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Some asymptotic properties of the local time of the uniform empirical process

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We study the almost sure asymptotic properties of the local time of the uniform empirical process. In particular, we obtain two versions of the law of the iterated logarithm for the integral of the square of the local time. It is interesting to note that the corresponding problems for the Wiener process remain open. Properties of L^p-norms of the local time are studied. We also characterize the joint asymptotics of the local time at a fixed level and the maximum local time.

Keywords: Brownian bridge; empirical process; local time

1. Introduction

Let $U_1, U_2, \ldots$ denote a sequence of independent random variables, all uniformly distributed on (0, 1). Define the uniform empirical process

$$\alpha_n(t) \stackrel{\text{def}}{=} n^{1/2}(F_n(t) - t), \qquad 0 \leqslant t \leqslant 1,$$

where $F_n(\cdot)$ is the empirical distribution function based on the first n observations, i.e.,

$$F_n(t) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{U_i \leqslant t\}}, \qquad 0 \leqslant t \leqslant 1.$$

We are interested in the (normalized) level crossings of the empirical process, defined by

$$L_1^x(\alpha_n) \stackrel{\text{def}}{=} n^{-1/2} \sum_{t \leqslant 1} \mathbb{1}_{\{\alpha_n(t) = x\}}, \qquad x \in \mathbb{R}.$$

From a statistical point of view, the study of such functionals of the empirical process is motivated by some nonparametric problems related to goodness-of-fit tests; see for example, Gaenssler and Gutjahr (1985).

It is easily checked (see Shorack and Wellner 1986, pp. 398–399) that $L_1^x(\alpha_n)$ is also the local time of α_n at x. Throughout the paper, for any stochastic process Z indexed by [0, 1] or $\mathbb{R}_+$, we write $L_t^x(Z)$ for the local time – whenever it is well defined – of Z at (level) x up to time t. More precisely, for any bounded Borel function f,

$$\int_0^t f(Z(s))\,\mathrm{d}s = \int_{\mathbb{R}} f(x) L_t^x(Z)\,\mathrm{d}x. \tag{1.1}$$

One of the first results in the literature concerning the local time of α_n is the following weak convergence, obtained by Dwass (1961):

$$\lim_{n\to\infty} \mathbb{P}(L_1^0(\alpha_n) > x) = \exp\left(-\frac{x^2}{2}\right), \qquad x > 0. \tag{1.2}$$

Since $\exp(-x^2/2)$ is also the tail distribution of the local time at 0 of a standard Brownian bridge, (1.2) confirms that the local time at 0 of α_n converges weakly to that of the Brownian bridge. Observe that this *cannot* be deduced, for example, from the strong approximation theorem of Komlós *et al.* (1975) for α_n, since $L_1^0(\alpha_n)$ is not a continuous functional of α_n. (However, there does exist a strong approximation of $L_1^0(\alpha_n)$ by the Brownian bridge local time; see (7.2) below.)

In this paper we are interested in strong limit theorems for the local time of α_n. We first recall two important results. For notational convenience, we write

$$\phi(n) \stackrel{\mathrm{def}}{=} (2\log\log n)^{1/2}, \tag{1.3}$$

throughout the paper.

Theorem A (Révész 1983). *Almost surely,*

$$\limsup_{n\to\infty} \frac{L_1^0(\alpha_n)}{\phi(n)} = 1. \tag{1.4}$$

Theorem B (Bass and Khoshnevisan 1995). *Let*

$$L_1^*(\alpha_n) \stackrel{\mathrm{def}}{=} \sup_{x\in\mathbb{R}} L_1^x(\alpha_n).$$

Then

$$\limsup_{n\to\infty} \frac{L_1^*(\alpha_n)}{\phi(n)} = 1 \quad \text{a.s.} \tag{1.5}$$

$$\liminf_{n\to\infty} (\log\log n)^{1/2} L_1^*(\alpha_n) = \sqrt{2}\pi \quad \text{a.s.} \tag{1.6}$$

Our first result concerns the *joint* asymptotics of $L_1^0(\alpha_n)$ and $L_1^*(\alpha_n)$.

Theorem 1.1. *Almost surely,*

$$\left\{ \left(\frac{L_1^0(\alpha_n)}{\phi(n)}, \frac{L_1^*(\alpha_n)}{\phi(n)} \right);\ n \geqslant 3 \right\}$$

is relatively compact, with limit set equal to

$$\mathscr{A} \stackrel{\text{def}}{=} \{(x,\, y) : 0 \leqslant x \leqslant y \leqslant 1\}.$$

Remark 1.1. (i) Expressions (1.4) and (1.5) say that $L_1^0(\alpha_n)$ and $L_1^*(\alpha_n)$ satisfy the same law of the iterated logarithm (LIL), whereas Theorem 1.1 confirms that

$$\limsup_{n\to\infty} \frac{L_1^*(\alpha_n) - L_1^0(\alpha_n)}{\phi(n)} = 1 \quad \text{a.s.} \tag{1.7}$$

This is satisfactory, since it is intuitively clear that $L_1^*(\alpha_n)$ may be far greater than $L_1^0(\alpha_n)$.

(ii) In light of (1.7), one may wonder if it is possible to obtain some information about the asymptotics of $L_1^x(\alpha_n) - L_1^0(\alpha_n)$. The corresponding problem is solved by Csörgő and Révész (1985) for the random walk, and by Csáki and Földes (1987) for the Wiener process; see Révész (1990, pp. 122–129) for an overview.

Our second result is about the integral of the square of the local time.

Theorem 1.2. *We have*

$$\limsup_{n\to\infty} \frac{1}{\phi(n)} \int_{-\infty}^{\infty} (L_1^x(\alpha_n))^2 \,\mathrm{d}x = \frac{1}{\sqrt{3}} \quad \text{a.s.},$$

$$\liminf_{n\to\infty} (\log\log n)^{1/2} \int_{-\infty}^{\infty} (L_1^x(\alpha_n))^2 \,\mathrm{d}x = \left(\frac{2|a_1|}{3}\right)^{3/2} \quad \text{a.s.},$$

where $a_1 < 0$ is the largest real zero of the Airy function $\mathrm{Ai}(\cdot)$.

Remark 1.2. It is interesting to note that the corresponding problems for the integral of the square of the Wiener local time are still open. In fact, let W denote a real-valued Wiener process; then

$$\limsup_{t\to\infty} \frac{1}{(2t^3 \log\log t)^{1/2}} \int_{-\infty}^{\infty} (L_t^x(W))^2 \,\mathrm{d}x = \overline{c} \in (0,\, \infty) \quad \text{a.s.}$$

$$\liminf_{t\to\infty} \frac{(\log\log t)^{1/2}}{t^{3/2}} \int_{-\infty}^{\infty} (L_t^x(W))^2 \,\mathrm{d}x = \underline{c} \in (0,\, \infty) \quad \text{a.s.}$$

However, the values of $\overline{c}$ and $\underline{c}$ are unknown. More discussions on this can be found in Khoshnevisan and Lewis (1998).

We also study the local time of α_n under the L^p-norm. The case $p = 3$ takes a particularly simple form.

Theorem 1.3. *We have*

$$\limsup_{n\to\infty} \frac{1}{\phi^2(n)} \int_{-\infty}^{\infty} (L_1^x(\alpha_n))^3 \,\mathrm{d}x = \frac{4}{\pi^2} \quad \text{a.s.}$$

$$\liminf_{n\to\infty} (\log\log n) \int_{-\infty}^{\infty} (L_1^x(\alpha_n))^3 \,\mathrm{d}x = \frac{9}{2} \quad \text{a.s.}$$

Theorem 1.4. *For* $p \geqslant 3$,

$$\limsup_{n\to\infty} \frac{1}{\phi^{p-1}(n)} \int_{-\infty}^{\infty} (L_1^x(\alpha_n))^p \,\mathrm{d}x = b\left(\frac{p-1}{2}\right) \quad \text{a.s.,}$$

where, for $q \geqslant 1$,

$$\begin{aligned} b(q) &\stackrel{\text{def}}{=} \frac{(q+1)^{q-1}}{q^q} \left(\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^{2q}}} \right)^{-2q} \\ &= (4q)^q (q+1)^{q-1} B^{-2q}\left(\frac{1}{2}, \frac{1}{2q}\right), \end{aligned} \tag{1.8}$$

with $B(\cdot, \cdot)$ *standing for the usual beta function.*

The rest of the paper is organized as follows. Section 2 is devoted to the study of some exact distributions related to the local time of the Brownian bridge. The main tool is Jeulin's theorem for the local time of the normalized Brownian excursion process, together with some well-known sample path decomposition theorems. The local time of the Brownian bridge is further investigated in Sections 3–6. In particular, we obtain in Section 3 the upper and lower tails of the integral of the square of the local time in question. The third and higher moments of the local time are studied in Sections 4 and 5, respectively. Section 6 concerns the joint tail of the local time at 0 and the maximum local time of the Brownian bridge. Theorems 1.1–1.4 are proved in Section 7. Finally, in Section 8, we briefly describe some asymptotic properties of the local time of the reflecting Brownian bridge and empirical process.

