Teklehaimanot Retta

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SOME CARDINAL GENERALIZATIONS OF PSEUDOCOMPACTNESS

TEKLEHAIMANOT RETTA, Addis Ababa

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1. INTRODUCTION

All spaces considered in this paper are assumed to be Tychonoff. A space X is said to be ***initially m-compact*** if every open cover of X of cardinality $\leqslant m$ has a finite subcover. Equivalently, X is initially m-compact if every filterbase of cardinality $\leqslant m$ has a nonvoid adherence. X is said to be ***weakly initially m-compact*** if every open cover of cardinality $\leqslant m$ has a finite subset with a dense union. X is called ***m-pseudocompact*** if every continuous image of X in $\mathbf{R}^m$ is compact. X is said to be ***m-pseudocompact in the sense of complete accumulation points*** (***mpcap*** for short) if every family of $\leqslant m$ open sets in X has a complete accumulation point, i.e., a point each neighbourhood of which meets k members of the family where k is the cardinality of the family.

When m is countable each of the properties of being weakly initially m-compact, m-pseudocompact, and mpcap is equivalent to being pseudocompact. See [7], [6], [2], and [5] for a discussion of initially m-compact, weakly initially m-compact, and m-pseudocompact spaces.

It is shown that if $m \geqslant c$ then the product of any collection of initially m-compact spaces is m-pseudocompact; that a regular closed set in an m-pseudocompact space and a perfect irreducible preimage of an m-pseudocompact space may fail to be m-pseudocompact. These statements are false in case m is countable. We also show that a weakly initially m-compact space is m-pseudocompact but that in general the converse is false. Further we show that the properties m-pseudocompactness and m-pseudocompactness in the sense of complete accumulation points are in general incomparable.

All undefined notation and terminology is as in [3].

2. m-PSEUDOCOMPACT SPACES

A *G_m-set* in X is an intersection of $\leqslant m$ open sets. A subset A of X is to be *G_m-dense* in X if every nonvoid G_m-set in X meets A. It is clear that A is G_m-dense in X if and only if it meets every nonvoid intersection of $\leqslant m$ zero sets in X. A family of sets is said to have the *m-intersection property* (*m.i.p.* for short) if every subset of $\leqslant m$ members has a nonempty intersection.

In the following theorem we collect some conditions equivalent to m-pseudocompactness. The equivalence of the conditions (a) and (d) is noted in [5]. The proof is left to the reader.

Theorem 1. *The following conditions on a space X are equivalent:*
(a) *Every zero set filter in X has the m.i.p.;*
(b) *every cozero cover of X of cardinality $\leqslant m$ has a finite subcover;*
(c) *every continuous image of X in a space of weight $\leqslant m$ is compact;*
(d) *X is m-pseudocompact;*
(e) *X is G_m-dense in $\beta(X)$.*

Corollary 1. *The product of any collection of m-pseudocompact spaces is m-pseudocompact iff it is pseudocompact.*

Proof. Necessity is obvious. To prove sufficiency, let $X = \pi X_i$, where X_i is m-pseudocompact for each i. Then X_i is G_m-dense in $\beta(X_i)$ for each i which implies that πX_i is G_m-dense in $\pi\beta(X_i)$. But $\pi\beta(X_i) = \beta(\pi X_i)$ by Glickberg's Theorem (see [4]) since πX_i is pseudocompact by assumption. Hence X is m-pseudocompact by Theorem 1 (e). □

Corollary 2. *If $m \geqslant c$ then the product of any collection of initially m-compact spaces is m-pseudocompact.*

Proof. Since an initially m-compact space is obviously m-pseudocompact it suffices to show, by Corollary 1, that the product is pseudocompact. It follows from Theorem 5 of [6] that an initially m-compact space is totally bounded if $m \geqslant c$. (Recall that a space is said to be *totally bounded* if the closure of every countable set is compact.) Since the product of any collection of totally bounded spaces is totally bounded and a totally bounded space is pseudocompact the assertion follows. □

Remark. It is well known that there are countably compact, i.e., initially ω-compact, spaces whose product is not pseudocompact. (See, for example, [3]). The above result shows that the corresponding result is not valid for $m \geqslant c$. The familiar examples of pseudocompact spaces whose product is not pseudocompact are subspaces of βD containing D where D is a discrete space. The result below shows that examples of this kind do not exist for $m \geqslant c$.

