Some Cases of Wave Motion due to a Submerged Obstacle.

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1. As far as I am aware, only one case of wave motion caused by a submerged obstacle has been worked out in any detail, namely the two-dimensional motion due to a circular cylinder; for this case, Prof. Lamb has given a solution applicable when the cylinder is of small radius and is at a considerable depth.* The method can be extended to bodies of different shape, and my object in this paper is to work out the simplest three-dimensional case, the motion of a submerged sphere.

The problem I have considered specially is the wave resistance of the submerged body. In the two-dimensional case, this is calculated by considerations of energy and work applied to the train of regular waves. But for a moving sphere the wave system is more complicated, like the well-known wave pattern for a moving point disturbance, and similar methods are not so easily applied; I have therefore calculated directly the horizontal resultant of the fluid pressure on the sphere. Before working out this case, the analysis for the circular cylinder is repeated, because it is necessary to carry the approximation a stage further than in Prof. Lamb's solution in order to verify that the resultant horizontal pressure on the cylinder is the same as the wave resistance obtained by the method of energy.

The stages in approximating to the velocity potential may be described in terms of successive images; the first stage ϕ_1 is the image of a uniform stream in the submerged body, the second stage ϕ_2 is the image of ϕ_1 in the free surface, the third ϕ_3 is the image of ϕ_2 in the submerged body, and so on. In order to keep the integrals convergent, a small frictional coefficient is introduced in the usual manner; after the calculations have been carried out, the coefficient is made zero. Further, the solution for uniform motion is built up so that expressions can be found for the velocity potential at any time after the starting of the motion, although only the final steady state has been studied in detail. The wave resistance of a sphere is found to have the form const. $\times\, \alpha^{3/2} e^{-\alpha/2}\, W_{1,1}(\alpha)$, in which α is $2gf/c^2$, with f the depth of the sphere and c its velocity; $W_{1,1}(\alpha)$ is a confluent hypergeometric function. In order to graph the wave resistance as a function of the velocity, expansions have been found for this particular variety of the function

* H. Lamb, 'Ann. di Matematica,' vol. 21, p. 237; also 'Hydrodynamics,' 4th ed., p. 401.

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$W_{k, m}(\alpha)$; it belongs to the logarithmic case for which a general expansion is not available.

In general form the graph of the resistance is very similar to that of the circular cylinder.

Circular Cylinder.

2. The steady state for uniform motion of the cylinder may be attacked directly, as in Prof. Lamb's solution, but we shall adopt his suggestion of building it up from simple oscillations. Take the axis of x in the free surface of the water, and the axis of y vertically upwards. A circular cylinder, of radius a, is making small oscillations parallel to Ox with velocity $c \cos \sigma t$, the axis of the cylinder being horizontal and perpendicular to Ox, and the mean position of the centre being the point $(0, -f)$. A first appproximation when the depth f is sufficiently large is found by ignoring the surface effect altogether and putting

$$\phi = ca^2 (x/r^2) e^{i\sigma t}; \qquad r^2 = x^2 + (y+f)^2. \tag{1}$$

This satisfies the boundary condition at the surface of the cylinder. For the next step, add a term X_1 to the velocity potential so as to satisfy the conditions at the free surface, but ignoring meantime the disturbance produced thereby at the surface of the cylinder. The term X_1 must be a potential function and it must satisfy the condition for deep water, namely, $\partial X_1/\partial y = 0$ for $y = -\infty$; these conditions are fulfilled by

$$X_1 = e^{i\sigma t} \int_0^\infty \alpha(\kappa) e^{\kappa y} \sin \kappa x \, d\kappa, \tag{2}$$

where α is a function of κ to be determined. This form is chosen because we can satisfy the conditions at the free surface by using an equivalent form for (1), since

$$x/r^2 = \int_0^\infty e^{-\kappa(y+f)} \sin \kappa x \, d\kappa; \qquad y+f > 0. \tag{3}$$

