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SOME ASYMPTOTIC RESULTS IN A MODEL OF POPULATION GROWTH[1]

I. A Class of Birth and Death Processes

BY BURTON SINGER

Columbia University

1. Introduction and summary. Motivated by ecological and genetic phenomena, Karlin and McGregor [3] introduced the following model to describe the continued formation and growth of mutant biological populations. It is assumed that a new mutant population arises at each event time of a stochastic process (referred to as the input process) $\{\nu(t), t > 0\}$ whose state space is the non-negative integers. Each new mutant population begins its evolution with a fixed number of elements and evolves according to the laws of a continuous time Markov chain $\mathscr{P}$ with stationary transition probability function

$$P_{i,j}(t) \qquad i, j = 0, 1, 2, \cdots; t \geqq 0.$$

We assume that all populations evolve according to the same Markov Chain and independent of one another. In terms of this structure, the basic question which we consider in this work can be formulated in the following manner:

(A) Given an input process $\{\nu(t), t > 0\}$ and the individual growing process $\mathscr{P}$, determine the asymptotic behavior as $t \to \infty$ of the mean and variance of $S(t) = \{$number of different sizes of mutant populations at time $t\}$ and determine the limit distribution as $t \to \infty$ of $S(t)$ appropriately normalized.

$S(t)$ is a special functional of the vector process

$$N(t) = \{N_0(t), N_1(t), N_2(t), \cdots\} \qquad t > 0$$

where $N_k(t) = \{$number of mutant populations with exactly k members at time $t\}$ and may be interpreted as a measure of the fluctuations in population size. We have restricted our considerations to this special case because it serves as a model problem for more general situations and possesses all the subtle difficulties of the general case.

The random variable $S(t)$ can also be identified as the number of distinct occupied states at time t among all Markov Chains which have begun their evolution up to that time. In the subsequent discussion we will refer to $\{S(t), t > 0\}$ as the "occupied states" process generated by the input process $\{\nu(t), t > 0\}$ and the Markov Chain $\mathscr{P}$. Without loss of generality we identify the state 0 as the initial state of all evolving Markov Chains and -1 as an absorbing state if absorption is possible.

In this paper we introduce "occupied states" processes generated by a class of

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null recurrent, transient, and absorbing barrier Birth and Death processes and a Poisson input process. The special feature of this class is that with the normalization

$$Y(t;u) = t^{-1}X(t^{\alpha}u), \qquad \alpha > 0$$

($\{X(t), t > 0\}$ is the growing process $\mathscr{P}$), the process $Y(t;u)$ converges weakly in the Markov sense as $t \to \infty$ to a Bessel diffusion (see C. Stone [8]). The main idea (also applicable to more general growing processes $\mathscr{P}$) is that one requires local limit theorems, and under some circumstances, specification of the rate of convergence of the transition density of $Y(t;u)$ to the density of the limiting diffusion in order to prescribe *exact* asymptotic formulas for $ES(t)$ and $\operatorname{Var} S(t)$ and to prove a central limit theorem for $S(t)$. The results of this paper are in sharp contrast with the asymptotic formulas for $ES(t)$ and $\operatorname{Var} S(t)$ which appear in the companion paper [6], where the growing process $\mathscr{P}$ is a general positive recurrent Markov chain and the input process remains Poisson.

Section 2 contains basic definitions, some intuitive discussion, and precise statements of the main results on asymptotic behavior of $ES(t)$ and $\operatorname{Var} S(t)$. In Section 3, we present detailed proofs of the theorems of Section 2, and we conclude with a central limit theorem for $S(t)$ in Section 4. The appendix contains some technical lemmas which are essential for asymptotic formulas that incorporate speed of convergence theorems.

2. Main results with discussion. A birth and death process is a stationary Markov process whose state space is the non-negative integers and whose transition probability matrix

$$P_{ij}(t) = \Pr\{X(t+s) = j \mid X(s) = i\}$$

satisfies the conditions

$$\begin{aligned} P_{ij}(t) &= \lambda_i t + o(t) && \text{if } j = i+1, \\ &= \mu_i t + o(t) && \text{if } j = i-1, \\ &= 1-(\lambda_i+\mu_i)t + o(t) && \text{if } j = i; \end{aligned}$$

as $t \to 0$ where $\lambda_i > 0$ for $i \geqq 0$, $\mu_i > 0$ for $i \geqq 1$ and $\mu_0 \geqq 0$. We restrict our consideration to birth and death processes satisfying

$$\text{(I)} \qquad \pi_n \sim Dn^{\gamma-1}, \qquad (\lambda_n \pi_n)^{-1} \sim Cn^{\beta-1}, \qquad n \to \infty$$

where $\pi_n = (\lambda_0 \lambda_1 \cdots \lambda_{n-1})/(\mu_1 \mu_2 \cdots \mu_n)$,

$C, D, \gamma, \beta+\gamma$ are positive constants and we assume $\beta < 0$ when $\mu_0 > 0$. When $\mu_0 = 0$ and (I) is satisfied, the birth and death process is null recurrent or transient according as $\beta \geqq 0$ or $\beta < 0$ respectively. For $\mu_0 > 0$ $\{X(t), t > 0\}$ is an absorbing barrier birth and death process where absorption occurs with positive probability strictly less than 1. The sample paths of these processes which do not get absorbed behave like a transient process whose parameters satisfy (I) with $\beta < 0$.

This distinction between null recurrent and transient Markov Chains does not, however, seem to be the deciding factor in determining the order of magnitude of the mean and variance of $S(t)$. There is a natural trichotomy depending on whether $\beta+\gamma$ is strictly greater than 1, equal to 1, or strictly less than 1 which leads to three distinct types of asymptotic formulas. Roughly speaking, the quantity $(\beta+\gamma)^{-1}$ is a measure of the rate at which individual particles move away from their initial state. The more rapidly an individual particle can reach a state n starting from 0, the greater the expected number of distinct occupied states among all particles in existence. In particular, if $T_{0,n} = \inf\{t: X(t) = n+1; X(0) = 0\}$, then subject to (I)

$$ET_{0,n} \sim Cn^{\beta+\gamma} \quad \text{as} \quad n \to \infty$$

where C is a constant independent of n. According to the previously mentioned trichotomy $ES(t)$ and $\operatorname{Var} S(t)$ have the most rapid growth for $\beta+\gamma < 1$, and the growth decreases for $\beta+\gamma = 1$ and $\beta+\gamma > 1$ respectively.

Another interpretation of the role of $\beta+\gamma$ and the rate at which individual particles spread out from their initial state can be seen if one considers that subject to hypotheses (I)

$$EM(t) \sim Kt^{(\beta+\gamma)^{-1}} \quad \text{as} \quad t \to \infty,$$

where

$$M(t) = \max_{0 \leqq u \leqq t} X(u),$$

and

$$\lim_{t\to\infty} \Pr\left(M(t)/t^{(\beta+\gamma)^{-1}} \leqq x\right) = G(x)$$

where $G(x)$ is a non-degenerate distribution function which we will describe in the proof of Theorem 2.3.

With these preliminaries at hand, our principal results take the following form.

THEOREM 2.1. *If* $0 < \beta+\gamma < 1$, *then*

$$ES(t) \sim t \tag{1}$$

and

$$\operatorname{Var} S(t) \sim t \quad \text{as} \quad t \to \infty. \tag{2}$$

It is assumed here and in Theorems 2.2–2.4, without loss of generality, that the Poisson input process has parameter $\lambda = 1$.

Note. (1) implies that we can expect as many distinct occupied states at time t as there are populations in existence.