Corollary 3. *Let $m \geqslant c$, let $\{D_i : i \in I\}$, be a collection of discrete spaces, and let $\{X_i : i \in I\}$ be a collection of m-pseudocompact spaces such that $D_i \subseteq X_i \subseteq \beta D_i$ for each i. Then πX_i is m-pseudocompact.*

Proof. It is clear from the proof of Corollary 2 that the product of m-pseudocompact spaces each containing a dense totally bounded subspace is m-pseudocompact. Let $A_i = \{p \in \beta D_i : p$ is in the closure of some countable subset of $D_i\}$. Then A_i is totally bounded for each i. Since each singleton set in A_i is a G_m-set in βD_i and X_i is G_m-dense in βX_i it follows that $A_i \subseteq X_i$. This concludes the proof. □

Example. Let m be uncountable. Then the space $X = [0,1]^m - \{p\}$ where p is any point of $[0,1]^m$ is k-pseudocompact for any $k < m$ but not k-pseudocompact for any $k \geqslant m$. Thus k-pseudocompactness is in general weaker than m-pseudocompactness if $k < m$.

3. Weakly initially m-compact spaces

Recall that X is weakly initially m-compact if every open cover of X of cardinality $\leqslant m$ has a finite subset with a dense union. Equivalently X is weakly initially m-compact if every open filter base in X of cardinality $\leqslant m$ has an adherence point.

Theorem 2.

(a) *A regular closed set in a weakly initially m-compact space is weakly initially m-compact.*

(b) *The preimage under a perfect irreducible map of a weakly initially m-compact space is weakly initially m-compact.*

(c) *A weakly initially m-compact space is m-pseudocompact.*

(d) *An extremally disconnected m-pseudocompact space is weakly initially m-compact.*

Proof. (a) The interiors in X of the members of an open filter base in a regular closed set form a filter base in X with the same adherence as the original filter base.

(b) Let $f : X \to Y$ be a perfect irreducible map from X onto a weakly initially m-compact space Y. Let $\mathbf{U}$ be an open filter base in X of cardinality $\leqslant m$. Then it follows from the closedness and irreducibility of f that $\mathbf{V} := \{\operatorname{int} f[U] : U \in \mathbf{U}\}$ is an open filter base in Y. Let y be an adherent point of $\mathbf{V}$ and let $K = f^{-1}[p]$. Then an easy compactness argument shows that K contains an adherence point of $\mathbf{U}$.

(c) Let X be a weakly initially m-compact space and let **F** be a zero set filter and let **E** be a subset **F** of cardinality $\leqslant m$. For each Z in **E** there is a countable family of cozero sets $\{C_n(Z)\colon n \in \mathbf{N}\}$ such that $\overline{C_{n+1}}(Z) \subseteq C_n(Z)$ for $n \in \mathbf{N}$ and $\bigcap C_n(Z) = Z$. Then the family $\{C_n(Z)\colon n \in \mathbf{N},\ Z \in \mathbf{E}\}$ is an open filter base in X of cardinality $\leqslant m$ whose adherence is the intersection of **E**. Hence X is m-pseudocompact by Theorem 1.

(d) Let X be an extremally disconnected m-pseudocompact space annd let **U** be an open filter base in X of cardinality $\leqslant m$. Let $\mathbf{V} = \{\overline{U}\colon U \in \mathbf{U}\}$. Then **V** is a family of zero (in fact clopen) sets in X whose intersection is nonvoid since X is m-pseudocompact. □

Examples. We now show that for $m \geqslant c$

(1) a regular closed set in an m-pseudocompact space need not be m-pseudocompact;

(2) an m-pseudocompact space need not be weakly initially m-compact;

(3) a perfect irreducible preimage of an m-pseudocompact space need not be m-pseudocompact.

R. M. Stephenson, Jr. and J. E. Vaughan [8], show that for each m and each discrete space D of cardinality m, there are weakly initially m-compact subspaces X and Y of βD containing D whose intersection (and so $X \times Y$) is not weakly initially m-compact. By Corollary 3, $X \times Y$ is m-pseudocompact. Thus an m-pseudocompact space need not be weakly initially m-compact. Let Z be the diagonal of $X \times Y$. Then Z is extremally disconnected and not weakly initially m-compact and hence not m-pseudocompact by Theorem 2. Since Z is a regular closed set in $X \times Y$ this shows that a regular closed set in an m-pseudocompact space need not be one. Finally let E be the Gleason cover of $X \times Y$, i.e., an extremally disconnected space which is mapped onto $X \times Y$ by a perfect irreducible map. Then E is not m-pseudocompact, since otherwise, it would be weakly initially m-compact, by Theorem 2(d), which is impossible. Hence a perfect irreducible preimage of an m-pseudocompact space need not be one.

4. m-PSEUDOCOMPACTNESS IN THE SENCE OF COMPLETE ACCUMULATION POINTS

W. W. Comfort and S. Negrepontis [1] define a space X to be *pseudo-(k,k)-compact*, where k is an infinite cardinal number, if for each family $\{U_i\colon i < k\}$ of nonvoid open sets indexed by ordinals less than k, there is $x \in X$ such that for each neighbourhood V of x $|\{i < k\colon U_i \cap V \neq 0\}| = k$. Recall that mpcap stands for m-pseudocompact in the sense of complete accumulation points.

Theorem 3.

