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SOME AUTOREGRESSIVE MOVING AVERAGE PROCESSES WITH GENERALIZED POISSON MARGINAL DISTRIBUTIONS

A. A. ALZAID AND M. A. AL-OSH

Department of Statistics & Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

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Abstract. Some simple models are introduced which may be used for modelling or generating sequences of dependent discrete random variables with generalized Poisson marginal distribution. Our approach for building these models is similar to that of the Poisson ARMA processes considered by Al-Osh and Alzaid (1987, *J. Time Ser. Anal.*, **8**, 261–275; 1988, *Statist. Hefte*, **29**, 281–300) and McKenzie (1988, *Adv. in Appl. Probab.*, **20**, 822–835). The models have the same autocorrelation structure as their counterparts of standard ARMA models. Various properties, such as joint distribution, time reversibility and regression behavior, for each model are investigated.

Key words and phrases: Generalized Poisson process, regression, time reversibility, quasi-binomial distribution, quasi-multinomial distribution, vector AR(1) process.

1. Introduction

In many problems which are Poissonian in nature the probability of occurrence of an event does not remain constant. Instead it is affected by the previous occurrences, thus resulting in unequal mean and variance in the data. For such type of data Consul and Jain (1973) introduced the generalized Poisson (GP) or Lagrangian distribution with probability function given by:

$$(1.1)\qquad P(X=x)=\begin{cases}\lambda(\lambda+\theta x)^{x-1}e^{-(\lambda+\theta x)}/x!, & x=0,1,2,\dots\\ 0 & \text{for}\quad x>m\quad\text{when}\quad\theta<0,\end{cases}$$

where $\lambda > 0$, $\max(-1, -\lambda/m) < \theta \le 1$, $m > 4$. The GP distribution was studied by many researchers. Janardan *et al.* (1979) have given many interesting biological applications of this model. The relevancy of the GP distribution in queueing theory is discussed by Consul and Shenton (1973) and Kumar (1981). The mean and the variance of the GP distribution are given by:

$$(1.2a)\qquad E(X)=\lambda/(1-\theta)$$

and

$$\text{(1.2b)} \qquad \mathrm{Var}(X) = \lambda/(1-\theta)^3.$$

From these expressions it is clear that unless $\theta = 0$, the variance can be greater or less than the mean according to whether θ is positive or negative. Consequently the GP distribution can be used as an alternative distribution for modelling counting processes whenever the index of dispersion (variance $\div$ mean) is not near one.

The purpose of this paper is to introduce some simple models for discrete time processes with GP marginal distribution. Our approach is parallel to that of Al-Osh and Alzaid (1987, 1988), Alzaid and Al-Osh (1988, 1990) and McKenzie (1985, 1988) which dealt with discrete time stationary processes with Poisson marginal distribution. In the development of the Poisson time series models, the binomial thinning is used instead of the scalar multiplication in the standard autoregressive moving average (ARMA) processes. Specifically, it has been assumed that an element of the Poisson process at time $t-1$, X_{t-1}, independently of the other elements and the time of the process has a constant probability, say α, of being retained to time t and probability, $1-\alpha = \bar{\alpha}$, of being deleted during the time interval $(t-1, t]$. This assumption implies that, given $X_{t-1} = x$, the number of retained elements to time t is a random variable having the binomial distribution with parameters (α, x). However, for many real data on counting processes, it seems logical to consider that the probability of retaining an element is not constant but might depend on the time or/and the number of elements already retained.

