y u^\varepsilon\right)$$

and since $u^\varepsilon$ converges strongly in $H^1(\mathscr{Q})$ we obtain

$$\limsup_{\varepsilon} \int_{\mathscr{Q}} \left(-\frac{1}{\varepsilon}\,\Delta_y u^\varepsilon\right)\left(\sum_{i,j} a_{ij} \frac{\partial^2 u^\varepsilon}{\partial x_i\,\partial x_j}\right) \leqslant \int_{\mathscr{Q}} \varphi \left(\sum_{i,j} \frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j}\right).$$

Since $u^\varepsilon$ satisfies (3) we have $\int_{\mathscr{O}} \varphi(x, \cdot) = 0$ a.e. in $\mathscr{O}$; thus

$$\int_{\mathscr{Q}} \varphi \left(\sum_{i,j} \frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j}\right) = 0 = \int \varphi \left(\sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x_i\,\partial x_j}\right).$$

This finishes our proof. □

REMARK III.4. If $\beta$ is convex or satisfies (34) it is still possible to prove some a priori estimates (independent of $\varepsilon$) and pass to the limit in order to obtain (14). However, the arguments used in passing to the limit involve some messy technical modifications and so we do not present them.

REMARK III.5. If in (32) one replaces $\beta((1/\varepsilon)(-\Delta_y u^\varepsilon))$ by $(1/\varepsilon)\beta(-\Delta_y u^\varepsilon)$ then under the same assumptions as in Theorem III.1 $u^\varepsilon$ converges to $u$, the solution of the linear averaged equation with coefficients $\bar{a}_{ij}$, $\bar{b}_i$, $\bar{c}$ and $\bar{f}$ as given in Remark III.2. The proof is similar to the proof of Proposition A.II.1.

## IV. – Study of NLAE equations.

In this section we briefly study the equation (14) with boundary condition (2). We continue to assume (4) and (15) and assume that $\gamma$ satisfies (26). We could consider different boundary conditions but we shall restrict ourselves here to Dirichlet conditions.

THEOREM IV.1. *Assume* (4), (15) *and* (26) *hold.*

i) *If $\gamma$ is convex then there is a unique solution $u$ of* (14) *and* (2) *in* $W^{2,\infty}(\mathcal{O})$. *In addition, if $\gamma \in C^\infty$, $a_{ij}, b_i, c, f \in C^\infty$ then $u \in C^\infty(\mathcal{O})$.*

ii) *If $\gamma$ satisfies* (34) *then there is a unique solution $u$ of* (14) *and* (2) *in $W^{2,p}(\mathcal{O})$ for $p < \infty$.*

iii) *If $a_{ij}$ do not depend on $y$ then is a unique solution $u$ of* (14) *and* (2) *in $H^2(\mathcal{O})$. In addition if $\gamma \in C^\infty$, $a_{ij}, b_i, c, f \in C^\infty$ then $u \in C^\infty$.*

iv) *There is at most one solution of* (14) *and* (2) *in* $W^{2,n}(\mathcal{O})$.

PROOF OF THEOREM IV. 1.

Proof of iv). Denote the $n\times n$ symmetric matrices by $S_n$ and define the following function on $S_n \times \mathbb{R}^n \times \mathbb{R} \times \bar{\mathcal{O}}$:

$$F(\xi_{ij}, p_i, t, x) \equiv \int_{\tilde{\mathcal{O}}} \gamma\Big\{f(x,\cdot) + \sum_{i,j} a_{ij}(x,\cdot)\xi_{ij} - \sum_i b_i(x,\cdot)p_i - c(x,\cdot)t\Big\}.$$

Thus (14) is equivalent to

$$F\left(\frac{\partial^2 u}{\partial x_i\,\partial x_j}, \frac{\partial u}{\partial x_i}, u, x\right) = 0 \quad \text{a.e. in } \mathcal{O}. \tag{14'}$$

We have $\partial F/\partial t \leqslant 0$ and $(\partial F/\partial\xi_{ij})\xi_i\xi_j \geqslant c_0\alpha|\xi|^2 > 0$. Part iv) is then just a consequence of [37].

Proof of i). With the preceding definition $F$ is convex and (14′) is an HJB equation. We now apply the results of [25] and [19]. Regularity is a consequence of [17].