Following the referee's advice, we emphasize that the present knowledge of the laws of local times for the Brownian and Bessel bridges, recently discussed in a unified way in Pitman (1999), plays an important part throughout our paper.

In the rest of the paper, we adopt the usual notation $a(u) \sim b(u)$, $u \to u_0$, to denote $\lim_{u\to u_0} a(u)/b(u) = 1$.

2. Local time of the Brownian bridge

We start by introducing the normalized excursion process. Let W be, as before, a Wiener process. Let

$$G \stackrel{\text{def}}{=} \sup\{t \leqslant 1 : W(t) = 0\}, \tag{2.1}$$

$$D \stackrel{\text{def}}{=} \inf\{t \geqslant 1 : W(t) = 0\}, \tag{2.2}$$

which represent respectively the left and right extremities of the excursion interval straddling time 1. The process

$$\left\{ \frac{|W(G + (D - G)t)|}{(D - G)^{1/2}};\, 0 \leqslant t \leqslant 1 \right\}$$

is called the normalized *Brownian excursion process*, cf. Chung (1976).

It was first observed by Chung (1976) and Kennedy (1976) that the supremum of the normalized excursion has the same distribution as the range of the Brownian bridge, which suggests that there would exist a close relationship between the two processes. It turns out that this is the case, as is revealed by the following theorem. For detailed surveys of Brownian path decompositions, see Bertoin and Pitman (1994), Biane (1993) and Yor (1995, Lecture 4).

Theorem C (Vervaat 1979). *Let* $\{\gamma(t);\, 0 \leqslant t \leqslant 1\}$ *be a standard Brownian bridge process, and* U *the almost surely unique location of the minimum of* γ*, that is, such that* $\gamma(U) = \inf_{0 \leqslant t \leqslant 1} \gamma(t)$*. Then* U *is uniformly distributed on* $(0, 1)$*. Furthermore,*

$$\rho(t) \stackrel{\text{def}}{=} \begin{cases} \gamma(t + U) - \gamma(U), & \textit{if } 0 \leqslant t \leqslant 1 - U, \\ \gamma(t + U - 1) - \gamma(U), & \textit{if } 1 - U \leqslant t \leqslant 1, \end{cases} \tag{2.3}$$

is distributed as a normalized Brownian excursion process, and is independent of the variable U*.*

Another deep result which we shall need is Jeulin's theorem for the local time of the excursion process.

Theorem D (Jeulin 1985, p. 264). *Let* $\{\rho(t);\, 0 \leqslant t \leqslant 1\}$ *denote a normalized excursion process. Define* $J(s) \stackrel{\text{def}}{=} \int_0^s L_1^x(\rho)\,\mathrm{d}x$ *for all* $s \geqslant 0$*. Then*

$$\{\tfrac{1}{2} L_1^{J^{-1}(t)}(\rho);\, 0 \leqslant t \leqslant 1\} \stackrel{\text{law}}{=} \{\rho(t);\, 0 \leqslant t \leqslant 1\}, \tag{2.4}$$

where '$\stackrel{\text{law}}{=}$' *stands for identity in law, and* J^{-1} *is the continuous inverse of* J*.*

From now on, γ and ρ denote respectively Brownian bridge and (normalized) excursion processes. Here is the main result of the section, which has several interesting consequences.

Theorem 2.1. *Let* $f \geqslant 0$ *and* $g \geqslant 0$ *be two Borel functions. Let* U *be uniformly distributed in* $(0, 1)$*, independent of the excursion process* ρ*. Then*

$$\left(\int_0^{\rho(U)} f(\tfrac{1}{2}L_1^x(\rho))L_1^x(\rho)\,\mathrm{d}x,\ \int_{\rho(U)}^{\infty} g(\tfrac{1}{2}L_1^x(\rho))L_1^x(\rho)\,\mathrm{d}x,\ \tfrac{1}{2}L_1^{\rho(U)}(\rho) \right) \tag{2.5}$$

has the same distribution as

$$\left(\int_0^U f(\rho(t))\,\mathrm{d}t,\ \int_U^1 g(\rho(t))\,\mathrm{d}t,\ \rho(U) \right). \tag{2.6}$$

Remark 2.1. As a less complete – but perhaps easier to memorize – statement, we underline that the occupation measures of the processes $\{\ell(x) \stackrel{\mathrm{def}}{=} \frac{1}{2}L_1^x(\rho);\ x \geqslant 0\}$ and $\{\rho(t);\ 0 \leqslant t \leqslant 1\}$ have the same distribution, that is, for every Borel function $f \geqslant 0$,

$$\int_0^\infty f(\ell(x))\,\mathrm{d}\langle \ell(\cdot)\rangle_x \stackrel{\mathrm{law}}{=} \int_0^1 f(\rho(t))\,\mathrm{d}t,$$

where, for any process Z, $\langle Z\rangle$ is the increasing process associated with Z. (Note that $\langle \rho\rangle_t = t$, whereas $\langle \ell(\cdot)\rangle_x = 2\int_0^x \ell(y)\,\mathrm{d}y$). For other pairs of processes with identical occupation measure laws, see Pitman and Yor (1998a).

Proof of Theorem 2.1. Let, as before, $J(s) = \int_0^s L_1^x(\rho)\,\mathrm{d}x$. Let $\{\Xi(t);\ t \geqslant 0\}$ be an arbitrary stochastic process, such that for each $t \geqslant 0$, $\Xi(t)$ is an $\mathbb{R}^2$-valued variable, measurable with respect to $\{\rho(s);\ 0 \leqslant s \leqslant 1\}$. For any Borel functions $H \geqslant 0$ and $K \geqslant 0$,

$$\begin{aligned}\mathbb{E}[H(\Xi(\rho(U)))K(L_1^{\rho(U)}(\rho))] &= \mathbb{E}\left[\int_0^1 H(\Xi(\rho(u)))K(L_1^{\rho(u)}(\rho))\,\mathrm{d}u\right] \\ &= \mathbb{E}\left[\int_0^\infty L_1^x(\rho)H(\Xi(x))K(L_1^x(\rho))\,\mathrm{d}x\right],\end{aligned}$$

by means of (1.1). According to the definition of J, the expression on the right-hand side is $\mathbb{E}[\int_0^\infty H(\Xi(x))K(L_1^x(\rho))\,\mathrm{d}J(x)]$. Since J is strictly increasing over $[0, \sup_{0\leqslant t\leqslant 1}\rho(t)]$, by a change of variable $x = J^{-1}(t)$, we obtain:

$$\begin{aligned}\mathbb{E}[H(\Xi(\rho(U)))K(L_1^{\rho(U)}(\rho))] &= \mathbb{E}\left[\int_0^1 H(\Xi(J^{-1}(t)))K(L_1^{J^{-1}(t)}(\rho))\,\mathrm{d}t\right] \\ &= \mathbb{E}[H(\Xi(J^{-1}(U)))K(L_1^{J^{-1}(U)}(\rho))].\end{aligned}$$