In our development of some ARMA models with GP marginal distribution, we will assume that the probability of retaining an element is not constant but is a linear function of the number of elements being retained. Specifically, given $X_{t-1} = n$ we will assume that the number of retained elements to time t has a quasi-binomial (QB) distribution with parameters (p, θ, n) i.e.

$$\text{(1.3)} \qquad P(Y=x) = pq\binom{n}{x}(p+x\theta)^{x-1}(q+(n-x)\theta)^{n-x-1}/(1+n\theta)^{n-1}, \qquad x = 0, 1, 2, \ldots, n$$

where $0 < q = 1-p < 1$ and θ is such that $n\theta < \min(p, q)$. Consul and Mittal (1975) introduced the QB distribution as an urn model. Shenton (1986) summarized some properties of the QB distribution and gave a list of references which dealt with this distribution. The mean and the variance of the QB distribution are given by:

$$\text{(1.4a)} \qquad E(Y) = np$$

and

$$\text{(1.4b)} \qquad \mathrm{Var}(Y) = pq\left[n^2 - \sum_{j=0}^{n-1}\frac{n!\theta^j}{(n-j-2)!(1+n\theta)^{j+1}}\right].$$

As in the case of the binomial distribution which tends to the Poisson distribution for large n such that $np \to \lambda$, the QB distribution tends to the GP distribution.

By Consul (1989), we have that the probability generating function (p.g.f.) of a random variable $X \sim \mathrm{GP}(\lambda, \theta)$ is $\phi_X(u) = e^{\lambda(t-1)}$ where $t = ue^{\theta(t-1)}$. Observe that for fixed $|\theta| < 1$ the function $te^{-\theta(t-1)}$ is increasing in t for $|t| \leq |\theta^{-1}|$. Therefore, it has an inverse $A_\theta(t)$ (say), i.e. $A_\theta(te^{-\theta(t-1)}) = t$. Therefore the p.g.f. $\phi_X(u)$ can be rewritten as

$$\phi_X(u) = e^{-\lambda(A_\theta(u)-1)}, \qquad |u| \leq 1. \tag{1.5}$$

An important property of the GP distribution which will play a major role in our development of the GP processes is summarized in the following lemma given in Consul (1975).

LEMMA 1.1. *Let the random variable Y be the sum of two random variables Y_1 and Y_2. Then Y has a $\mathrm{GP}(\lambda, \theta)$ distribution, and the conditional distribution of $Y_1 \mid Y = n$ is $\mathrm{QB}(p, \theta/\lambda, n)$ if and only if Y_1 and Y_2 are independent such that $Y_1 \sim \mathrm{GP}(p\lambda, \theta)$ and $Y_2 \sim \mathrm{GP}(q\lambda, \theta)$.*

Let $\{S(n) : n = 0, 1, \ldots\}$ be a sequence of random variables such that $S(n) \sim \mathrm{QB}(p, \theta/\lambda, n)$ for some p and θ. Such random variables will be called throughout the paper quasi-binomial operators. This is analogous to the terminology of the binomial thinning mentioned above. Using Lemma 1.1, we have the following corollary.

COROLLARY 1.1. *If $X \sim \mathrm{GP}(\lambda, \theta)$ and is independent of $\{S(n)\}$ then $S(X) \sim \mathrm{GP}(p\lambda, \theta)$.*

The present paper contains four additional sections. The following section presents a first order autoregressive process with GP marginal distribution. Section 3 discusses the corresponding first order moving average process. Extension of this process to a higher order moving average process is given in Section 4. The last section presents a GP process with ARMA correlation structure. Properties of each of these models, such as autocorrelation behaviour, joint distribution and regression, are discussed and compared with those of the corresponding Gaussian processes.

2. The first-order autoregressive process

The first-order generalized Poisson (GPAR(1)) process is built using a sequence $\{\epsilon_t\}$ of independent identically distributed (i.i.d.) $\mathrm{GP}(q\lambda, \theta)$ random variables. The GPAR(1) process is defined by:

$$X_t = S_t(X_{t-1}) + \epsilon_t \qquad t = 1, 2, \ldots \tag{2.1}$$

where $\{S_t(\cdot) : t = 1, 2, \ldots\}$ is a sequence of independent and identically quasi-binomial operators $(p, \theta/\lambda, \cdot)$ independent of $\{\epsilon_t\}$. The process $\{X_t\}$, defined by (2.1), is a Markov chain with transition matrix given by:

$$\begin{aligned} p_{ij} =& P(X_{t+1} = j \mid X_t = i) \\ =& \sum_{k=0}^{\min\{i,j\}} \binom{i}{k} \frac{pq\lambda}{\lambda + i\theta} \left[\frac{p\lambda + k\theta}{\lambda + i\theta}\right]^{k-1} \left[\frac{q\lambda + (i-k)\theta}{\lambda + i\theta}\right]^{i-k-1} \\ & \cdot \lambda q[\lambda q + \theta(j-k)]^{j-k-1} e^{-\lambda q - \lambda\theta(j-k)}/(j-k)!. \end{aligned} \tag{2.2}$$