THEOREM 2.2. *If* $\beta+\gamma = 1$, *then*

$$ES(t) \sim C_1 t \tag{3}$$

$$\operatorname{Var} S(t) \sim C_2 t \tag{4}$$

where

$$C_1 = \int_0^\infty \left(1 - \exp\left[-\int_0^1 (1-q_0) g(0, w; u)\, du\right] dw,\right.$$

$$C_2 = \int_0^\infty \left(\exp\left[-\int_0^1 (1-q_0) g(0, w; u)\, du\right] - \exp\left[-2\int_0^1 (1-q_0) g(0, w; u)\, du\right]\right) dw,$$

and $g(0, w; u) = d_1(w^{\gamma-1})u^{-\gamma/(\beta+\gamma)}\exp[-d_2u^{-1}w^{\beta+\gamma}]$ *is the transition density corresponding to a Bessel diffusion with initial state* 0, *present state* w, *and time parameter* u. The constants d_1 and d_2 are given explicitly in the statement of Theorem 2.5, and q_0 = probability of eventual absorption starting from state 0 for a Birth and Death process satisfying (I).

THEOREM 2.3. *If* $\beta+\gamma > 1$, *then*

(5) $$t^{-[(\beta+\gamma)^{-1}+\varepsilon]}ES(t) \to 0 \quad as \quad t \to \infty, \qquad \forall \varepsilon > 0$$

(6) $$t^{-[(\beta+\gamma)^{-1}]}ES(t) \to \infty \quad as \quad t \to \infty,$$

(7) $$t^{-[(\beta+\gamma)^{-1}+\varepsilon]}\operatorname{Var} S(t) \to 0 \quad as \quad t \to \infty \qquad \forall \varepsilon > 0.$$

Exact asymptotic formulas analogous to (1)–(4) for processes satisfying the condition $\beta+\gamma > 1$ are not known in general. The principal difficulty in establishing such results is the requirement of a delicate estimate of the rate of convergence of the transition density of the normalized Birth and Death process $c^{-1}X(c^{\alpha}t)$ as $c \to \infty$ to the transition density of the limiting diffusion process $X_0(t)$.

The appropriate formulas should be of the form

(8) $$ES(t) \sim t^{(\beta+\gamma)^{-1}}L_1(t), \qquad t \to \infty$$

(9) $$\operatorname{Var} S(t) \sim t^{(\beta+\gamma)^{-1}}L_2(t), \qquad t \to \infty$$

where $L_1(t)$ and $L_2(t)$ are slowly varying functions in the sense of Karamata, $L_1(t) \to \infty$, $t \to \infty$, and $L_2(t) \to 0$, $t \to \infty$. An example of this kind of result is contained in Theorem 2.4 which is a special case of $\beta+\gamma = 2$. We defer a discussion of the class of Birth and Death processes for which (8) and (9) hold, and the accompanying rate of convergence theorems, to a separate work. For a particular case, consider the process

$$X(2t) = \textstyle\sum_{k=0}^{m(2t)} Z_k$$

where

$$Z_k = +1 \quad \text{with probability } \tfrac{1}{2},$$
$$= -1 \quad \text{with probability } \tfrac{1}{2}, \qquad k = 0, 1, 2, \cdots$$

are independent of each other and of $m(t)$, a Poisson process with parameter $\lambda = 1$. Then introduce the absolute value process $Y(t) = |X(2t)|$.

THEOREM 2.4. *If* $\{S(t), t > 0\}$ *is the "occupied states" process generated by a Poisson input process with parameter* $\lambda = 1$ *and the absolute value process* $Y(t)$ *described above, then*

(10) $$ES(t) \sim (2t\log t)^{\frac{1}{2}} \qquad t \to \infty$$

(11) $$\operatorname{Var} S(t) \sim (c_3 2^{\frac{1}{2}}\log 2)(t^{\frac{1}{2}}/\log^{\frac{1}{2}} t) \qquad t \to \infty$$

where $$c_3 = \textstyle\int_0^{\infty} e^{-1/y}(2y^{-2}-y^{-3})\,dy.$$

NOTE. In the context of conditions (I), the absolute value process $Y(t)$ is a Birth and Death process with $\pi_n = 2, (\lambda_n \pi_n)^{-1} = \frac{1}{2}$ for $n = 1, 2 \cdots$. Thus $D = 2, C = \frac{1}{2}$, $\beta = \gamma = 1$.

The slowly varying functions which appear in (8) and (9) have an interesting interpretation in terms of a related infinite urn scheme. In particular, suppose that at each event time of a Poisson process one ball is thrown at an infinite array of cells and has probability p_k of hitting the kth cell, $k = 0, 1, 2, \cdots$. If $\hat{S}(t)$ denotes the number of occupied cells at time t, then

$$E\hat{S}(t) = \textstyle\sum_{k=0}^{\infty} (1 - e^{-tp_k}) \tag{12}$$

and

$$\operatorname{Var} \hat{S}(t) = \textstyle\sum_{k=0}^{\infty} (e^{-tp_k} - e^{-2tp_k}). \tag{13}$$

The relevance of these formulas for the "occupied states" process appears if we make the identification

$$p_k = \textstyle\int_0^1 g(0, k; u)\, du. \tag{14}$$

Then, to establish (8) one uses the speed of convergence, as well as a local limit theorem for large deviations, to show that

$$ES(t) \sim t^{(\beta+\gamma)^{-1}} \textstyle\sum_{k=0}^{\infty} (1 - \exp(-t^{[1-(\beta+\gamma)^{-1}]} \int_0^1 g(0, k; u)\, du)). \tag{15}$$

The infinite sum in (15) is just the expected number of occupied cells at time $t^{[1-(\beta+\gamma)^{-1}]}$ in the urn scheme described above. Now we introduce the function

$$\alpha(t) = \max\{k : p_k \geqq t^{-1}\}, \tag{16}$$

and notice that for the special identification (14),

$$\alpha(t) = L(t) \quad \text{slowly varying as} \quad t \to \infty. \tag{17}$$

By Theorem 1 of Karlin [5], condition (17) implies that

$$E\hat{S}(t) \sim L(t), \qquad t \to \infty.$$

Hence,

$$\begin{aligned} ES(t) &\sim t^{(\beta+\gamma)^{-1}} L(t^{1-(\beta+\gamma)^{-1}}) \\ &= t^{(\beta+\gamma)^{-1}} L_1(t), \qquad t \to \infty. \end{aligned}$$

To establish (9), the speed of convergence and large deviation form of a local limit theorem are used to show

$$\operatorname{Var} S(t) \sim t^{(\beta+\gamma)^{-1}} J(t) \tag{18}$$

where

$$\begin{aligned} J(t) = \textstyle\int_0^{\infty} [&\exp(-t^{[1-(\beta+\gamma)^{-1}]} \textstyle\int_0^1 g(0, y; u)\, du) \\ &- \exp(-2t^{[1-(\beta+\gamma)^{-1}]} \textstyle\int_0^1 g(0, y; u)\, du)]\, dy. \end{aligned}$$

The integral in (18) cannot, however, be replaced by $\operatorname{Var} \hat{S}(t^{[1-(\beta+\gamma)^{-1}]})$ where the infinite urn scheme has probabilities satisfying (14). To see the distinction between these two expressions it is convenient to rewrite the above integral as

(19) $$\int_0^\infty e^{-1/y}(2y^{-2}-y^{-3})((yt)^{-1}\int_0^{yt}(\alpha^*(2x)-\alpha^*(x))\,dx)\,dy,$$

(replacing $t^{[1-(\beta+\gamma)^{-1}]}$ by t) and to rewrite $\operatorname{Var}\hat{S}(t)$ as

(20) $$\int_0^\infty e^{-1/y}(2y^{-2}-y^{-3})((yt)^{-1}\int_0^{yt}(\alpha(2x)-\alpha(x))\,dx)\,dy,$$

where

$$\alpha^*(x) = p^{-1}(x^{-1}),$$
$$p(y) = \int_0^1 g(0,y;u)\,du,$$
$$\alpha(x) = [\alpha^*(x)],$$

($[a]$ = integer part of a).