This means that $(\Xi(\rho(U)),\ L_1^{\rho(U)}(\rho))$ has the same distribution as $(\Xi(J^{-1}(U)),\ L_1^{J^{-1}(U)}(\rho))$. In particular, taking

$$\Xi(t) \stackrel{\text{def}}{=} \left(\int_0^t f(\tfrac{1}{2}L_1^x(\rho))L_1^x(\rho)\,\mathrm{d}x,\ \int_t^\infty g(\tfrac{1}{2}L_1^x(\rho))L_1^x(\rho)\,\mathrm{d}x \right)$$

$$= \left(\int_0^{J(t)} f(\tfrac{1}{2}L_1^{J^{-1}(s)}(\rho))\,\mathrm{d}s,\ \int_{J(t)}^1 g(\tfrac{1}{2}L_1^{J^{-1}(s)}(\rho))\,\mathrm{d}s \right),$$

it follows that, the $\mathbb{R}^3$-valued variable in (2.5) has the same distribution as

$$\left(\int_0^U f(\tfrac{1}{2}L_1^{J^{-1}(s)}(\rho))\,\mathrm{d}s,\ \int_U^1 g(\tfrac{1}{2}L_1^{J^{-1}(s)}(\rho))\,\mathrm{d}s,\ \tfrac{1}{2}L_1^{J^{-1}(U)}(\rho) \right),$$

which, according to Theorem D, is distributed as

$$\left(\int_0^U f(\rho(s))\,\mathrm{d}s,\ \int_U^1 g(\rho(s))\,\mathrm{d}s,\ \rho(U) \right).$$

This completes the proof of the theorem. □

We present a few applications (which certainly are not exhaustive) of Theorem 2.1. The first confirms that the study of the distribution of additive functionals of $L(\gamma)$ can be reduced to that of the corresponding problems for the excursion process.

Corollary 2.2. *Let γ be a Brownian bridge. For any Borel function h: $\mathbb{R}_+ \mapsto \mathbb{R}_+$, such that $h(0) = 0$,*

$$\int_{-\infty}^{\infty} h(L_1^x(\gamma))\,\mathrm{d}x \stackrel{\text{law}}{=} \frac{1}{2}\int_0^1 \frac{h(2\rho(t))}{\rho(t)}\,\mathrm{d}t. \tag{2.7}$$

In particular, for any $p > 0$,

$$\int_{-\infty}^{\infty} (L_1^x(\gamma))^p\,\mathrm{d}x \stackrel{\text{law}}{=} 2^{p-1}\int_0^1 (\rho(t))^{p-1}\,\mathrm{d}t. \tag{2.8}$$

Remark 2.2. Let

$$S \stackrel{\text{def}}{=} \sup_{0\leqslant t\leqslant 1} \gamma(t), \qquad I \stackrel{\text{def}}{=} \inf_{0\leqslant t\leqslant 1} \gamma(t). \tag{2.9}$$

An equivalent formulation of (2.7) is:

$$\int_I^S h(L_1^x(\gamma))\,\mathrm{d}x \stackrel{\text{law}}{=} \frac{1}{2}\int_0^1 \frac{h(2\rho(t))}{\rho(t)}\,\mathrm{d}t, \tag{2.10}$$

which now holds for any Borel function h: $\mathbb{R}_+ \mapsto \mathbb{R}_+$.

Proof of Corollary 2.2. It is an immediate consequence of Theorem C that the processes

$$\{L_1^{x+\gamma(U)}(\gamma);\ x \geqslant 0\} \quad \text{and} \quad \{L_1^x(\rho);\ x \geqslant 0\}$$

have the same distribution, where, as before, U is the location of the minimum of γ. In particular, if $h(0) = 0$,

$$\int_{-\infty}^{\infty} h(L_1^x(\gamma))\,\mathrm{d}x \stackrel{\text{law}}{=} \int_0^{\infty} h(L_1^x(\rho))\,\mathrm{d}x. \tag{2.11}$$

On the other hand, by Theorem 2.1, for any non-negative Borel function f,

$$\int_0^{\infty} f(\tfrac{1}{2}L_1^x(\rho))L_1^x(\rho)\,\mathrm{d}x \stackrel{\text{law}}{=} \int_0^1 f(\rho(t))\,\mathrm{d}t.$$

This, jointly considered with (2.11), yields (2.7). $\square$

The next result plays a key role in our proof of Theorem 1.1.

Corollary 2.3. *Let γ be a Brownian bridge,*

$$(L_1^*(\gamma),\, L_1^0(\gamma)) \stackrel{\text{law}}{=} (2(S-I),\, -2I), \tag{2.12}$$

where S and I are as in (2.9). *In particular, for any $x>0$,*

$$\mathbb{P}(L_1^*(\gamma)<x) = 1 - 2\sum_{k=1}^{\infty}(k^2x^2-1)\exp\left(-\frac{k^2x^2}{2}\right) \tag{2.13}$$

$$= \frac{\sqrt{128\pi^5}}{x^3}\sum_{k=1}^{\infty} k^2\exp\left(-\frac{2k^2\pi^2}{x^2}\right). \tag{2.14}$$

Consequently,

$$\log\mathbb{P}(L_1^*(\gamma)<y) \sim -\frac{2\pi^2}{y^2}, \qquad y\to 0, \tag{2.15}$$

$$\log\mathbb{P}(L_1^*(\gamma)>y) \sim -\frac{y^2}{2}, \qquad y\to\infty. \tag{2.16}$$

Proof. Fix $p>0$. Take $f(x)=g(x)=x^p$ in Theorem 2.1 to see that

$$\left(2^{-p}\int_0^{\infty}(L_1^x(\rho))^{p+1}\,\mathrm{d}x,\, \tfrac{1}{2}L_1^{\rho(U)}(\rho)\right) \stackrel{\text{law}}{=} \left(\int_0^1 \rho^p(t)\,\mathrm{d}t,\, \rho(U)\right).$$

Raising the first variables on both sides to the power of $1/p$, and then letting p go to infinity, we obtain:

$$\left(\tfrac{1}{2}\sup_{x\geqslant 0}L_1^x(\rho),\, \tfrac{1}{2}L_1^{\rho(U)}(\rho)\right) \stackrel{\text{law}}{=} \left(\sup_{0\leqslant t\leqslant 1}\rho(t),\, \rho(U)\right).$$

Multiplying both sides by 2 and applying Theorem C yields (2.12).

The exact distribution of the range of the Brownian bridge is well known, (see Csörgő and Révész 1981, p. 164; Kuiper 1960; Chung 1976): for $x>0$,

$$\mathbb{P}(S - I \leqslant x) = 1 - 2\sum_{k=1}^{\infty}(4k^2x^2 - 1)\exp(-2k^2x^2), \tag{2.17}$$

which, in view of (2.12), implies (2.13). The expression in (2.14) immediately follows from Poisson's summation formula (see Feller 1966, p. 630). □

Remark 2.3. The distribution function of the range of a Brownian bridge (cf. (2.17)), and hence also that of $L_1^*(\gamma)$ (cf. (2.13)), is related to the Jacobi theta function. For probabilistic interpretations of this famous function (and of the Riemann zeta function) in terms of Brownian motion, we refer to Biane *et al.* (1999), Biane and Yor (1987), Chung (1976), Csáki (1979), Csáki and Mohanty (1981; 1986), Csörgő and Horváth (1997, p. 102), Deheuvels (1985), Smith and Diaconis (1988), Williams (1990), and Yor (1997, Chapter 11).

Remark 2.4. (i) We can choose various functions f and g in Theorem 2.1 to obtain many identities in law, which hold **jointly**. For example, together with Theorem C, we immediately see that the random variable

$$\int_0^1 \mathbb{1}_{\{\gamma(t)>0\}}\,\mathrm{d}t$$

is uniformly distributed on (0, 1), independent of any variable of the form $\int_{\mathbb{R}} f(L_1^x(\gamma))\,\mathrm{d}x$. In particular, it is independent of $(S - I, \sup_{x\in\mathbb{R}} L_1^x(\gamma))$. This kind of independence is explained and extended by Chaumont (1998).