This Markov chain is ergodic and hence there is a unique stationary distribution. Therefore, the $GP(\lambda, \theta)$ distribution is the unique stationary distribution. Consequently if $X_0 \sim GP(\lambda, \theta)$ we get a strictly stationary GPAR(1) process with distribution given by (1.1).

The mean and the variance of GPAR(1) process are given by (1.2). The autocorrelation function $\rho_X(k)$ may be derived easily using (2.1) and it is of the form

$$(2.3) \qquad \rho_X(k) = \operatorname{corr}(X_t, X_{t-k}) = p^{|k|}, \qquad k = 0, \pm 1, \pm 2, \dots.$$

Thus the behaviour of the autocorrelation function of the GPAR(1) process is similar to that of the AR(1) process except that it is always nonnegative due to the fact that p is a nonnegative number.

From (2.2) and (2.3), it follows that the parameters λ and θ describe the marginal distribution of the process while p independently describes the correlation structure of the process.

As an illustration of the considerable differences between the GPAR(1) process and the corresponding PAR(1) process, Fig. 1 displays one realization of length 100 for each process. In both processes the mean and the autocorrelation are kept the same as 2 and .5 respectively. Figure 1(a) displays the realization of the PAR(1) ($\theta = 0$) process whereas Fig. 1(b) displays the GPAR(1) ($\theta = 1/2$) process. From the figures it is clear that the GPAR(1) process has more variability than the PAR(1) process. This is expected since the variance of the GPAR(1) process is twice that of the PAR(1) process.

The joint distribution of X_{t+1} and X_t for the stationary GPAR(1) process can be obtained by utilizing (2.2). Instead we will compute the joint p.g.f. of X_{t+1} and X_t which is:

$$\begin{aligned}\phi_{X_{t+1},X_t}(u,v) &= E(u^{X_{t+1}} v^{X_t}) \\ &= E(u^{S(X_t)+\epsilon_{t+1}} v^{X_t}) \\ &= E(u^{\epsilon_{t+1}}) \cdot E(u^{S(X_t)} v^{S(X_t)+X_t-S(X_t)}) \\ &= E(u^{\epsilon_{t+1}}) \cdot E((uv)^{S(X_t)}) \cdot E(v^{X_t-S(X_t)})\end{aligned}$$

where in the last step we have used Lemma 1.1. Observe that Lemma 1.1 implies $\epsilon_{t+1} \overset{d}{=} X_t - S(X_t)$ where $\overset{d}{=}$ stands for equality in distribution. Therefore

$$(2.4) \qquad \phi_{X_{t+1},X_t}(u,v) = \exp[\lambda q(A_\theta(u) + A_\theta(v) - 2) + \lambda p(A_\theta(uv) - 1)].$$

The above p.g.f. is symmetric in u and v and hence the joint distribution of X_{t+1} and X_t is symmetric. Also, since the joint distribution of any finite set of X_t's can be obtained from (2.4) and the marginal GP distribution, the GPAR(1) process is time reversible. As an application of this observation, it follows from (2.1), (1.2a) and (1.4a) that:

$$(2.5) \qquad \begin{aligned} E(X_{n+1} \mid X_n = x) &= E(X_n \mid X_{n+1} = x) \\ &= px + q\lambda/(1-\theta), \qquad x = 0, 1, \dots. \end{aligned}$$

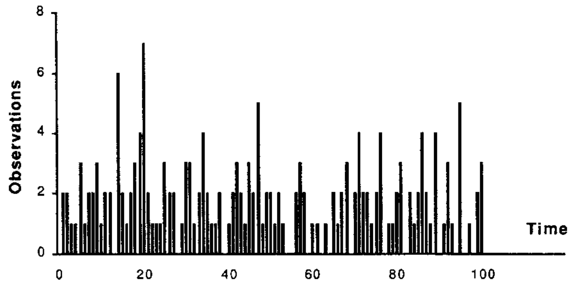

Fig. 1(a). Poisson AR(1) process ($\lambda = 2, \theta = 0$).