The asymptotic behavior of $J(t)$ reduces to asymptotic behavior of

(21) $$I^*(x) = x^{-1}\int_0^x(\alpha^*(2w)-\alpha^*(w))\,dw \quad \text{as} \quad x\to\infty$$

while asymptotic behavior of $\operatorname{Var}\hat{S}(t)$ is determined by that of

$$I(x) = x^{-1}\int_0^x(\alpha(2w)-\alpha(w))\,dw.$$

These two expressions, however, are not asymptotic to each other for $\beta+\gamma > 1$. In fact for the absolute value process of Theorem 2.4, we have $I^*(x) \sim \log 2\log^{-\frac{1}{2}} x$, $x\to\infty$,

and $$I(x) \sim e^{\Omega(x)}, \qquad x\to\infty,$$

where $$\Omega(x) = -\delta(x)\log^{\frac{1}{2}} x + \tfrac{1}{4}\delta^2(x),$$
$$\delta(x) = (2\log^{\frac{1}{2}} x + O(\log\log x\log^{-\frac{1}{2}} x)) \pmod 1.$$

Thus the final formula (9) is determined in general by showing that $I^*(x) \sim L(x)$, $x\to\infty$ where $L(x)$ is slowly varying and converges to 0 as $x\to\infty$.

We conclude this section with statements of the local limit theorems which are required in the proofs of Theorems 2.1–2.4. For a discussion and detailed proofs of these and other local limit theorems see B. Singer [7].

THEOREM 2.5. (i) *If the infinitesimal parameters of a Birth and Death process satisfy* (I) *with* $\alpha = \beta/(\beta+\gamma) > 0$ *or* $\alpha \leqq 0$ *and* $t^j P_{00}(t)$ *monotone for t sufficiently large,* $-j < \alpha \leqq -(j-1)$, $j = 1,2,\cdots$ *then*

(22) $$tP_{[xt],[wt]}(t^{\beta+\gamma}u) \to g(x,w;u), \qquad t\to\infty$$

where $g(x,w;u)$ *is the transition density of a Bessel diffusion with the explicit formula*

$$g(x,w;u) = DK_0 w^{\gamma-1}\int_0^\infty e^{-su} I(-sx^{\beta+\gamma})I(-sw^{\beta+\gamma})s^{-\alpha}\,ds,$$
$$I(s) = \Gamma(1-\alpha)[CDs(\beta+\gamma)^{-2}]^{\alpha/2} I_{-\alpha}(2(CDs)^{\frac{1}{2}}(\beta+\gamma)^{-1}),$$

(23) $$K_0 = (\beta+\gamma)^{2\alpha-1}[C^{\alpha-1}D^\alpha\Gamma^2(1-\alpha)]^{-1} \quad \textit{for} \quad \alpha > 0,$$
$$= (-1)^j[\Gamma(1-\alpha)\Gamma(j+\alpha)]^{-1}\cdot C(\beta+\gamma)^{-1}$$
$$\cdot\int_0^\infty y^{j+\alpha-1} d^j/dy^j(I^{-2}(y))\,dy$$

for $-j < \alpha \leqq -(j-1), j = 1, 2, \cdots$. The convergence is uniform in x, w, u where $0 < x, w, u \leqq M < \infty$.

NOTE. An integral form of (i) for $\alpha > 0$ was proved by W. Studden, [9].

(ii) Subject to the hypotheses of (i)

$$tP_{i,[tw]}(t^{\beta+\gamma}u) \to g(0, w; u) \quad \text{for} \quad \mu_0 = 0, \qquad t \to \infty$$

$$\to (1-q_i)g(0, w; u) \quad \text{for} \quad \mu_0 > 0 \quad \text{and} \quad \alpha < 0.$$

$$q_i = \frac{\mu_0 \sum_{n=i}^{\infty} (\lambda_n \pi_n)^{-1}}{1 + \mu_0 \sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1}}$$

= probability of eventual absorption starting from state i.

$$g(0, w; u) = d_1 w^{\gamma-1} u^{-\gamma/(\beta+\gamma)} \exp(-d_2 u^{-1} w^{\beta+\gamma})$$

where

$$d_1 = DK_0 \Gamma(1-\alpha),$$

$$d_2 = CD(\beta+\gamma)^{-2}.$$

For the special example of $\beta+\gamma = 2$ corresponding to the absolute value process described previously, we require the rate of convergence estimate given by

THEOREM 2.6. *If* $Y(t) = \left|\sum_{k=0}^{m(2t)} Z_k\right|$ *where*

$$Z_k = +1 \quad \textit{with probability} \quad \tfrac{1}{2},$$

$$\quad -1 \quad \textit{with probability} \quad \tfrac{1}{2}, \qquad k = 0, 1, 2, \cdots$$

are independent of each other and of $m(t)$, *a Poisson process with parameter* $\lambda = 1$, *then*

$$\left|t^{\frac{1}{2}} P_{0,j}(tu) - g(0, t^{-\frac{1}{2}}j; u)\right| \leqq t^{-(1-4\alpha)} M g(0, t^{-\frac{1}{2}}j; u)$$

for t *sufficiently large, where* $j \in [C_3 t^{\frac{1}{2}} L_3(t),\ C_4 t^{\frac{1}{2}+\alpha}], u \geqq L_4(t), 0 < \alpha < \frac{1}{8}$, C_3 *and* C_4 *are arbitrary positive constants,* M *a positive constant independent of* t *and* u, *and* $L_i(t)$ *are slowly varying functions such that* $L_i(t) \to 0, t \to \infty$ *for* $i = 3, 4$, *and*

$$g(0, t^{-\frac{1}{2}}j; u) = (\pi u)^{-\frac{1}{2}} \exp(-j^2/4tu).$$

REMARK. In contrast to Theorem 2.5, this is also an assertion about large deviations since for values of j of the form $[xt^{-\frac{1}{2}}]$, x is no longer restricted to a bounded region independent of t. The full power of Theorem 2.6 is required to justify the asymptotic formulas (10) and (11).

3. Technical details. To establish Theorems 2.1–2.4 we first require exact analytical expressions for $ES(t)$ and $\operatorname{Var} S(t)$. The assumption of a Poisson input process with parameter $\lambda = 1$ immediately implies that $\{N_k(t)\}_{k=0}^{\infty}$ are independent Poisson random variables with parameters $\int_0^t P_{0,k}(\tau)\, d\tau$, $k = 0, 1, 2, \cdots$, respectively (e.g. Karlin [4] or Karlin and McGregor [3]). Then setting

$$X_k(t) = 1 \quad \text{if} \quad N_k(t) > 0,$$

$$= 0 \quad \text{if} \quad N_k(t) = 0,$$

we may write

(24) $S(t) = \sum_{k=0}^{\infty} X_k(t), \quad ES(t) = \sum_{k=0}^{\infty} (1-\exp(-\int_0^t P_{0,k}(\tau)\,d\tau)),$

(25) $\operatorname{Var} S(t) = \sum_{k=0}^{\infty} (\exp(-\int_0^t P_{0,k}(\tau)\,d\tau) - \exp(-2\int_0^t P_{0,k}(\tau)\,d\tau)).$

Since the Local Limit Theorem 2.5 requires the variable u to be bounded away from zero, we show that it is enough to consider the asymptotic behavior of

(26) $E_\delta S(t) = \sum_{k=0}^{\infty} (1-\exp(-t\int_\delta^1 P_{0,k}(tu)\,du))$

and

(27) $V_\delta S(t) = \sum_{k=0}^{\infty} (\exp(-t\int_\delta^1 P_{0,k}(tu)\,du) - \exp(-2t\int_\delta^1 P_{0,k}(tu)\,du))$

in Theorems 2.1 and 2.2, $\delta > 0$ is arbitrarily small and independent of t.