(ii) From (2.10) we deduce:

$$S - I \overset{\text{law}}{=} \frac{1}{2}\int_0^1 \frac{\mathrm{d}t}{\rho(t)}.$$

Therefore, (2.13) and (2.14) also express the distribution function of $\int_0^1 \mathrm{d}t/\rho(t)$. For further discussions on this, see Biane and Yor (1987), Chung (1976) and Pitman and Yor (1996). We also mention Chung's identity in law: if $\tilde{\gamma}$ denotes an independent copy of γ,

$$\sup_{0\leqslant t\leqslant 1}\gamma^2(t) + \sup_{0\leqslant t\leqslant 1}\tilde{\gamma}^2(t) \overset{\text{law}}{=} \sup_{0\leqslant t\leqslant 1}\rho^2(t);$$

see Chung (1976), Yor (1997, p. 16).

(iii) The identities (2.13) and (2.14) were previously proved by Bass and Khoshnevisan (1995) using the Ray–Knight theorem. Not surprisingly, their consequences (2.15)–(2.16) played an essential part in the proof of Theorem B above (for more details, see Bass and Khoshnevisan 1995; see also Khoshnevisan 1992, 1993).

3. Tail probabilities for the square integral

Recall that γ is a Brownian bridge. This section is devoted to the study of the upper and lower tails of the variable $\int_{-\infty}^{\infty}(L_1^x(\gamma))^2\,\mathrm{d}x$.

Theorem 3.1. *We have*

$$\log \mathbb{P}\left(\int_{-\infty}^{\infty} (L_1^x(\gamma))^2 \,\mathrm{d}x < y\right) \sim -\frac{8|a_1|^3}{27y^2}, \qquad y \to 0^+, \tag{3.1}$$

$$\log \mathbb{P}\left(\int_{-\infty}^{\infty} (L_1^x(\gamma))^2 \,\mathrm{d}x > y\right) \sim -\frac{3}{2}y^2, \qquad y \to \infty, \tag{3.2}$$

where $a_1 < 0$ is, as before, the largest real zero of the Airy function $\mathrm{Ai}(\cdot)$.

We need the following result which relates the tail behaviour of a non-negative random variable with its moment generating function.

Lemma 3.2. *Let X be an almost surely non-negative random variable. Assume*

$$\lim_{n\to\infty} \frac{[\mathbb{E}(X^n)]^{1/n}}{n} = a, \tag{3.3}$$

for some constant $a \in (0, \infty)$. Then

$$\log \mathbb{P}(X > x) \sim -\frac{x}{a\mathrm{e}}, \qquad x \to \infty.$$

Proof. That $\limsup_{x\to\infty} x^{-1} \log \mathbb{P}(X > x) \leqslant -1/(a\mathrm{e})$ immediately follows from Chebyshev's inequality and Stirling's formula. The lower bound, which needs more care, can be proved using Laplace's method. We only outline the proof, and refer to the proof of Lemma 2.7 in Marcus and Rosen (1994) for full details.

Let $\delta > 0$. For all sufficiently small $\varepsilon > 0$, we have

$$(1+\delta)\log\frac{1-\varepsilon}{(1+\delta)\mathrm{e}} > \max\left(-(1-\varepsilon) + (1+\delta)\log\frac{1}{(1+\delta)^2},\ (1+\delta)\log\frac{1+\varepsilon}{\mathrm{e}}\right). \tag{3.4}$$

For large $x > 0$, define $y = (1+\delta)^2 x$ and $n = [(1+\delta)x/(a\mathrm{e})]$. Then

$$\begin{aligned} \mathbb{P}(X > x) &\geqslant \frac{\mathbb{E}(X^n)}{y^n} - \frac{1}{y^n}\int_0^x u^n \,\mathrm{d}\mathbb{P}(X \leqslant u) - \frac{1}{y^n}\int_x^\infty u^n \,\mathrm{d}\mathbb{P}(X \leqslant u) \\ &\stackrel{\text{def}}{=} A_1 - A_2 - A_3. \end{aligned}$$

By (3.3),

$$A_1 \geqslant \frac{((1-\varepsilon)an)^n}{y^n} = \exp\left(n \log\frac{(1-\varepsilon)an}{y}\right).$$

On the other hand, by integration by parts and the upper bound $\limsup_{z\to\infty} z^{-1} \log \mathbb{P}(X > z) \leqslant -1/(a\mathrm{e})$,

$$
\begin{aligned}
A_2 &\leqslant \frac{n}{y^n}\int_0^x u^{n-1}\mathbb{P}(X>u)\,\mathrm{d}u \\
&\leqslant \frac{n}{y^n}\int_0^x u^{n-1}c\exp\left(-\frac{(1-\varepsilon)u}{a\mathrm{e}}\right)\mathrm{d}u \\
&\leqslant \frac{cn}{y^n}x^n\exp\left(-\frac{(1-\varepsilon)x}{a\mathrm{e}}\right),
\end{aligned}
$$

where c denotes a finite constant. We also have

$$
\begin{aligned}
A_3 &\leqslant \frac{1}{y^{n+[\delta n]}}\int_y^\infty u^{n+[\delta n]}\,\mathrm{d}\mathbb{P}(X\leqslant u) \\
&\leqslant \frac{\mathbb{E}(X^{n+[\delta n]})}{y^{n+[\delta n]}} \\
&\leqslant \left(\frac{(1+\varepsilon)a(n+[\delta n])}{y}\right)^{n+[\delta n]}.
\end{aligned}
$$

In view of (3.4), $A_1 \geqslant 3A_2$ and $A_1 \geqslant 3A_3$. This gives

$$
\liminf_{x\to\infty}\frac{1}{x}\log\mathbb{P}(X>x)\geqslant\frac{1+\delta}{a\mathrm{e}}\log\frac{1-\varepsilon}{(1+\delta)\mathrm{e}},
$$

which yields the lower bound in the lemma by sending ε and δ to 0 (in this order). □

Proof of Theorem 3.1. Let ρ be as before a (normalized) excursion process. The exact distribution of $\int_0^1\rho(t)\,\mathrm{d}t$ has been determined by several authors, among them Darling (1983), Louchard (1984), Groeneboom (1989) and Takács (1991; 1992). For all $y>0$,

$$
\mathbb{P}\left(\int_0^1\rho(t)\,\mathrm{d}t<y\right)=\frac{2^{7/6}}{9y^{7/3}}\sum_{k=1}^\infty a_k^2U\left(\frac{1}{6},\frac{4}{3};\frac{2|a_k|^3}{27y^2}\right)\exp\left(-\frac{2|a_k|^3}{27y^2}\right), \tag{3.5}
$$

where $U(a, b; x)$ is the confluent hypergeometric function, and $0>a_1>a_2>\ldots$ are the real zeros of the Airy function $\mathrm{Ai}(\cdot)$.

According to Abramowitz and Stegun (1965, p. 508), for fixed a and b, $U(a, b; x)\sim x^{-a}$ ($x\to\infty$), whereas as k goes to infinity, $|a_k|$ behaves like a constant multiple of $k^{2/3}$ (Abramowitz and Stegun 1965, p. 450). Consequently,

$$
\log\mathbb{P}\left(\int_0^1\rho(t)\,\mathrm{d}t<y\right)\sim-\frac{2|a_1|^3}{27y^2},\qquad y\to 0^+.
$$

Since by (2.8),

$$
\int_{-\infty}^{\infty}(L_1^x(\gamma))^2\,\mathrm{d}x\overset{\text{law}}{=}2\int_0^1\rho(t)\,\mathrm{d}t, \tag{3.6}
$$

this yields (3.1).

It remains to check (3.2). Curiously, the exact distribution (3.5) does not seem to immediately yield the upper tail of $\int_0^1 \rho(t)\,\mathrm{d}t$. However, the moments of this variable are also estimated in Takács (1991; 1992):

$$\mathbb{E}\left[\left(\int_0^1 \rho(t)\,\mathrm{d}t\right)^n\right] \sim 3\sqrt{2}n\left(\frac{n}{12\mathrm{e}}\right)^{n/2}, \qquad n \to \infty. \tag{3.7}$$

Applying Lemma 3.2 gives

$$\log \mathbb{P}\left(\int_0^1 \rho(t)\,\mathrm{d}t > y\right) \sim -6y^2, \qquad y \to \infty.$$

Together with (3.6), this implies (3.2), hence the theorem. $\square$

Remark 3.1. In the proof of Theorem 3.1, that $y^{-2}\log \mathbb{P}(\int_0^1 \rho(t)\,\mathrm{d}t > y)$ has a non-denegerate limit (as y goes to infinity) follows from a large-deviation result for general Gaussian processes; see Azencott (1980, p. 57). However, the identification of the limit is easier using (3.7).