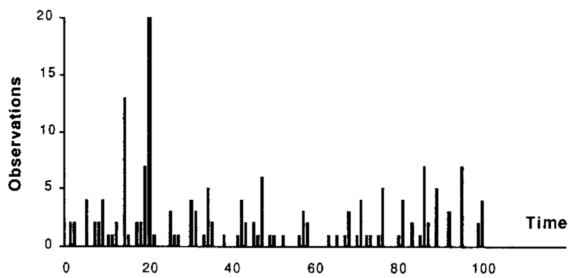

Fig. 1(b). Generalized Poisson AR(1) process ($\lambda = 2, \theta = 0.5$).

This shows that the regression behaviour in the GPAR(1) process is similar to that of Gaussian AR(1) process. However as in the Poisson AR(1) process the similarity does not extend to higher order moments.

3. The first-order moving average process

The generalized Poisson first order moving average (GPMA(1)) process is constructed in essentially the same way as the GPAR(1) process discussed in the previous section. Let $\{\epsilon_t\}$ be a sequence of i.i.d. random variables such that $\epsilon_t \sim \mathrm{GP}(\lambda^*, \theta)$ and let $\{S_t(\cdot)\}$ be a sequence of i.i.d. random operators independent of $\{\epsilon_t\}$ such that $S_t(n) \sim \mathrm{QB}(p^*, \theta/\lambda^*, n)$. We define the GPMA(1) process $\{X_t\}$ by

$$X_{t+1} = S_{t+1}(\epsilon_t) + \epsilon_{t+1}, \qquad t = 0, 1, \dots. \tag{3.1}$$

It is obvious from (3.1) that $\{X_t\}$ is a stationary process and X_{t+r} and X_t are independent if $|r| > 1$. Also $S_{t+1}(\epsilon_t)$ and ϵ_{t+1} are independent by assumption. This implies by consecutive applications of Lemma 1.1 that the marginal distribution of X_t is $\mathrm{GP}(\lambda, \theta)$ where $\lambda = (1 + p^*)\lambda^*$.

The autocorrelation function for the GPMA(1) process may be determined directly and it is of the form

$$\rho_X(k) = \begin{cases} p, & |k| = 1 \\ 0, & |k| > 1. \end{cases} \tag{3.2}$$

where $p = p^*(1+p^*)^{-1}$.

The fact that $p^* \geq 0$ implies $0 \leq \rho_X(1) \leq .5$ which is a full possible range of positive correlation. This is because for the ordinary first order moving average $|\rho_X(1)| \leq 0.5$.

Figure 2 displays realizations of the GPMA(1) and the PMA(1) processes. Similar remarks as those of AR(1) processes hold.

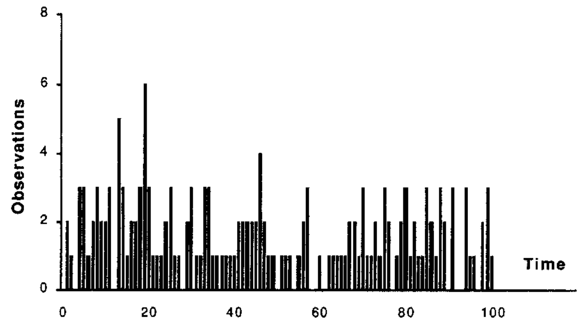

Fig. 2(a). Poisson MA(1) process ($\lambda = 2, \theta = 0$).