To this end choose $\delta > 0$ arbitrary and fix it. Then

$$\begin{aligned} 0 &\leqq t^{-1}(ES(t) - E_\delta S(t)) \\ &= t^{-1}\sum_{k=0}^{\infty} \exp(-t\int_\delta^1 P_{0,k}(tu)\,du)(1-\exp(-t\int_0^\delta P_{0,k}(tu)\,du)) \\ &\leqq \sum_{k=0}^{\infty} \int_0^\delta P_{0,k}(tu)\,du = \delta. \end{aligned}$$

A similar argument shows that $t^{-1}|\operatorname{Var} S(t) - V_\delta S(t)| \leqq 3\delta$.

Proof of Theorem 2.1. Choose $\varepsilon > 0$ sufficiently small that

(28) $\int_0^\varepsilon g(0,w;u)\,dw < \delta$ uniformity in $u \in [0,1]$,

and c sufficiently large that

(29) $\int_c^\infty g(0,w;u)\,dw < \delta, \qquad \forall\, u \in [\delta, 1].$

Bring in the inequalities

(30) $1 \geqq t^{-1}ES(t) \geqq t^{-1}I(t)$

where

(31) $I(t) = \sum_{k=[\varepsilon t^{(\beta+\gamma)^{-1}}]}^{[ct^{(\beta+\gamma)^{-1}}]} (1-\exp(-t\int_\delta^1 P_{0,k}(tu)\,du)).$

We will show that $t^{-1}I(t) \geqq 1-\delta^*$ for t sufficiently large and δ^* arbitrary.

To this end we first apply the elementary inequality $1-e^{-x} \geqq x - \frac{1}{2}x^2$ for $x > 0$ in (31) to obtain the lower bound

(32) $$\begin{aligned} I(t) \geqq{}& t\int_\delta^1 \Pr\{[\varepsilon t^{(\beta+\gamma)^{-1}}] \leqq X(tu) \leqq [ct^{(\beta+\gamma)^{-1}}] \mid X(0) = 0\}\,du \\ &- \tfrac{1}{2}t^2 \sum_{k=[\varepsilon t^{(\beta+\gamma)^{-1}}]}^{[ct^{(\beta+\gamma)^{-1}}]} \int_\delta^1\int_\delta^1 P_{0,k}(tu_1)P_{0,k}(tu_2)\,du_1\,du_2. \end{aligned}$$

Using (28), (29), and the integral form of Theorem 2.5, we have

(33) $\Pr\{X(tu) > ct^{(\beta+\gamma)^{-1}} \mid X(0) = 0\} \leqq \delta_1 + \delta,$

(34) $\Pr\{X(tu) < \varepsilon t^{(\beta+\gamma)^{-1}} \mid X(0) = 0\} \leqq \delta_1 + \delta$

where $\delta_1 > 0$ is arbitrary and the inequalities hold for t large, say $t > T(\delta_1)$. Substituting (33) and (34) in the first term of (32) we have

(35) $$t\int_\delta^1 \Pr\{\varepsilon t^{(\beta+\gamma)^{-1}} \leqq X(tu) \leqq Ct^{(\beta+\gamma)^{-1}} \mid X(0) = 0\}\, du \geqq t(1-2(\delta_1+\delta))(1-\delta).$$

For an upper bound on the second term in (32) we use the local limit Theorem 2.5 and a standard estimate to write

(36) $$\begin{aligned} &\tfrac{1}{2}t^2 \textstyle\sum_{k=[\varepsilon t^{(\beta+\gamma)^{-1}}]}^{[ct^{(\beta+\gamma)^{-1}}]} \int_\delta^1\int_\delta^1 P_{0,k}(tu_1)P_{0,k}(tu_2)\, du_1\, du_2 \\ &\leqq t^{2-2/(\beta+\gamma)}(1+\delta_1)^2 \int_{[\varepsilon t^{(\beta+\gamma)^{-1}}-1]}^{[ct^{(\beta+\gamma)^{-1}}]} \int_\delta^1\int_\delta^1 g(0, x/t^{(\beta+\gamma)^{-1}}; u_1) \\ &\times g(0, x/t^{(\beta+\gamma)^{-1}}; u_2)\, du_1\, du_2\, dx. \end{aligned}$$

Introduce the change of variable $x = t^{(\beta+\gamma)^{-1}}w$. Then the integrals in (36) become

(37) $$t^{2-(\beta+\gamma)^{-1}}(1+\delta_1)^2 \int_0^c\int_\delta^1\int_\delta^1 g(0,w;u_1)g(0,w;u_2)\, du_1\, du_2\, dw = At^{2-(\beta+\gamma)^{-1}}.$$

Combining (35)–(37) we have

$$t^{-1}I(t) \geqq (1-2(\delta_1+\delta))(1-\delta) - At^{1-(\beta+\gamma)^{-1}}.$$

Since $\beta+\gamma < 1$, the second term vanishes as $t \to \infty$, and the proof of (1) is complete.

To verify (2) we simply write $V_\delta S(t)$ as

$$V_\delta S(t) = \textstyle\sum_{k=0}^\infty (1-\exp(-2t\int_\delta^1 P_{0,k}(tu)\, du)) - \sum_{k=0}^\infty (1-\exp(-t\int_\delta^1 P_{0,k}(tu)\, du))$$

and apply the proof of (1) to each term to obtain $\operatorname{Var} S(t) \sim 2t - t = t$ as $t \to \infty$.

PROOF OF THEOREM 2.2. We decompose $E_\delta S(t)$ into three terms as

$$\begin{aligned} E_\delta S(t) &= \textstyle\sum_{k<[\varepsilon t]} + \sum_{k=[\varepsilon t]}^{[ct]} + \sum_{k>[ct]} (1-\exp(-t\int_\delta^1 P_{0,k}(tu)\, du)), \\ &= I_1(t) + I_2(t) + I_3(t) \end{aligned}$$

where $\varepsilon > 0$ is arbitrarily small and c is arbitrarily large but independent of t.

$t^{-1}I_1(t) < \varepsilon$ trivially, since each term in this sum is bounded above by 1.

$$\begin{aligned} t^{-1}I_3(t) &< \textstyle\int_\delta^1 \Pr\{X(tu) > ct \mid X(0) = 0\}\, du \leqq \int_\delta^1 ((\int_c^\infty g(0,w;u)\, dw) + \delta_1)\, du \\ &< \delta_2 + \delta_1 \end{aligned}$$

where $\delta_2 > 0$ is chosen arbitrarily and then c is large enough that

$$\int_c^\infty g(0,w;u)\, dw < \delta_2, \qquad \forall u \in [\delta, 1]$$

$\delta_1 > 0$ is chosen arbitrarily, and the integral form of Theorem 2.5 implies that for $t > T(\delta_1)$

$$\left|\Pr\{X(tu) > ct \mid X(0) = 0\} - \int_c^\infty g(0,w;u)\, dw\right| < \delta_1.$$

Thus the growth of $ES(t)$ is subsumed in the second term, $I_2(t)$. Rewrite $I_2(t)$ as

(38) $$I_2(t) = \textstyle\sum_{k=[\varepsilon t]}^{[ct]} (1-\exp(-\int_\delta^1 g(0, t^{-1}k; u)\, du)) + e_1(t)$$

where

$$e_1(t) = \sum_{k=[\varepsilon t]}^{[ct]} \exp\left(-\int_\delta^1 g(0, t^{-1}k; u)\, du\right)$$
$$\left(1 - \exp\left(-\int_\delta^1 (tP_{0,k}(tu) - g(0, t^{-1}k; u))\, du\right)\right).$$