4. Tails of the third moment

The tail probabilities of the third moment of the local time of the Brownian bridge bear the following simple form:

Theorem 4.1. *If γ is a Brownian bridge,*

$$\log \mathbb{P}\left(\int_{-\infty}^{\infty} (L_1^x(\gamma))^3\,\mathrm{d}x < y\right) \sim -\frac{9}{2y}, \qquad y \to 0^+, \tag{4.1}$$

$$\log \mathbb{P}\left(\int_{-\infty}^{\infty} (L_1^x(\gamma))^3\,\mathrm{d}x > y\right) \sim -\frac{\pi^2 y}{8}, \qquad y \to \infty. \tag{4.2}$$

In order to prove Theorem 4.1, we need the following preliminary result.

Lemma 4.2. *Fix $m \geqslant 2$. Let $\xi_1, \xi_2, \ldots, \xi_m$ denote independent non-negative variables, such that for any $1 \leqslant i \leqslant m$,*

$$\log \mathbb{P}(\xi_i > y) \sim -ay, \qquad y \to \infty, \tag{4.3}$$

for some constant $a > 0$. Then

$$\log \mathbb{P}(\xi_1 + \ldots + \xi_m > y) \sim -ay, \qquad y \to \infty.$$

Proof of Lemma 4.2. Only the upper bound needs checking. By induction, we only have to treat the case $m = 2$. According to (4.3), for $\varepsilon > 0$, there exists a constant C_ε, depending on ε, such that

$$\mathbb{P}(\xi_i \geqslant x) \leqslant C_\varepsilon \, \mathrm{e}^{-(1-\varepsilon)ax},$$

for $i = 1$ or 2, and all $x \geqslant 0$. Therefore,

$$\begin{aligned}
\mathbb{P}(\xi_1 + \xi_2 > y) &\leqslant \mathbb{P}(\xi_1 > y) + \mathbb{P}(\xi_1 + \xi_2 > y,\, \xi_1 \leqslant y) \\
&\leqslant C_\varepsilon \, \mathrm{e}^{-(1-\varepsilon)ay} - \int_0^y \mathbb{P}(\xi_2 > y - x) \, \mathrm{d}_x \mathbb{P}(\xi_1 \geqslant x) \\
&\leqslant C_\varepsilon \, \mathrm{e}^{-(1-\varepsilon)ay} - C_\varepsilon \int_0^y \mathrm{e}^{-(1-\varepsilon)a(y-x)} \, \mathrm{d}_x \mathbb{P}(\xi_1 \geqslant x).
\end{aligned}$$

By integration by parts, the integral expression on the right-hand side is

$$\begin{aligned}
&\mathbb{P}(\xi_1 \geqslant y) - \mathrm{e}^{-(1-\varepsilon)ay} - (1-\varepsilon)a \int_0^y \mathrm{e}^{-(1-\varepsilon)a(y-x)} \mathbb{P}(\xi_1 \geqslant x) \, \mathrm{d}x \\
&\geqslant -\mathrm{e}^{-(1-\varepsilon)ay} - C_\varepsilon a \int_0^y \mathrm{e}^{-(1-\varepsilon)a(y-x)} \, \mathrm{e}^{-(1-\varepsilon)ax} \, \mathrm{d}x \\
&= -\mathrm{e}^{-(1-\varepsilon)ay} - C_\varepsilon a y \, \mathrm{e}^{-(1-\varepsilon)ay}.
\end{aligned}$$

Assembling these pieces yields $\limsup_{y\to\infty} y^{-1} \log \mathbb{P}(\xi_1 + \xi_2 > y) \leqslant -(1-\varepsilon)a$. This completes the proof of the lemma by sending ε to 0^+. $\square$

Proof of Theorem 4.1. The Laplace transform of $\int_0^1 \rho^2(t) \, \mathrm{d}t$ is well known (see, for example, Pitman and Yor 1982, p. 432): for all $\lambda > 0$,

$$\mathbb{E}\left[\exp\left(-\lambda \int_0^1 \rho^2(t) \, \mathrm{d}t\right)\right] = \left(\frac{\sqrt{2\lambda}}{\sinh \sqrt{2\lambda}}\right)^{3/2}. \tag{4.4}$$

This, combined with an exponential-type Tauberian theorem (see Bingham *et al.* 1987, Theorem 4.12.9), yields the following lower tail:

$$\log \mathbb{P}\left(\int_0^1 \rho^2(t) \, \mathrm{d}t < y\right) \sim -\frac{9}{8y}, \qquad y \to 0^+.$$

In view of (2.8), this is equivalent to (4.1).

It is also possible to prove (4.2) from (4.4) by means of analytic continuation and a sophisticated version of the Tauberian theorem. However, there is an easier way to handle the problem. According to Williams's identification (see, for example, Rogers and Williams 1987, pp. 88–89), ρ can be realized as a standard three-dimensional Bessel bridge, that is,

$$\sqrt{\gamma^2 + \tilde{\gamma}^2 + \hat{\gamma}^2} \text{ is an excursion process,} \tag{4.5}$$

where $\tilde{\gamma}$ and $\hat{\gamma}$ denote two independent copies of the Brownian bridge γ. The exact distribution function of $\int_0^1 \gamma^2(t) \, \mathrm{d}t$ is explicitly known (see, for example, Csörgő and Révész 1981, p. 43): for $y > 0$,

$$\mathbb{P}\left(\int_0^1 \gamma^2(t)\,\mathrm{d}t \leqslant y\right) = 1 - \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\int_{(2k-1)\pi}^{2k\pi} \frac{\exp(-yt^2/2)}{\sqrt{t|\sin t|}}\,\mathrm{d}t,$$

from which it immediately follows that

$$\log \mathbb{P}\left(\int_0^1 \gamma^2(t)\,\mathrm{d}t > y\right) \sim -\frac{\pi^2 y}{2}, \qquad y \to \infty.$$

By (2.8), $\int_{\mathbb{R}}(L_1^x(\gamma))^3\,\mathrm{d}x$ is distributed as $4\int_0^1 \rho^2(t)\,\mathrm{d}t$. This, together with (4.5) and Lemma 4.2, yields (4.2), hence the theorem. □

5. Upper tails of higher moments

This section is devoted to the study of the upper tail of $L_1^x(\gamma)$ under the L^p-norm, where γ is, as before, a Brownian bridge.

Theorem 5.1. *For any $p \geqslant 3$,*

$$\log \mathbb{P}\left(\int_{-\infty}^{\infty} (L_1^x(\gamma))^p\,\mathrm{d}x > y\right) \sim -\frac{y^{1/q}}{2b^{1/q}(q)}, \qquad y \to \infty, \tag{5.1}$$

where $q \stackrel{\text{def}}{=} (p-1)/2$, and $b(q)$ is as in (1.8).

Proof. Clearly, $q \stackrel{\text{def}}{=} (p-1)/2 \geqslant 1$. For any process Z indexed by [0, 1], write its L^q-norm under the Lebesgue measure over [0, 1] as $\|Z\|_q \stackrel{\text{def}}{=} (\int_0^1 |Z(t)|^q\,\mathrm{d}t)^{1/q}$. Let $\tilde{\gamma}$ and $\hat{\gamma}$ denote two independent copies of γ. Define

$$\rho(t) \stackrel{\text{def}}{=} \sqrt{\gamma^2(t) + \tilde{\gamma}^2(t) + \hat{\gamma}^2(t)}, \qquad 0 \leqslant t \leqslant 1,$$

which, according to (4.5), is an excursion process. By the triangular inequality,

$$\|\gamma^2\|_q \leqslant \|\rho^2\|_q \leqslant \|\gamma^2\|_q + \|\tilde{\gamma}^2\|_q + \|\hat{\gamma}^2\|_q. \tag{5.2}$$

Assume for the moment that we could show

$$\log \mathbb{P}\big(\|\gamma^2\|_q > y\big) \sim -\frac{2}{b^{1/q}(q)}\,y, \qquad y \to \infty; \tag{5.3}$$

then by (5.2) and Lemma 4.2, we would also have

$$\log \mathbb{P}\big(\|\rho^2\|_q > y\big) \sim -\frac{2}{b^{1/q}(q)}\,y, \qquad y \to \infty.$$

Using (2.8), this would complete the proof of Theorem 5.1.