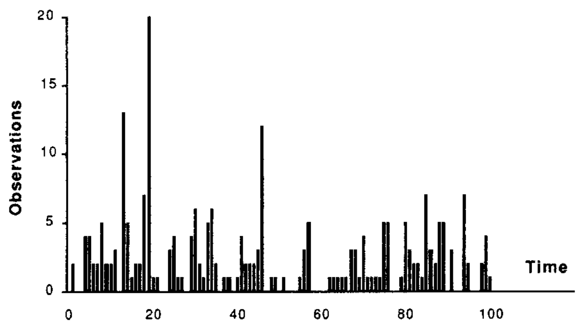

Fig. 2(b). Generalized Poisson MA(1) process ($\lambda = 2, \theta = 0.5$).

Unlike the GPAR(1) process, the GPMA(1) process is not Markovian. However, the joint p.g.f. of the GPMA(1) process can be computed easily using (3.1). The bivariate p.g.f. of X_{t+1} and X_t is given by

$$(3.3)\quad G_{X_{t+1},X_t}(u_1,u_2) = \exp[\lambda^*(A_\theta(u_1)+A_\theta(u_2)-2)+\lambda^*p^*(A_\theta(u_1u_2)-1)]$$
$$= \exp[\lambda q(A_\theta(u_1)+A_\theta(u_2)-2)+\lambda p(A_\theta(u_1u_2)-1)].$$

Comparison of (3.3) and (2.4) shows that the bivariate distribution in the GPMA(1) process is similar to that of the GPAR(1) process. An immediate consequence of this observation leads, in view of (2.5), to

$$E(X_{t+1} \mid X_t = x) = E(X_t \mid X_{t+1} = x) = px + \frac{q\lambda}{1-\theta}.$$

The joint p.g.f. of any finite set of consecutive observations can be obtained by the procedure that yielded (3.3). Thus the joint p.g.f. of $(X_t, X_{t-1}, \dots, X_{t-r+1})$ is given by

$$\phi(u_1, \dots, u_r) = \exp\left[\lambda q \sum_{i=1}^{r} (A_\theta(u_i) - 1) + \lambda p \sum_{i=1}^{r} (A_\theta(u_i u_{i+1}) - 1)\right]. \tag{3.4}$$

It is obvious from (3.4) that the GPMA(1) process is time-reversible. Also from (3.4) we can compute the distribution of the total counts occurring during time $(t - r + 1, t]$ i.e. $T_r = \sum_{i=1}^{r} X_{t-r+i}$. The p.g.f. of T_r can be obtained by setting $u_i = u$, $i = 1, \dots, r$ in (3.4). This gives

$$E(u^{T_r}) = \exp[\lambda q r (A_\theta(u) - 1) + \lambda p (r - 1)(A_\theta(u^2) - 1)].$$

This implies that T_r is distributed like the sum of two independent random variables say $Z + 2Y$ where $Z \sim \mathrm{GP}(\lambda r, \theta)$ and $Y \sim \mathrm{GP}((r - 1)\lambda p, \theta)$. Consequently the behaviour of the Poisson MA(1) process extends to the GPMA(1) process which is in contrast to the standard Gaussian MA(1) process in which $\sum X_i$ still has Gaussian distribution.

4. Higher order moving average processes

A generalized Poisson moving average process of higher order can be constructed by extending the GPMA(1) process in an obvious way. Thus, the GPMA(q) process is defined by

$$X_t = \epsilon_t + \sum_{i=1}^{q} S_{t,i}(\epsilon_{t-1}),$$

where $\{\epsilon_t\}$ is a sequence of i.i.d $\mathrm{GP}(\lambda, \theta)$ random variables and $\{\boldsymbol{S}_t = (S_{t,1}, \dots, S_{t,q})'\}$ is an independent sequence of i.i.d. random vector operators with the $S_{t,i}(m) \sim \mathrm{QB}(p_i, m, \theta)$. By Lemma 1.1, the above conditions on $\{\epsilon_t\}$ and $\{\boldsymbol{S}_t\}$ are sufficient to determine the marginal distribution of X_t as $\mathrm{GP}(\mu, \theta)$ where $\mu = \lambda \left(\sum_{i=1}^{q} p_i + 1\right)$. However, the conditions on $\{\boldsymbol{S}_t\}$ are not sufficient to determine the joint distribution of the process unless the joint distribution of $\boldsymbol{S}_t$ is given. One approach is to assume the components of $\boldsymbol{S}_t$ being independent. Such an assumption has been made by McKenzie (1988) to define a MA(q) process with Poisson marginal distribution. Alternatively one may assume a certain kind of dependence among the components of $\boldsymbol{S}_t$ as suggested by Al-Osh and Alzaid (1988) in their definition of Poisson MA(q) process. The latter approach results in some complication in calculating the joint distribution and hence in some structural properties of the process. Therefore we assume, here, that the components of $\boldsymbol{S}_t$ are independent. Under this assumption we get the autocorrelation function as