Proof if ii). If $\gamma$ satisfies (34) then clearly

$$F(\xi_{ij}, p_i, t, x) = -c_0\Big\{-\sum_{i,j}\bar{a}_{ij}(x)\xi_{ij} + \sum_i \bar{b}_i(x)p_i + \bar{c}(x)t\Big\} + \Phi(\xi_{ij}, p_i, t, x)$$

where $\bar{a}_{ij}$, $\bar{b}_i$, $\bar{c}$ are as defined previously and $\Phi$ is uniformly bounded (because of (34)). Then ii) follows from the results of [15].

Proof of iii). If $a_{ij}$ do not depend on $y$ we know by the preceding section that there exists a solution $u$ of (14) in $H^2(\mathcal{O})$. Let us now prove the uniqueness of this solution.

Let $u$ and $v$ be two solutions of (14) or equivalently (14′). From (14′) we find

$$\begin{aligned}-\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i\,\partial x_j}(u-v)&\\ +\Big(\int_0^1 \frac{\partial F}{\partial p_i}\big(\theta(D^2u, Du, u) + (1-\theta)(D^2v, Dv, v); x\big)\,d\theta\Big)&\frac{\partial}{\partial x_i}(u-v)\\ +\Big(\int_0^1 \frac{\partial F}{\partial t}\big(\theta(D^2u, Du, u) + (1-\theta)(D^2v, Dv, v); x\big)\,d\theta\Big)&(u-v) = 0 \quad \text{in } \mathcal{O}\end{aligned}$$

and

$$u - v = 0 \quad \text{on } \partial\mathcal{O}\,.$$

By (26) and the assumption on the coefficients $b_i$ and $c$ we find that $\partial F/\partial p_i$ and $\partial F/\partial t$ are both uniformly bounded and by bootstrapping $u - v \in W^{2,p}(\mathcal{O})$ for $p < \infty$ and uniqueness follows from [5].

We turn our attention to the regularity problem now. It will be sufficient to prove $C^{2,\alpha}$ regularity assuming that everything is smooth. We first remark that there are coefficients $\tilde{b}_i \in L^\infty(\mathcal{O})$ and $\tilde{c} \in L^\infty(\mathcal{O})$ such that

$$-\sum_{i,j} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x) + \sum_i \tilde{b}_i(x)\frac{\partial u}{\partial x_i}(x) + \tilde{c}(x)u(x) = \tilde{f}(x) \quad \text{in } \mathcal{O}$$

where $\tilde{f} \in L^\infty(\mathcal{O})$. Thus $u \in W^{2,p}(\mathcal{O})$ for $p < \infty$. Then

$$f(x, y) - \sum_i b(x, y) \frac{\partial u}{\partial x_i}(x) - c(x, y) u(x)$$

is in $C^{0,\alpha}(\overline{\mathcal{Q}})$ for $0 < \alpha < 1$. Denote this function by $h(x, y)$. We have

$$\int_{\tilde{\mathcal{O}}} \gamma \left( h(x, y) + \sum_{i,j} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \, \partial x_j}(x) \right) dy = 0 \quad \text{in } \mathcal{O}$$

but if $t(x)$ is the unique value such that

$$\int_{\tilde{\mathcal{O}}} \gamma(h(x, y) + t(x)) \, dy = 0 \quad \text{for } x \in \mathcal{O}$$

then

$$- \sum_{i,j} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \, \partial x_j}(x) - t(x)$$

and $t(x) \in C^{0,\alpha}(\overline{\mathcal{O}})$, so by the Schauder estimates $u \in C^{2,\alpha}$. □

Remark IV.1. Part ii) of Theorem IV.1 only needs $f \in L^\infty(\mathcal{Q})$, $a_{ij} \in C(\overline{\mathcal{Q}})$, $b_i \in L^\infty(\mathcal{Q})$, $c \in L^\infty_+(\mathcal{Q})$, (4) and (33). For part iv) if we only have $f \in L^\infty(\mathcal{Q})$, $a_{ij} \in C(\overline{\mathcal{Q}})$, $b_i \in L^\infty(\mathcal{Q})$, $c \in L^\infty_+(\mathcal{Q})$, (4) and

(26′) $\gamma \in C(\mathbb{R})$, $\gamma$ is strictly increasing on $\mathbb{R}$ and $\gamma(\mathbb{R}) = \mathbb{R}$.

Then the existence and uniqueness results in $H^2(\mathcal{O})$ still hold; in addition $u \in W^{2,p}(\mathcal{O})$ for $p < \infty$. If $\gamma$ is $C^1$ and $\gamma'$ is bounded away from zero then the regularity statement is still true.

Remark IV.2. The existence of any solution, $u$, of (14) and (2) under assumptions (4), (15) and (26) is an open question.

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