(a) *A regular closed set in a mpcap space is mpcap.*

(b) *A perfect irreducible preimage of an mpcap space is mpcap.*

(c) *An initially m-compact space is mpcap.*

(d) *A space is mpcap iff it is pseudo-(k, k)-compact for each $k \leqslant m$.*

(e) *Let $D \subseteq X \subseteq \beta D$, where D is discrete. Then X is mpcap iff every infinite subset of D of cardinality $\leqslant m$ has a complete accumulation point in X.*

Proof. The proofs of parts (a) and (b) are similar to those of the corresponding parts of Theorem 2.

(c) Recall that a space X is intially m-compact iff each infinite subset of cardinality $\leqslant m$ has a complete accumulation point. (See for example, [7].) Let X be an initially m-compact space and let $\mathbf{U}$ be an infinite family of open sets of cardinality $k \leqslant m$. Let $\mathbf{U} = \{U_i : i < k\}$ be a one to one indexing of $\mathbf{U}$. Let $f: k \to D$ be such that $f(i) \in U_i$, for each $i \in k$. Let $A = \{f(i): i \in k\}$ and for each $a \in A$ let $c(a) = |f^{-1}[a]|$. If $|A| = k$ or if $c(a) = k$ for some $a \in A$ then $\mathbf{U}$ has a complete accumulation point. So assume that $|A| < k$ and $c(a) < k$ for each $a \in A$. We may assume that $|A| = \operatorname{cf} k$, the cofinality of k. We can define a function $g: \operatorname{cf} k \to A$ such that $c(g(i)) < c(g(j))$ whenever $i < j < k$ and such that $\{c(g(i)) : i < \operatorname{cf} k\}$ is cofinal with k. We may assume that g is onto. Let x be a complete accumulation point of A, let V be a neighbourhood of x and let $B = V \cap A$. Then $|B| = \operatorname{cf} k$. Hence $\sum\{c(b): b \in B\} = k$. Hence $|\{i < k: V \cap U_i \neq 0\}| = k$. Hence x is complete accumulation point of $\mathbf{U}$.

(d) The proof is similar to that of part (c) and is left to the reader.

(e) If X is mpcap and A is an infinite subset of D of cardinality $\leqslant m$ then it must have a complete accumulation point since A is a union of singleton open sets of cardinality $\leqslant m$. Conversely let $\mathbf{U}$ be an infinite family of nonvoid open sets of cardinality $\leqslant m$. Let $\mathbf{U} = \{U_i : i < k\}$ be a one to one indexing of $\mathbf{U}$ where $k \leqslant m$. Let $f: k \to D$ be such that $f(i) \in U_i$ for $i < k$. Proceed as in part (c) above. □

Examples. Let $m \geqslant c$. We give examples to show that

(1) an mpcap space need not be m-pseudocompact (and hence not weakly initially m-compact);

(2) an m-pseudocompact space need not be mpcap;

(3) the product of two mpcap spaces need not be mpcap.

Let D be a discrete space and let $p \in \beta D$. The type of p, $T(p) := \{\overline{f}(p): \overline{f}$ is a mapping from βD to βD whose restriction to D is a permutation of $D\}$. Let $n(p) = \min\{|A|: A \in p\}$. For each infinite cardinal $k \leqslant |D|$, let $p(k)$ be an ultrafilter on D such that $n(p(k)) = k$. Let $X = D \cup \bigcup\{T(p(k)): k \leqslant m$ and $k \leqslant |D|\}$. Then any infinite subset of D of cardinality of k has a complete accumulation point in $T(p(k))$. Hence X is mpcap by Theorem 3.

In particular if D is countable and p is any free ultrafilter on D then $D \cup T(p)$ is mpcap for any m. If $m \geqslant c$ then the space is not m-pseudocompact since any m-pseudocompact subset of βD containing D is βD itself.

Let $m \geqslant c$ and let D be a discrete space of cardinality m. There exist weakly initially m-pseudocompact spaces X and Y of βD containing D such that $X \cap Y$ contains no uniform ultrafilter (an ultrafilter each member of which has cardinality m). (See [8].) Then $X \cap Y$ is not mpcap since D has no complete accumulation point in $X \cap Y$. Hence $X \times Y$ is not mpcap since the diagonal of $X \times Y$, which is a regular closed set in $X \times Y$, is not mpcap. Hence the product of two mpcap spaces need not be mpcap. We also see that an m-pseudocompact space need not be mpcap.

I conclude this discussion with two questions:

(1) Are there m-pseudocompact spaces whose product is not m-pseudocompact where $m > \omega$?

(2) Are weakly initially m-compact spaces necessarily mpcap?

In connection with question 2, we note that if D is a discrete space and $D \subseteq X \subseteq \beta D$ and X is m-pseudocompact (and hence weakly initially m-compact) then it is mpcap by Theorem 4.

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Author's address: Mathematics Department, Addis Ababa University, P. O. Box 1176, Addis Ababa, Ethiopia.