$$\rho_X(k) = \sum_{i=0}^{q-k} p_i p_{i+k} \Big/ \sum_{i=0}^{q} p_i$$

where $p_0 = 1$. This is a close analogue of the usual autocorrelation function of the MA(q) process and it is of the same form as that of the Poisson MA(q) process introduced by McKenzie (1988). Therefore the autocorrelation function does not play any role in the determination of the marginal distribution of the process. On the other hand, the mean and the variance give indication whether the process is more likely to be Poisson ($E(X) = \text{Var}(X)$) or generalized Poisson ($E(X) \neq \text{Var}(X)$).

5. A $(p, p-1)$ autoregressive moving average process

Realization of some processes depends not only on their immediate past, as in GPAR(1) case, but also on previous realization. In this section we shall introduce a generalized Poisson process with ARMA($p, p-1$) correlation structure.

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_p)'$ be a vector of nonnegative components such that $\sum_{i=1}^{p} \alpha_i \leq 1$ and let θ be a nonnegative number. Now for every integer $n \geq 0$, define (in distribution) the vector random operator $\boldsymbol{S}(n)$ on the set of nonnegative integers by

$$(5.1) \qquad P(\boldsymbol{S}(n) = \boldsymbol{x}) = \binom{n}{\boldsymbol{x}} \frac{\prod_{i=0}^{p} \alpha_i (\alpha_i + x_i\theta/\lambda)^{x_i - 1}}{(1 + n\theta/\lambda)^{n-1}},$$

where $\boldsymbol{x} \equiv (x_1, \ldots, x_p)$, $\binom{n}{\boldsymbol{x}} = n!/\prod_{i=0}^{p} x_i!$, $\alpha_0 = 1 - \sum_{i=1}^{n} \alpha_i$ and $x_0 = n - \sum_{i=1}^{p} x_i$. Since the univariate marginal distribution of $\boldsymbol{S}(n)$ is quasi-binomial, we will call a distribution of the form (5.1) quasi-multinomial distribution with vector parameter $(\theta, \boldsymbol{\alpha}, n)$ (denoted by QMB$(\theta, \boldsymbol{\alpha}, n)$) (see Consul and Mittal (1977) for more details about this distribution).

Now let X be GP(λ, θ). Then $\boldsymbol{S}(X)$ has a multiple generalized Poisson distribution, that is

$$(5.2) \qquad P(\boldsymbol{S}(X) = x) = \prod_{i=1}^{p} \frac{(\alpha_i\lambda + \theta x_i)^{x_i - 1}}{x_i!} \exp\left[-\sum_{i=1}^{p} (\alpha_i\lambda + \theta x_i)\right], \qquad x_i = 0, 1, 2, \ldots, \quad i = 1, \ldots, p.$$

Let $\{\boldsymbol{S}_t(n) \equiv (S_{1,t}(n), \ldots, S_{p,t}(n)) : n \geq 0\}_{t=0}$ be a sequence of independent classes of independent random vectors such that for every m and $n \geq 0$, $\boldsymbol{S}_t$ has QMB(θ, α, n) distribution. Furthermore, let $\{\epsilon_t\}$ be a sequence of iid random variables independent of $\{\boldsymbol{S}_t(n)\}$ such that ϵ_t has GP$(\alpha_0\lambda, \theta)$ distribution. Assuming that the joint distribution of $(X_0, X_1, \ldots, X_{p-1})$ is given and that it is independent of $\{\boldsymbol{S}_t(n), \epsilon_t\}$, then we define the process $\{X_t\}$ by

$$(5.3) \qquad X_t = \sum_{i=1}^{p} S_{i,t-i}(X_{t-i}) + \epsilon_t, \qquad t = p, p+1, \ldots.$$