It remains to prove (5.3). There exists a finite positive constant $c(q)$, depending only on q, such that

$$\log \mathbb{P}\big(\|\gamma^2\|_q > y\big) \sim -c(q)y, \qquad y \to \infty. \tag{5.4}$$

This follows from a well-known large-deviation result (Azencott 1980, p. 62), or from the general theory for Gaussian measures (Fernique 1997, p. 39; Ledoux and Talagrand 1991, p. 59). Therefore, the proof of (5.3) is reduced to showing

$$c(q) = \frac{2}{b^{1/q}(q)}. \tag{5.5}$$

The Gaussian theory does give the exact value of the constant $c(q)$, in the form of an extreme value of some functional in a Gaussian space. However, in our setting, we do not need to do any technical computation in order to determine the value of $c(q)$. Indeed, according to the strong approximation theorem of Komlós *et al.* (1975), possibly in an enlarged probability space, there exist a coupling for α_n and a sequence of independent Brownian bridges $\{\gamma_k\}_{k\geqslant 1}$, such that

$$\sup_{0\leqslant t\leqslant 1}\left|\alpha_n(t) - \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\gamma_k(t)\right| = \mathscr{O}\left(\frac{(\log n)^2}{\sqrt{n}}\right) \text{ a.s.}$$

Applying (5.4) and the usual Borel–Cantelli argument, we obtain:

$$\limsup_{n\to\infty}\frac{\|(\alpha_n)^2\|_q}{\log\log n} = \frac{1}{c(q)} \quad \text{a.s.} \tag{5.6}$$

On the other hand, the Finkelstein (1971) functional LIL confirms that

$$\limsup_{n\to\infty}\frac{\|(\alpha_n)^2\|_q}{2\log\log n} = \sup_{f\in\mathbb{S}, f(1)=0}\|f^2\|_q \stackrel{\text{def}}{=} d(q), \text{ a.s.,} \tag{5.7}$$

where $\mathbb{S} = \{f\colon f(t) = \int_0^t \dot{f}(s)\,\mathrm{d}s, \int_0^1 \dot{f}^2(s)\,\mathrm{d}s \leqslant 1\}$ is Strassen's set. Comparing (5.6) and (5.7) yields $c(q) = 1/(2d(q))$. To compute $d(q)$, recall that, according to Strassen (1964),

$$\tilde{d}(q) \stackrel{\text{def}}{=} \sup_{f\in\mathbb{S}}\|f^2\|_q = b^{1/q}(q),$$

where $b(q)$ is as in (1.8). Since $\{f\in\mathbb{S}\colon f(1)=0\}$ is a compact subset (in the space of continuous functions on [0, 1] endowed with the uniform topology), the 'sup' expression in (5.7) is attained by some function, say f_*, with $f_*\in\mathbb{S}$ and $f_*(1)=0$. By symmetry, $f_*(t) = f_*(1-t)$. Let $\tilde{f}(t) \stackrel{\text{def}}{=} 2f_*(t/2)$ for $t\in[0, 1]$. Then $\tilde{f}\in\mathbb{S}$ and realizes $\sup_{f\in\mathbb{S}}\|f^2\|_q$, that is

$$\tilde{d}(q) = \|\tilde{f}^2\|_q = 4\|f_*^2\|_q,$$

which yields

$$d(q) = \|f_*^2\|_q = \frac{\tilde{d}(q)}{4} = \frac{b^{1/q}(q)}{4}.$$

Since $c(q) = 1/(2d(q))$, we obtain (5.5). This completes the proof. □

6. Joint tail

In order to prove Theorem 1.1, we need the joint tail behaviour of $L_1^0(\gamma)$ (γ being a Brownian bridge) and $L_1^*(\gamma) \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}} L_1^x(\gamma)$. Here is the main result of this section.

Theorem 6.1. *Fix $0 < x_1 < x_2 < y_1 < y_2 < 1$. There exists $n_0 = n_0(x_1, x_2, y_1, y_2) < \infty$ such that, for all $n \geqslant n_0$,*

$$\mathbb{P}\left(x_1 \leqslant \frac{L_1^0(\gamma)}{\phi(n)} \leqslant x_2,\ y_1 \leqslant \frac{L_1^*(\gamma)}{\phi(n)} \leqslant y_2 \right) \geqslant \frac{1}{\log n}, \tag{6.1}$$

where ϕ is the function defined in (1.3).

The main ingredient in the proof of Theorem 6.1 is the following estimate. Recall (S, I) from (2.9).

Lemma 6.2. *As a and b go to infinity,*

$$\log \mathbb{P}(S > a, |I| > b) \sim -2(a+b)^2. \tag{6.2}$$

Proof of Lemma 6.2. The upper bound in the lemma is easy. Indeed, $\mathbb{P}(S > a, |I| > b) \leqslant \mathbb{P}(S + |I| > a + b)$, whereas from (2.17), it is easily seen that $\log \mathbb{P}(S + |I| > a + b) \sim -2(a+b)^2$ (for $a + b \to \infty$). This yields the desired upper bound in (6.2).

To verify the lower bound, we use the representation $\gamma(t) = W(t) - tW(1)$ (for $0 \leqslant t \leqslant 1$), where W is a standard Wiener process. Fix $0 < \varepsilon < 1$. For $a > 1$ and $b > 1$, let us assume $a \leqslant b$ without loss of generality, to see that

$$\begin{aligned}
\mathbb{P}(S > a, |I| > b) &\geqslant \mathbb{P}\big(\sup_{0\leqslant t\leqslant 1}(W(t) - tW(1)) > a,\ \inf_{0\leqslant t\leqslant 1}(W(t) - tW(1)) < -b,\ |W(1)| < \varepsilon\big) \\
&\geqslant \mathbb{P}\big(\sup_{0\leqslant t\leqslant 1} W(t) > (1+\varepsilon)a,\ \inf_{0\leqslant t\leqslant 1} W(t) < -(1+2\varepsilon)b,\ |W(1)| < \varepsilon\big) \\
&\geqslant \mathbb{P}((1+\varepsilon)a < W(u) < (1+2\varepsilon)a, \\
&\qquad -(1+5\varepsilon)b < W(v) < -(1+2\varepsilon)b,\ -\varepsilon < W(1) < \varepsilon),
\end{aligned}$$

for any $0 < u < v < 1$. By the Markov property,

$$\begin{aligned}
\mathbb{P}(S > a, |I| > b) \geqslant\ & \mathbb{P}((1+\varepsilon)a < W(u) < (1+2\varepsilon)a) \\
& \times \mathbb{P}(-(1+3\varepsilon)(a+b) < W(v-u) < -(1+2\varepsilon)(a+b)) \\
& \times \inf_{-(1+5\varepsilon)b < x < -(1+2\varepsilon)b} \mathbb{P}(-\varepsilon < W(1-v) < \varepsilon \,|\, W(0) = x).
\end{aligned}$$

We now choose $u = a/2(a+b)$ and $v = u + 1/2$. By Mill's ratio for Gaussian tails, for any $\varepsilon_1 > 0$, when $a \to \infty$ and $b \to \infty$,

$$\log \mathbb{P}(S > a, |I| > b) \geqslant -(1+\varepsilon_1)[(1+\varepsilon)^2 a(a+b) + (1+3\varepsilon)^2(a+b)^2 + (1+6\varepsilon)^2 b(a+b)]$$

$$\geqslant -2(1+\varepsilon_1)(1+6\varepsilon)^2(a+b)^2,$$

which yields the lower bound in (6.2) since ε and ε_1 can be as close to 0 as possible. Lemma 6.2 is proved. □

Proof of Theorem 6.1. By Corollary 2.3,

$$(L_1^*(\gamma) - L_1^0(\gamma),\, L_1^0(\gamma)) \overset{\text{law}}{=} (2S, 2|I|). \tag{6.3}$$