Routine estimates applied to the first term in (38) allow us to write this sum as

$$\int_{\varepsilon t}^{ct} \left(1 - \exp\left(-\int_\delta^1 g(0, t^{-1}x; u)\, du\right)\right) dx + O(1). \tag{39}$$

Introduce the change of variable $x = wt$ in the integral in (39), and we obtain

$$t^{-1} \sum_{k=[\varepsilon t]}^{[ct]} \left(1 - \exp\left(-\int_\delta^1 g(0, t^{-1}k; u)\, du\right)\right) \tag{40}$$
$$\to \int_\varepsilon^c \left(1 - \exp\left(-\int_\delta^1 g(0, w; u)\, du\right)\right) dw \qquad \text{as} \quad t \to \infty.$$

For the error term $e_1(t)$ in (38) we use the Local Limit Theorem 2.5 to assert that for arbitrary $c^{-1}\varepsilon_1 = \varepsilon^* > 0 \; \exists \; T(\varepsilon_1, c)$ such that $t > T(\varepsilon_1, c)$ implies

$$|tP_{0,k}(tu) - g(0, t^{-1}k; u)| < \varepsilon^*. \tag{41}$$

Using (41) in $e_1(t)$ yields the inequalities

$$e_1(t) \leqq \sum_{k=[\varepsilon t]}^{[ct]} \exp\left(-\int_\delta^1 g(0, t^{-1}k; u)\, du\right)(1 - \exp(-\varepsilon^*)) < \varepsilon^* ct. \tag{42}$$

and

$$e_1(t) \geqq -\sum_{k=[\varepsilon t]}^{[ct]} \exp\left(-\int_\delta^1 g(0, t^{-1}k; u)\, du\right) \exp(\varepsilon^*)(1 - \exp(-\varepsilon^*)) \tag{43}$$
$$\geqq -\varepsilon_1^* ct \exp(\varepsilon^*).$$

Then (42) and (43) imply that $t^{-1}e_1(t) \to 0$ as $t \to \infty$.

Since $\varepsilon > 0$ is arbitrarily small and c is arbitrarily large, we have

$$E_\delta S(t) \sim t \int_0^\infty \left(1 - \exp\left(-\int_\delta^1 g(0, w; u)\, du\right)\right) dw, \qquad t \to \infty$$

and the proof of (3) is complete.

To show that $\operatorname{Var} S(t) \sim C_2 t$, $t \to \infty$, we rewrite $V_\delta S(t)$ as in the proof of Theorem 2.1 and apply the asymptotic formula (3) to each term. □

For ease of exposition we divide the proof of Theorem 2.3 into two lemmas.

LEMMA 3.1. *If* $\beta + \gamma > 1$, *then* $\forall\, \varepsilon > 0$.

$$ES(t)t^{-[(\beta+\gamma)^{-1}+\varepsilon]} \to 0 \quad as \quad t \to \infty; \tag{5}$$

$$\operatorname{Var} S(t)t^{-[(\beta+\gamma)^{-1}+\varepsilon]} \to 0 \quad as \quad t \to \infty. \tag{7}$$

PROOF. Let $M_i(t) = \max_{0 \leqq s \leqq t} X_i(s)$ where $X_i(s)$, $i = 1, 2, \cdots$ are independent copies of Birth and Death processes satisfying the hypotheses (I) with $\beta + \gamma > 1$. Then notice that

$$S(t) \leqq \max_{1 \leqq i \leqq \nu(t)} \{M_i(t)\}. \tag{44}$$

$\nu(t)$ is the Poisson input process with parameter $\lambda = 1$.

Thus

$$ES(t)t^{-[(\beta+\gamma)^{-1}+\varepsilon]} \leqq E_{\nu(t)} E\left(\max_{1 \leqq i \leqq \nu(t)} (M_i(t)t^{-[(\beta+\gamma)^{-1}+\varepsilon)}) \mid \nu(t)\right). \tag{45}$$

Now define $G_t(u) = \Pr\{M(t)t^{-(\beta+\gamma)^{-1}} \leqq u\}$. Then we have

$$E(\max_{1\leqq i\leqq \nu(t)}(M_i(t)t^{-[(\beta+\gamma)^{-1}+\varepsilon]}) \mid \nu(t)=k)$$

$$\leqq \delta + \int_\delta^\infty (1-G_t^k(t^\varepsilon u)\,du$$

$$\leqq \delta + k\int_\delta^\infty (1-G_t(t^\varepsilon u))\,du \tag{46}$$

$$\leqq \delta + k\int_\delta^\infty du \int_{t^\varepsilon u}^\infty dG_t(y)$$

$$\leqq \delta + k(r-1)^{-1}\delta^{-(r-1)}t^{-\varepsilon r}\int_0^\infty y^r\,dG_t(y)$$

where $\delta > 0$ is arbitrary. Choose $r > \varepsilon^{-1}$ and fix it. Substituting (46) in the inequality (45) we obtain

$$ES(t)t^{-[(\beta+\delta)^{-1}+\varepsilon]} \leqq \delta + [(r-1)\delta^{(r-1)}t^{(\varepsilon r-1)}]^{-1}\int_0^\infty y^r\,dG_t(y). \tag{47}$$

Now $G_t(x)$ is a distribution function having moments of arbitrary order, and it converges to a non-degenerate distribution function $G_\infty(x)$ at continuity points of $G_\infty(\cdot)$. The limit distribution is known explicitly in the sense that

$$G_\infty(x) = 1 - F(x^{-(\beta+\gamma)}) \tag{48}$$

where the Laplace transform of $F(\cdot)$ is given by

$$\int_0^\infty e^{-sx}\,dF(x) = (I(s))^{-1} \tag{49}$$

and

$$I(s) = \Gamma(\gamma(\beta+\gamma)^{-1})[CDs(\beta+\gamma)^{-2}]^{\beta/2(\beta+\gamma)} I_{-\beta/(\beta+\gamma)}(2(CDs)^{\frac{1}{2}}(\beta+\gamma)^{-1}).$$

See, e.g., Karlin and McGregor [2]. The convergence of G_t to a limit distribution and the fact that for every $r > 0$, $t \in (0, \infty)$, x^r is uniformly integrable with respect to $\{G_t(x)\}$, implies the inequality

$$ES(t)t^{-[(\beta+\gamma)^{-1}+\varepsilon]} \leqq \delta K[t^{r\varepsilon-1}(r-1)\delta^{(r-1)}]^{-1} \tag{50}$$

where K is a constant independent of t. Letting $t \to \infty$ in (50) establishes (5).

To verify (7) rewrite $\operatorname{Var} S(t)$ as

$$\operatorname{Var} S(t) = \sum_{k=0}^\infty (1-\exp(-2\int_0^t P_{0,k}(\tau)d\tau)) - \sum_{k=0}^\infty (1-\exp(-\int_0^t P_{0,k}(\tau)\,d\tau))$$

and apply the above proof of (5) to both sums. □

LEMMA 3.2. *If* $\beta+\gamma > 1$, *then*

$$ES(t)t^{-(\beta+\gamma)^{-1}} \to \infty \quad as \quad t \to \infty. \tag{6}$$

PROOF. Invoking the Local Limit Theorem 2.5 we have for arbitrary δ_1 such that $0 < \delta_1 < 1$,

$$t^{(\beta+\gamma)^{-1}}P_{0,k}(tu) \geqq g(0, kt^{-(\beta+\gamma)^{-1}}; u)(1-\delta_1), \tag{51}$$

where $k \in [c_3 t^{(\beta+\gamma)^{-1}}, c_4 t^{(\beta+\gamma)^{-1}}]$, $t >$ some $T(\delta_1)$, and $0 < c_3 < c_4 < \infty$.