The process $\{X_t\}$, as defined by (5.3), is a generalization of the Poisson ARMA(p, $p-1$) of Alzaid and Al-Osh (1990) where the thinning operations $\alpha_i o$ are replaced with the random operations $S_i(\cdot)$. In fact the Poisson ARMA$(p, p-1)$ corresponds to the case $\theta = 0$. The autoregressive structure of the process (5.2) is due to the dependence of X_t on X_{t-i} for $i = 1, \dots, p$. The moving average dependence is a result of the dependence between the components of $\boldsymbol{S}_{t-i}(X_i)$ which appear in different lags of time. The moving average dependence implies that $\{X_t\}$ is not p-Markovian which is a typical property of the ordinary AR(p) process. This results in difficulty in direct calculations of the joint distribution of the process. This problem can be overcome by considering the vector process $\{\boldsymbol{Z}_m\}$ where

$$\boldsymbol{Z}_t = \left(X_t, \sum_{i=2}^{p} S_{i,t+1-i}(X_{t+1-i}), \sum_{i=3}^{p} S_{i,t+2-i}(X_{t+2-i}), \dots, S_{p,t-1}(X_{t-1}) \right)' .$$

Since the distribution of X_t is marginal distribution of that of $\boldsymbol{Z}$, it follows that the joint distribution of the process $\{X_t\}$ is a marginal distribution of the joint distribution of the process $\{\boldsymbol{Z}_t\}$. Following an argument similar to that of Alzaid and Al-Osh (1990), it can be shown that the process $\{\boldsymbol{Z}_m\}$ is Markovian with the multiple generalized Poisson distribution

$$P(\boldsymbol{Z} = \boldsymbol{z}) = \prod_{i=1}^{p} \frac{(w_{i-1}\lambda + \theta z_i)^{z_i - 1}}{z_i!} \exp\left[- \sum_{i=1}^{p} (w_{i-1}\lambda + \theta z_i) \right]$$

$$w_0 = 1, \quad w_i = \sum_{j=1}^{i} \alpha_j w_{i-j}, \quad z_i = 0, 1, \dots, \quad i = 1, \dots, p$$

as the limiting distribution of the process. This in turn implies that the process $\{X_n\}$ has GP(θ, λ) limiting distribution.

In fact if one assumes that the initial distribution and limiting distribution of $\{\boldsymbol{Z}\}$ are identical, a stationary process would be obtained. A typical situation in which this happens is when $X_0, X_1, \dots, X_{p-1}$ are iid with GP(θ, λ) distribution.

Finally the autocovariance function $\gamma(k)$ for the stationary process $\{X_m\}$ can also be computed following an argument similar to that of Alzaid and Al-Osh (1990). This results in

$$\gamma(k) = \sum_{i=1}^{p} \alpha_i \gamma(k - i) + \sum_{i=k+1}^{p} \gamma(S_i, k - i) + \delta_k(0) \frac{\alpha_0 \lambda}{(1 - \theta)^3} \tag{5.4}$$

where $\delta_k(0) = 1$ if $k = 0$ and 0 if $k \neq 0$ and $\gamma(S_i, k - i)$ is such that

$$\gamma(S_i, -l) = \sum_{j=1}^{l-1} \alpha_j \gamma(S_i, j - l) + \delta_{i-l}(0) \frac{\alpha_i \lambda}{(1 - \theta)^3}, \qquad l = 2, \dots, p$$

with

$$\gamma(S_i, -1) = \delta_{i-1}(0) \frac{\alpha_i \lambda}{(1 - \theta)^3}, \qquad i = 1, \dots, p.$$

It is clear from (5.4) that the behaviour of $\gamma(k)$ is the same as that of ARMA$(p, p - 1)$.

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