Now fix $0 < x_1 < x_2 < y_1 < y_2 < 1$. Without loss of generality, we can assume $y_1 - x_1 < y_2 - x_2$ (otherwise, we certainly will have $x_2 + y_1 - y_2 > 0$ and can replace x_1 by $x_2 + (y_1 - y_2)/2$). In view of (6.3), the probability term on the left-hand side of (6.1) is greater than or equal to

$$\mathbb{P}\left(y_1 - x_1 \leqslant \frac{L_1^*(\gamma) - L_1^0(\gamma)}{\phi(n)} \leqslant y_2 - x_2,\, x_1 \leqslant \frac{L_1^0(\gamma)}{\phi(n)} \leqslant x_2 \right)$$

$$= \mathbb{P}\left(\frac{y_1 - x_1}{2} \leqslant \frac{S}{\phi(n)} \leqslant \frac{y_2 - x_2}{2}, \frac{x_1}{2} \leqslant \frac{|I|}{\phi(n)} \leqslant \frac{x_2}{2} \right),$$

which, according to Lemma 6.2, is greater than $\exp(-(1+\varepsilon)y_1^2\phi^2(n)/2)$ for any fixed $\varepsilon > 0$ and sufficiently large n. Since $y_1 < 1$, we can choose ε such that $(1+\varepsilon)y_1^2 < 1$. This completes the proof. □

7. Proof of Theorem 1.1

The proofs of Theorems 1.1–1.4 are based on the corresponding tail estimates evaluated in Sections 3–6, together with the usual Borel–Cantelli argument. The latter is quite similar to the argument in Bass and Khoshnevisan (1995), who provide in full detail the proof of Theorem B. Hence, we give the proof of Theorem 1.1, and we feel free to omit the rest of the proofs. The key ingredients in the Borel–Cantelli argument are the following Facts 7.1–7.3 (all of which can be found in Bass and Khoshnevisan 1995), and the usual LIL for the uniform empirical process α_n (see, for example, Csörgő and Révész 1981, p. 157).

Fact 7.1. *Fix $0 < \mu < \frac{1}{4}$. Possibly in an enlarged probability space, there exists a coupling for α_n and a sequence of Brownian bridges $(\gamma_n)_{n \geqslant 1}$, such that for all sufficiently large n,*

$$\mathbb{P}\left(\sup_{x \in \mathbb{R}} |L_1^x(\alpha_n) - L_1^x(\gamma_n)| \geqslant n^{-\mu} \right) \leqslant n^{-2}. \tag{7.1}$$

Consequently, as n goes to infinity,

$$\sup_{x \in \mathbb{R}} |L_1^x(\alpha_n) - L_1^x(\gamma_n)| = \mathcal{O}(n^{-\mu}) \quad \text{a.s.} \tag{7.2}$$

Fact 7.2. *Let* $\{Z_1(t);\ 0 \leqslant t \leqslant 1\}$ *and* $\{Z_2(t);\ 0 \leqslant t \leqslant 1\}$ *be adapted stochastic processes. For any* $b > 0$,

$$\sup_{x\in\mathbb{R}} |L_1^x(Z_1) - L_1^x(Z_2)| \leqslant \sup_{0\leqslant t\leqslant 1} \frac{|Z_1(t) - Z_2(t)|}{b^2} + \sum_{j=1}^{2} \sup_{x,y\in\mathbb{R}, |x-y|\leqslant b} |L_1^x(Z_j) - L_1^y(Z_j)|.$$

Fact 7.3. *Let* γ *be a Brownian bridge. There exists a universal constant* $C > 0$ *such that for all* $0 < b < 1$ *and* $\lambda > 0$,

$$\mathbb{P}\left(\sup_{x,y\in\mathbb{R}, |x-y|\leqslant b} |L_1^x(\gamma) - L_1^y(\gamma)| \geqslant \sqrt{b}\big(2\sqrt{\log(1/b)} + \lambda\big) \right) \leqslant C \exp\left(-\frac{\lambda^2}{C} \right). \tag{7.3}$$

Fact 7.4. *The following LIL holds:*

$$\limsup_{n\to\infty} \sup_{0\leqslant t\leqslant 1} \frac{|\alpha_n(t)|}{\phi(n)} = \frac{1}{2} \quad \text{a.s.},$$

where ϕ *is defined in* (1.3).

Proof of Theorem 1.1. In view of (1.4) and (1.5) (though our proof outlined below would also yield Theorems A and B, with only a few modifications), the only part to check is that any $(x, y) \in \mathscr{A}$ is a limit point of $(L_1^0(\alpha_n)/\phi(n),\ L_1^*(\alpha_n)/\phi(n))$. Without loss of generality, we can assume $0 < x < y < 1$. Fix $\delta > 0$ so small that

$$0 < x - 3\delta < x + 3\delta < y - 3\delta < y + 3\delta < 1.$$

Define $n(k) \stackrel{\text{def}}{=} k^{17k}$. Recall from Section 1 that α_n is the empirical process based on the first n observations of an independent and identically distributed sampling $\{U_i\}_{i\geqslant 1}$. Let

$$\Delta(k) \stackrel{\text{def}}{=} n(k) - n(k-1),$$

$$\tilde{\alpha}_k{}^{(t)} \stackrel{\text{def}}{=} \frac{1}{\sqrt{\Delta(k)}} \sum_{i=n(k-1)+1}^{n(k)} (\mathbb{1}_{\{U_i\leqslant t\}} - t), \qquad 0 \leqslant t \leqslant 1.$$

Observe that $\tilde{\alpha}_k$ is the empirical process based on the observations $(U_{n(k-1)+1}, \cdots, U_{n(k)})$. For each k, the process $\tilde{\alpha}_k$ is distributed as $\alpha_{\Delta(k)}$. Write $L_1^*(\tilde{\alpha}_k) \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}} L_1^x(\tilde{\alpha}_k)$, and consider the measurable events

$$E_k \stackrel{\text{def}}{=} \left\{ x - 2\delta \leqslant \frac{L_1^0(\tilde{\alpha}_k)}{\phi(n(k))} \leqslant x + 2\delta,\ y - 2\delta \leqslant \frac{L_1^*(\tilde{\alpha}_k)}{\phi(n(k))} \leqslant y + 2\delta \right\}.$$

By (7.1), for all sufficiently large k,

$$\mathbb{P}(E_k) \geqslant \mathbb{P}\left(x - \delta \leqslant \frac{L_1^0(\gamma)}{\phi(n(k))} \leqslant x + \delta,\ y - \delta \leqslant \frac{L_1^*(\gamma)}{\phi(n(k))} \leqslant y + \delta \right) - \frac{1}{\Delta^2(k)}$$

$$\geqslant \frac{1}{\log n(k)} - \frac{1}{\Delta^2(k)},$$

the last inequality following from Theorem 6.1. This yields $\sum_k \mathbb{P}(E_k) = \infty$. Since the events (E_k) are independent, we can apply the Borel–Cantelli lemma to see that, almost surely, there are infinitely many k satisfying:

$$x - 2\delta \leqslant \frac{L_1^0(\tilde{\alpha}_k)}{\phi(n(k))} \leqslant x + 2\delta, \qquad y - 2\delta \leqslant \frac{L_1^*(\tilde{\alpha}_k)}{\phi(n(k))} \leqslant y + 2\delta. \tag{7.4}$$

Now we want to show that for all sufficiently large k, $L_1^x(\tilde{\alpha}_k)$ is 'very close' to $L_1^x(\alpha_{n(k)})$, uniformly in x. The idea is to apply Fact 7.2 to the processes $Z_1 \stackrel{\text{def}}{=} \tilde{\alpha}_k$ and $Z_2 \stackrel{\text{def}}{=} \alpha_{n(k)}$. First, observe that

$$\sqrt{n(k)}\alpha_{n(k)} = \sqrt{n(k-1)}\alpha_{n(k-1)} + \sqrt{\Delta(k)}\tilde{\alpha}_k.$$