Using the inequality (51) we obtain the lower bound

$$ES(t)t^{(\beta+\gamma)^{-1}} \geqq t^{(\beta+\gamma)^{-1}} \sum_{k=[c_3 t^{(\beta+\gamma)^{-1}}]}^{[c_4 t^{(\beta+\gamma)^{-1}}]} (1-\exp(-t\int_\delta^1 P_{0,k}(tu)\,du))$$

$$\geqq t^{-(\beta+\gamma)^{-1}} \sum_{k=[c_3 t^{(\beta+\gamma)^{-1}}]}^{[c_4 t^{(\beta+\gamma)^{-1}}]} (1-\exp(-t^{1-(\beta+\gamma)^{-1}}(1-\delta_1)\int_\delta^1 g(0,kt^{-(\beta+\gamma)^{-1}};u)\,du)).$$

Since $\forall\, \varepsilon > 0$

$$\liminf_{t\to\infty} (1-\exp(-t^{1-(\beta+\gamma)^{-1}}(1-\delta_1)\int_\delta^1 g(0,kt^{-(\beta+\gamma)^{-1}};u)\,du))$$

$$\geqq 1-\varepsilon, \qquad k \in [c_3 t^{(\beta+\gamma)^{-1}}, \qquad c_4 t^{(\beta+\gamma)^{-1}}]$$

we have $\liminf_{t\to\infty} ES(t)t^{-(\beta+\gamma)^{-1}} \geqq (c_4-c_3)(1-\varepsilon)$ where c_4 is an arbitrarily large positive constant and c_3 is an arbitrarily small positive constant. □

PROOF OF THEOREM 2.4 (Equation 10). Let

$$E_L S(t) = \sum_{k=0}^{\infty} (1-\exp(-t\int_{L(t)}^1 P_{0,k}(tu)\,du))$$

where $L(t) \to 0$, $t\to\infty$ and is slowly varying.

For our purposes we will choose $L(t) = (\frac{1}{2}\varepsilon \log^{-1} t)^{24}$. Introduce $E\hat{S}(t) = \sum_{k=0}^{\infty}(1-\exp(-tp_k))$ where $p_k = \int_0^1 g(0,k;u)\,du$ and write

$$\frac{ES(t)}{(2t\log t)^{\frac{1}{2}}} = \frac{ES(t)}{E_L S(t)} \cdot \frac{E_L S(t)}{t^{\frac{1}{2}} E\hat{S}(t^{\frac{1}{2}})} \cdot \frac{E\hat{S}(t^{\frac{1}{2}})}{(2\log t)^{\frac{1}{2}}}.$$

To verify that each of these ratios converges to 1 as $t\to\infty$ we break the remainder of the proof into three principle steps.

(i) In accordance with the discussion in Section 2, $E\hat{S}(t)$ is the expected number of occupied cells at time t in an urn scheme where one ball is thrown at each event time of a Poisson process at an infinite array of cells with probability

$$p_k = \int_0^1 g(0,k;u)\,du, \qquad k = 0,1,2,\cdots$$

of hitting the kth cell. By Theorem 1 in Karlin [5], asymptotic behavior of $E\hat{S}(t)$ is determined by asymptotic behavior of $\alpha(x) = [\alpha^*(x)] = \max\{k : p_k \geqq x^{-1}\}$ where $\alpha^*(x) = p^{-1}(x^{-1})$ and $p(y) = \int_0^1 g(0,y;u)\,du$. From the proof of Lemma A.3 in the appendix, $\alpha(x) \sim 2(\log x)^{\frac{1}{2}}$, $x\to\infty$. Hence, $E\hat{S}(t) \sim 2(\log t)^{\frac{1}{2}}$, $t\to\infty$ or $E\hat{S}(t^{\frac{1}{2}})/(2\log t)^{\frac{1}{2}} \to 1$, $t\to\infty$.

(ii)

$$E_L S(t) = \sum_{k<[\varepsilon(t/\log t)^{1/2}]} + \sum_{k=[\varepsilon(t/\log t)^{1/2}]}^{[ct^{1/2+\alpha}]} + \sum_{k>[ct^{1/2+\alpha}]} (1-\exp(-t\int_{L(t)}^1 P_{0,k}(tu)\,du))$$

$$= I_1(t)+I_2(t)+I_3(t), \qquad 0<\alpha<\tfrac{1}{8},$$

$$I_1(t) \leqq \varepsilon(t/\log t)^{\frac{1}{2}}$$

since each term is bounded above by 1.

Using the integral form of Theorem 2.6 we have

$$I_3(t) \leqq t\int_{L(t)}^1 \Pr\{X(tu) > ct^{\frac{1}{2}+\alpha} \mid X(0)=0\}\,du$$

$$\leqq Kt\int_{L(t)}^1 du \int_{ct^{1/2+\alpha}}^{\infty} g(0,w;u)\,dw \tag{52}$$

$$\leqq K^* t^{3/2-\alpha} \exp(-\tfrac{1}{4}c^2 t^{2\alpha}).$$

Thus

$$I_1(t)/(t\log t)^{\frac{1}{2}} \to 0, \qquad t \to \infty$$

$$I_3(t)/(t\log t)^{\frac{1}{2}} \to 0, \qquad t \to \infty$$

and the asymptotic behavior of $E_L S(t)$ is incorporated in $I_2(t)$.

Now

$$I_2(t) = \sum_{k=[\varepsilon(t/\log t)^{1/2}]}^{[ct^{1/2+\alpha}]}(1-\exp(-t^{\frac{1}{2}}\int_{L(t)}^{1} g(0,t^{-\frac{1}{2}}k;u)\,du))+e_1(t)$$

where

(53) $$e_1(t) = \sum_{k=[\varepsilon(t/\log t)^{1/2}]}^{[c_1 t^{1/2+\alpha}]} \exp(-t^{\frac{1}{2}}\int_{L(t)}^{1} g(0,t^{-\frac{1}{2}}k;u)\,du) \times [1-\exp(-t^{\frac{1}{2}}\int_{L(t)}^{1}(t^{\frac{1}{2}}P_{0,k}(tu)-g(0,t^{-\frac{1}{2}}k;u))\,du)].$$

Routine estimates allow us to rewrite the first term in $I_2(t)$ as

$$\int_{\varepsilon(t/\log t)^{1/2}}^{ct^{1/2+\alpha}}(1-\exp(-t^{\frac{1}{2}}\int_{L(t)}^{1} g(0,t^{-\frac{1}{2}}x;u)\,du))\,dx+O(1).$$

Introducing the change of variable $x = wt^{\frac{1}{2}}$ and approximating the integral by a sum we have

(54) $$I_2(t) = t^{\frac{1}{2}}E\hat{S}(t^{\frac{1}{2}})+O(t^{\frac{1}{2}})+e_1(t).$$

By Lemma A.2 in the appendix $|e_1(t)| \leqq K_1 t^{4\alpha}$, $0<\alpha<\frac{1}{8}$.
Hence $I_2(t)/(t^{\frac{1}{2}}E\hat{S}(t^{\frac{1}{2}})) \to 1$, $t\to\infty$.

(iii) Finally

$$ES(t)/(E_L S(t)) = 1+[ES(t)-E_L S(t)]/E_L S(t)$$

and $0 \leqq [ES(t)-E_L S(t)]/(E_L S(t)) \leqq [ES(t)-E_L S(t)]/(t/\log t)^{\frac{1}{2}} \to 0$, $t\to\infty$, by Lemma A.1. Thus $ES(t) \sim (2t\log t)^{\frac{1}{2}}$ and the proof of (10) is complete. ▯

PROOF OF THEOREM 2.4 (Equation 11). Let

$$\hat{V}(t) = \int_0^\infty (\exp(-t\int_0^1 g(0,y;u)\,du)-\exp(-2t\int_0^1 g(0,y;u)\,du))\,dy$$

and

$$V_L(t) = \sum_{k=0}^{\infty}(\exp(-t\int_{L(t)}^1 P_{0,k}(tu)\,du)-\exp(-2t\int_{L(t)}^1 P_{0,k}(tu)\,du)).$$

Now decompose Var $S(t)$ into three terms as in the proof of Formula (10), denoting them respectively by $V_1(t)$, $V_2(t)$, and $V_3(t)$. Thus we have

$$\frac{\operatorname{Var} S(t)}{t^{\frac{1}{2}}f(t)} = \frac{V_1(t)}{t^{\frac{1}{2}}f(t)}+\frac{V_2(t)}{V_L(t)}\cdot\frac{V_L(t)}{t^{\frac{1}{2}}\hat{V}(t^{\frac{1}{2}})}\cdot\frac{\hat{V}(t^{\frac{1}{2}})}{f(t)}+\frac{V_3(t)}{f(t)t^{\frac{1}{2}}}$$

where

$$f(t) = c_3(2/\log t)^{\frac{1}{2}}\log 2, \qquad c_3 = \int_0^\infty e^{-y^{-1}}(2y^{-2}-y^{-3})\,dy.$$

The first term is less than ε since each component is bounded above by 1. The last term vanishes as $t\to\infty$ using the estimates $0 \leqq I_3(t) \leqq K^* t^{3/2-\alpha}\exp(-\frac{1}{4}c^2t^{2\alpha})$ and $0 \leqq V_3(t) \leqq \sum_{k>Ct^{1/2+\alpha}}(1-\exp(-2t\int_0^1 P_{0,k}(tu)\,du))$.

Finally $\hat{V}(t^{\frac{1}{2}})/f(t) \to 1$, $t \to \infty$, by Lemma A.3. $V_L(t)/[t^{\frac{1}{2}}\hat{V}(t^{\frac{1}{2}})] \to 1, t \to \infty$, by rewriting $V_L(t)$ as

$$V_L(t) = \sum_{k=0}^{\infty}(1-\exp(-2t\int_{L(t)}^{1} P_{0,k}(tu)\,du) - \sum_{k=0}^{\infty}(1-\exp(-t\int_{L(t)}^{1} P_{0,k}(tu)\,du))$$

and applying the estimates of the previous proof to write $V_L(t) = t^{\frac{1}{2}}\,\hat{V}(t^{\frac{1}{2}}) + O(e_1(t)) + o(1)$, where $e_1(t)$ is given by (53).

Since $\operatorname{Var} S(t)/V_L(t) \to 1$, $t \to \infty$, by Lemma A.1, the proof of Theorem 2.4 is complete.

NOTE. The speed of convergence and large deviation parts of Theorem 2.6 are essential for the estimate on $e_1(t)$ given by Lemma A.2. A large deviation theorem is also used in Lemma A.1, and a generalization of this argument to general Birth and Death processes satisfying (I) with $\beta+\gamma > 1$ is required for formulas like (8) and (9).

4. A central limit theorem. If we consider "occupied states" processes generated by arbitrary continuous time Markov Chains with stationary transition probabilities and a Poisson input process, recall that

$$\{N_k(t)\} = \{\text{number of chains in state } k \text{ at time } t\}_{k=0}^{\infty}$$

are independent Poisson random variables with parameters

$$\int_0^t P_{0,k}(\tau)\,d\tau, \qquad k = 0,1,2,\cdots,$$

respectively.
Thus

$$X_k(t) = 1 \quad \text{if} \quad N_k(t) > 0,$$
$$= 0 \quad \text{if} \quad N_k(t) = 0, \qquad k = 0,1,2,\cdots,$$

are independent uniformly bounded random variables and

$$S(t) = \sum_{k=0}^{\infty} X_k(t) < \infty \quad \text{with probability} \quad 1 \tag{55}$$

for all finite t. The family $\{X_k(t)\}$, $t > 0$, $k = 0,1,2,\cdots$ forms an infinite array of independent random variables satisfying (55) and an imitation of the proof of the Central Limit Theorem for finite arrays of independent random variables using the Lindeberg conditions yields

$$\Pr\{[S(t)-ES(t)]/(\operatorname{Var} S(t))^{\frac{1}{2}} \leqq x\} \to \Phi(x) \quad \text{as} \quad t \to \infty \tag{56}$$

provided $\operatorname{Var} S(t) \to \infty$ as $t \to \infty$.

The last condition; namely, $\operatorname{Var} S(t) \to \infty$, $t \to \infty$ is satisfied for all processes considered in this work. Thus we have in particular,

THEOREM 4.1. *If $\{S(t), t > 0\}$ is an "occupied states" process generated by Birth and Death processes satisfying* (I) *and a Poisson input process, then*

$$\Pr\{[S(t)-ES(t)]/(\operatorname{Var} S(t))^{\frac{1}{2}} \leqq x\} \to \Phi(x) \quad \text{as} \quad t \to \infty.$$

For central limit theorems involving other input and growing processes we refer the reader to [6]. In general $S(t)$ is a sum of dependent random variables which, nevertheless, is asymptotically normal for wide classes of input and growing processes. For examples where a central limit theorem is not valid in the context of positive recurrent Markov Chains, consider the infinite urn scheme described previously with $\{p_k\}_{k=0}^{\infty}$ such that $\limsup_{k\to\infty} p_{k+1}/p_k < 1$. Then $\operatorname{Var}\hat{S}(t)$ is bounded, and there is no convergence of the appropriately normalized occupied cells process to a normal distribution. For an extensive discussion of this question in the context of the infinite urn scheme, see [5].

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APPENDIX

Throughout this section we refer to the absolute value process of Theorem 2.4. Lemmas A.1, A.2, and A.3, however, indicate the kind of arguments which are essential in general for exact asymptotic formulas when $\beta+\gamma > 1$.

LEMMA A.1.

$$[ES(t)-E_L S(t)]/(t/\log t)^{\frac{1}{2}} \to 0, \qquad t\to\infty$$

where

$$E_L S(t) = \textstyle\sum_{k=0}^{\infty}(1-\exp(-t\int_{L(t)}^{1} P_{0,k}(tu)\,du))$$

$$L(t) = (\tfrac{1}{2}\varepsilon \log^{-1} t)^{24}.$$

PROOF. Let $L_4(t) = (\log t)^{-\frac{1}{2}}$ and $L_3(t) = (\log t)^{-2}$.

Then

$$0 \leqq ES(t)-E_L S(t)$$

$$\leqq \varepsilon t^{\frac{1}{2}}L_4(t)+\textstyle\sum_{k>\varepsilon t^{1/2}L_4(t)}(1-\exp(-t\int_0^{L(t)}P_{0,k}(tu)\,du)) \tag{57}$$

$$\leqq \varepsilon t^{\frac{1}{2}}L_4(t)+t\textstyle\int_0^{L(t)}\Pr\{X(tu) > \varepsilon t^{\frac{1}{2}}L_4(t) \mid X(0)=0\}\,du.$$

Recall that $X(tu) = |Y(2(tu))|$ where, for $k = 0, 1, 2, \cdots$

$$Y(t) = \textstyle\sum_{k=0}^{m(t)} Z_k;$$

$$Z_k = +1 \quad \text{with probability} \quad \tfrac{1}{2},$$

$$= -1 \quad \text{with probability} \quad \tfrac{1}{2},$$

are independent of each other and of the Poisson process $m(t)$.