Therefore,

$$\sup_{0\leqslant t\leqslant 1} |\tilde{\alpha}_k(t) - \alpha_{n(k)}(t)| \leqslant \frac{\sqrt{n(k)} - \sqrt{\Delta(k)}}{\sqrt{\Delta(k)}} \sup_{0\leqslant t\leqslant 1} |\alpha_{n(k)}(t)| + \frac{\sqrt{n(k-1)}}{\sqrt{\Delta(k)}} \sup_{0\leqslant t\leqslant 1} |\alpha_{n(k-1)}(t)|$$

$$\leqslant k^{-8} \sup_{0\leqslant t\leqslant 1} |\alpha_{n(k)}(t)| + k^{-8} \sup_{0\leqslant t\leqslant 1} |\alpha_{n(k-1)}(t)|.$$

Applying the LIL for α_n (see Fact 7.4) gives that (almost surely) for all large k,

$$\sup_{0\leqslant t\leqslant 1} |\tilde{\alpha}_k(t) - \alpha_{n(k)}(t)| \leqslant k^{-7}. \tag{7.5}$$

Now we study the oscillations of the local times of $\tilde{\alpha}_k$ and $\alpha_{n(k)}$. Fix $0 < \mu < \frac{1}{4}$. By (7.1) and (7.3),

$$\mathbb{P}\left(\sup_{x,y\in\mathbb{R},|x-y|\leqslant k^{-3}} |L_1^x(\alpha_{n(k)}) - L_1^y(\alpha_{n(k)})| \geqslant k^{-1} \right)$$

$$\leqslant n^{-2}(k) + \mathbb{P}\left(\sup_{x,y\in\mathbb{R},|x-y|\leqslant k^{-3}} |L_1^x(\gamma) - L_1^y(\gamma)| \geqslant k^{-1} - 2n^{-\mu}(k) \right)$$

$$\leqslant n^{-2}(k) + k^{-3}$$

$$\leqslant k^{-2}. \tag{7.6}$$

Similarly, we have

$$\mathbb{P}\left(\sup_{x,y\in\mathbb{R},|x-y|\leqslant k^{-3}} |L_1^x(\alpha_{\Delta(k)}) - L_1^y(\alpha_{\Delta(k)})| \geqslant k^{-1} \right) \leqslant k^{-2}. \tag{7.7}$$

Recall that $\tilde{\alpha}_k$ has the same law as $\alpha_{\Delta(k)}$. Applying the Borel–Cantelli lemma to (7.7) and (7.6) yields that almost surely, for all large k,

$$\sup_{x,y\in\mathbb{R},|x-y|\leqslant k^{-3}} |L_1^x(\tilde{\alpha}_k) - L_1^y(\tilde{\alpha}_k)| \leqslant k^{-1},$$

$$\sup_{x,y\in\mathbb{R},|x-y|\leqslant k^{-3}} |L_1^x(\alpha_{n(k)}) - L_1^y(\alpha_{n(k)})| \leqslant k^{-1}.$$

Combining these two inequalities with (7.5), and applying Fact 7.2 to $b = k^{-3}$, we obtain that, for all large k,

$$\sup_{x\in\mathbb{R}} |L_1^x(\tilde{\alpha}_k) - L_1^x(\alpha_{n(k)})| \leqslant \frac{3}{k}.$$

In view of (7.4), we have proved that, with probability one, there are infinitely many n such that

$$x - 3\delta \leqslant \frac{L_1^0(\alpha_n)}{\phi(n)} \leqslant x + 3\delta, \qquad y - 3\delta \leqslant \frac{L_1^*(\alpha_n)}{\phi(n)} \leqslant y + 3\delta.$$

This completes the proof. □

8. Local time of the reflecting empirical process

We start with the local time of the reflecting Brownian bridge. Let W be a Wiener process, and let G denote the last zero of W before time 1 (see (2.1)). The process

$$\left\{ \frac{|W(G + (1-G)t)|}{(1-G)^{1/2}};\ 0 \leqslant t \leqslant 1 \right\}$$

is referred to by Chung (1976) as the *Brownian meander process*.

It is observed by Kennedy (1976) that the supremum of the meander is distributed as $2\sup_{0\leqslant t\leqslant 1}|\gamma(t)|$, where γ is a Brownian bridge. A pathwise explanation to this (*à la* Vervaat) is provided by Biane and Yor (1987); see also Bertoin and Pitman (1994).

The following analogue of Jeulin's theorem (Theorem D) for the local time of the reflecting Brownian bridge is known.

Theorem E (Biane and Yor 1987). *Let* $K(s) \overset{\mathrm{def}}{=} \int_0^s L_1^x(|\gamma|)\,\mathrm{d}x$ *for all* $s \geqslant 0$*; then*

$$\{\tfrac{1}{2}L_1^{K^{-1}(t)}(|\gamma|);\ 0 \leqslant t \leqslant 1\}$$

is distributed as a Brownian meander process.

Remark 8.1. For a unified approach to Theorems E and D, as well as for some extensions, we refer to Carmona *et al.* (1999), Pitman (1999).

From Theorem E, we can easily deduce the following identity in law, which is the counterpart of Corollary 2.3:

$$(L_1^*(|\gamma|),\, L_1^0(|\gamma|)) \stackrel{\text{law}}{=} \left(2 \sup_{0\leqslant t\leqslant 1} m(t),\, 2m(1)\right), \tag{8.1}$$

where $L_1^*(|\gamma|) \stackrel{\text{def}}{=} \sup_{x\geqslant 0} L_1^x(|\gamma|)$, and $\{m(t);\, 0 \leqslant t \leqslant 1\}$ denotes a meander process.

The joint law of $\sup_{0\leqslant t\leqslant 1} m(t)$ and $m(1)$ is determined by the following 'Gauss transform': let $\mathscr{N}$ denote a Gaussian $\mathscr{N}(0, 1)$ variable, independent of the meander process m; then according to Pitman and Yor (1998b), for all $y > x > 0$,

$$\frac{\partial^2}{\partial x \partial y} \mathbb{P}\left(|\mathscr{N}| \sup_{0\leqslant t\leqslant 1} m(t) < y,\, |\mathscr{N}| m(1) < x\right) = \frac{\sinh x}{(\sinh y)^2}. \tag{8.2}$$

Unfortunately, we have not succeeded in obtaining accurate asymptotics of the joint tail of $L_1^*(|\gamma|)$ and $L_1^0(|\gamma|)$ from (8.1)–(8.2).

If we are only interested in the variable $L_1^*(|\gamma|)$, then (8.1) confirms that it has the same distribution as twice the supremum of the meander. The latter having been explicitly evaluated by Chung (1976) and Kennedy (1976), we arrive at:

Theorem 8.1. *For any* $x > 0$,

$$\begin{aligned}\mathbb{P}(L_1^*(|\gamma|) < x) &= \frac{\sqrt{32\pi}}{x} \sum_{k=1}^{\infty} \exp\left(-\frac{2(2k-1)^2\pi^2}{x^2}\right) \\ &= 1 - 2\sum_{k=1}^{\infty} (-1)^{k+1} \exp\left(-\frac{k^2x^2}{8}\right).\end{aligned}$$

In particular,

$$\log \mathbb{P}(L_1^*(|\gamma|) < y) \sim -\frac{2\pi^2}{y^2}, \qquad y \to 0^+,$$

$$\log \mathbb{P}(L_1^*(|\gamma|) > y) \sim -\frac{y^2}{8}, \qquad y \to \infty.$$

We also have the following LILs for the maximum local time of the reflecting empirical process, which is to be compared with Theorem B. Note that the local time at 0 of the reflecting empirical process is easy, since it is twice that of the original empirical process.

Theorem 8.2. *Let* α_n *be a uniform empirical process, and let*

$$L_1^*(|\alpha_n|) \stackrel{\text{def}}{=} \sup_{x\geqslant 0} L_1^x(|\alpha_n|).$$

Then

$$\limsup_{n\to\infty} \frac{L_1^*(|\alpha_n|)}{\phi(n)} = 2 \quad \text{a.s.},$$

$$\liminf_{n\to\infty} (\log\log n)^{1/2} L_1^*(|\alpha_n|) = \sqrt{2}\pi \quad \text{a.s.},$$

where ϕ *is defined in* (1.3).

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