Now decompose the above integral into two terms as

$$\textstyle\int_0^{L(t)} = \int_0^{L_3(t)t^{-1/4}} + \int_{L_3(t)t^{-1/4}}^{L(t)}.$$

For the first integral we apply Chebyshev's inequality to the integrand to obtain

$$(58)\qquad \int_0^{L_3(t)t^{-1/4}} \Pr\{|Y(2tu)| > \varepsilon t^{\frac{1}{2}} L_4(t)\}\, du \leqq L_3{}^2(t)/[\varepsilon^2 t^{\frac{1}{2}} L_4{}^2(t)] \to 0, \qquad t \to \infty.$$

For $L_3(t)t^{-\frac{1}{4}} \leqq u \leqq L(t)$ we have

$$\Pr\{|Y(2tu)| > \varepsilon t^{\frac{1}{2}} L_4(t)\} \leqq \Pr\{Y(2tu)/(2tu)^{\frac{1}{2}} > \varepsilon L_4(t)/(2^{\frac{1}{2}} u^{\frac{1}{2}-\lambda})\}$$

where λ is chosen sufficiently close to $\frac{1}{2}$ that we can apply the asymptotic formula for large deviations $\Pr\{Y(n)\cdot n^{-\frac{1}{2}} > x\}/(1-\Phi(x)) \to 1$, $x \to \infty$, where $xn^{-1/6} \to 0$, $n \to \infty$, and

$$1-\Phi(x) \sim (2\pi)^{-\frac{1}{2}} x^{-1} \exp(-x^2/2), \qquad x \to \infty.$$

With $\lambda = 11/24$ we have

$$\Pr\{Y(2tu)/(2tu)^{\frac{1}{2}} > \varepsilon L_4(t)/(2^{\frac{1}{2}} u^{1/24})\}$$
$$\leqq C^*((\pi)^{\frac{1}{2}} \varepsilon L_4(t))^{-1} u^{1/24} \exp(-\varepsilon^2 L_4{}^2(t)/4u^{1/12})$$

and

$$(59)\qquad t\int_{L_3(t)t^{-1/4}}^{L(t)} \Pr\{Y(2tu) > \varepsilon t^{\frac{1}{2}} L_4(t)\}\, du$$
$$\leqq C^*(L(t))^{1/24}(\varepsilon(\pi)^{\frac{1}{2}} L_4(t))^{-1} \to 0, \qquad t \to \infty.$$

Substituting (58) and (59) in (57) we have

$$ES(t) - E_L S(t) \leqq \varepsilon t^{\frac{1}{2}} L_4(t) + t^{\frac{1}{2}} L_3{}^2(t)(\varepsilon^2 L_4{}^2(t))^{-1} + 2\hat{C}(L(t))^{1/24}(L_4(t))^{-1},$$

and the proof is complete. □

LEMMA A.2.

$$|e_1(t)| \leqq K_1 t^{4\alpha}, \qquad 0 < \alpha < \tfrac{1}{8}$$

where

$$e_1(t) = \sum_{k=[\varepsilon(t/\log t)^{1/2}]}^{[ct^{1/2+\alpha}]} \exp(-t^{\frac{1}{2}} \textstyle\int_{L(t)}^1 g(0, t^{-\frac{1}{2}}k; u)\, du)$$
$$\times [(1 - \exp(-t^{\frac{1}{2}} \textstyle\int_{L(t)}^1 (t^{\frac{1}{2}} P_{0,k}(tu) - g(0, t^{-\frac{1}{2}}k; u))\, du].$$

PROOF. Using the Local Limit Theorem 2.6 we have

$$e_1(t) \leqq \sum (1 - \exp(-Mt^{-(\frac{1}{2}-4\alpha)} \textstyle\int_{L(t)}^1 g(0, t^{-\frac{1}{2}}k; u)\, du))$$
$$(60)\qquad \leqq Mt^{-(\frac{1}{2}-4\alpha)} \int_{L(t)}^1 \sum g(0, t^{-\frac{1}{2}}k; u)\, du$$
$$\leqq M^* t^{-(\frac{1}{2}-4\alpha)} \int_{L(t)}^1 du \left(\int_{\varepsilon(t/\log t)^{1/2}}^{ct^{1/2+\alpha}} g(0, t^{-\frac{1}{2}}x; u)\, dx + O(1)\right)$$

where both of the above sums are over integers in the range $[\varepsilon(t/\log t)^{\frac{1}{2}}] \leqq k \leqq [ct^{\frac{1}{2}+\alpha}]$. Let $x = wt^{\frac{1}{2}}$ in the integral in (60). Then

$$e_1(t) \leqq \hat{M} t^{-(\frac{1}{2}-4\alpha)} \int_0^1 du\, (t^{\frac{1}{2}} \textstyle\int_0^\infty g(0, w; u)\, dw + O(1)) \leqq K_1 t^{4\alpha}, \qquad 0 < \alpha < \tfrac{1}{8}.$$

Similar estimates yield a lower bound $e_1(t) \geqq -K_1 t^{4\alpha}$.

LEMMA A.3.

$$\hat{V} = \int_0^\infty (\exp(-tp(y)) - \exp(-2tp(y)))\,dy$$
$$\sim \log 2 \log^{-\frac{1}{2}} t \int_0^\infty e^{-1/y}(2y^{-2} - y^{-3})\,dy, \qquad t \to \infty$$

where $p(y) = \int_0^1 g(0, y; u)\,du$.

PROOF. Define

$$\alpha^*(x) = \sup\{y : p(y) \geqq x^{-1}\},$$
$$= 0 \quad \text{if} \quad \{y : p(y) \geqq x^{-1}\} = \phi,$$
$$= p^{-1}(x^{-1}).$$

Then we may rewrite $\hat{V}(t)$ as

$$\hat{V}(t) = \int_0^\infty (\exp(-t/x) - \exp(-2t/x))\,d\alpha^*(x);$$
$$= \int_0^\infty e^{-1/y}(2y^{-2} - y^{-3})((yt)^{-1}\int_0^{yt}(\alpha^*(2w) - \alpha^*(w))\,dw)\,dy,$$

after two integrations by parts.

To obtain an explicit formula for $\alpha^*(x)$ we integrate $p(y)$ by parts to obtain

$$(61) \qquad p(y) = 4y^{-2}\pi^{-\frac{1}{2}}\exp(-y^2/4)$$
$$\times (1 - 4y^{-2}\cdot\tfrac{3}{2} + 4y^{-2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\int_1^\infty z^{-\frac{7}{2}}\exp(-y^2(z-1)/4)\,dz),$$
$$= [c_3 y^{-2}\exp(-y^2/4)](1 + h(y)),$$

where $h(y) = O(y^{-2})$; $c_3 = 4\pi^{-\frac{1}{2}}$.

Setting $p(y) = x^{-1}$ in (61) we apply a standard asymptotic argument (see, e.g., de Bruijn [1]) to solve this equation for

$$y = \alpha^*(x) = 2(\log x)^{\frac{1}{2}} - \log^{-\frac{1}{2}} x \log\log c_3 x + \log^{-\frac{1}{2}} x \log c_4 + O(\log^{-\frac{3}{2}} x(\log\log x)^2)$$

where $c_4 = \pi^{-\frac{1}{2}}$. Then $\alpha^*(2x) - \alpha^*(x) = \log 2 \log^{-\frac{1}{2}} x + O(\log^{-1} x)$

and

$$(62) \qquad x^{-1}\int_0^x (\alpha^*(2w) - \alpha^*(w))\,dw \sim \log 2 \log^{-\frac{1}{2}} x, \qquad x \to \infty.$$

Substituting (62) in the above formula for $\hat{V}(t)$ and applying routine estimates, we obtain

$$\hat{V}(t) \sim \log 2 \log^{-\frac{1}{2}} t^{-\frac{1}{2}} \int_0^\infty e^{-1/y}(2y^{-2} - y^{-3})\,dy. \qquad \square$$

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