The surface elevation is expressed similarly by

$$\eta = e^{i\sigma t} \int_0^\infty \beta(\kappa) \sin \kappa x \, d\kappa. \tag{4}$$

In order to keep the various integrals convergent, we assume that the liquid has a slight amount of friction proportional to velocity; in the sequel the results are simplified by making the frictional coefficient μ tend to zero. In these circumstances the pressure equation is

$$p/\rho = \text{const.} - gy + \mu\phi - \tfrac{1}{2} q^2. \tag{5}$$

Hence the conditions at the free surface are, neglecting the square of the velocity,

$$-gy + \mu\phi = \text{const.}; \qquad -\partial\phi/\partial y = \partial\eta/\partial t.$$

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Here ϕ is the velocity potential after (2) has been added to (1); thus the equations for α and β are

$$\left.\begin{aligned} ca^2\kappa e^{-\kappa f} - \kappa\alpha &= i\sigma\beta \\ i\sigma ca^2 e^{-\kappa f} + i\sigma\alpha - g\beta + \mu ca^2 e^{-\kappa f} + \mu\alpha &= 0 \end{aligned}\right\} \tag{6}$$

From these we obtain the expressions for X_1 and η, namely

$$X_1 = ca^2 e^{i\sigma t}\int_0^\infty \frac{g\kappa + \sigma^2 - i\mu\sigma}{g\kappa - \sigma^2 + i\mu\sigma}\, e^{-\kappa(f-y)}\sin\kappa x\, d\kappa, \tag{7}$$

$$\eta = ca^2 e^{i\sigma t}\int_0^\infty \frac{2\kappa(\mu + i\sigma)}{g\kappa - \sigma^2 + i\mu\sigma}\, e^{-\kappa f}\sin\kappa x\, d\kappa. \tag{8}$$

The expression for X_1 can be divided into two parts

$$X_1 = -ca^2 e^{i\sigma t}\int_0^\infty e^{-\kappa(f-y)}\sin\kappa x\, d\kappa - 2ca^2 e^{i\sigma t}\int_0^\infty \frac{g\kappa e^{-\kappa(f-y)}\sin\kappa x\, d\kappa}{\sigma^2 - i\mu\sigma - g\kappa}. \tag{9}$$

If we regard X_1 as the image of the oscillating cylinder in the free surface, we see from the form of the first integral in (9) that part of the image is a negative doublet at the image point $(0, f)$. We obtain next the velocity potential of the motion produced by a sudden small displacement of the cylinder, and we take this to be equivalent to a momentary doublet of constant strength. Suppose then that at a time τ a doublet is suddenly created, maintained constant for a time $\delta\tau$, and then annihilated. The velocity potential at any subsequent time t is given by a Fourier synthesis of the preceding results for an oscillating cylinder, and we have

$$\phi = \frac{\delta\tau}{\pi}\int_0^\infty e^{i\sigma(t-\tau)}[\phi]\, d\sigma, \tag{10}$$

where $[\phi]$ is the sum of (1) and (9), omitting the factor $e^{i\sigma t}$.

Carrying out this integration for the value of ϕ in (1) and for the first part of (9) gives simply the momentary doublet at the centre of the cylinder and the negative doublet at the image point. These doublets last for a short time $\delta\tau$; the subsequent fluid motion is contributed by the second part of (9). For this we have to evaluate the real part of

$$\int_0^\infty \frac{e^{i\sigma(t-\tau)}}{\sigma^2 - i\mu\sigma - g\kappa}\, d\sigma; \qquad t-\tau > 0. \tag{11}$$

We obtain the value by contour integration; further we simplify the result by neglecting μ^2. We shall make μ zero ultimately, but we must retain it sufficiently to keep the integrals convergent; however, at one or two stages, superfluous terms may be omitted when it is clear that the final limiting values will not be affected. We find for (11) the value

$$-\pi\kappa V e^{-\frac{1}{2}\mu(t-\tau)}\sin\{\kappa V(t-\tau)\},$$

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writing V for $\sqrt{(g/\kappa)}$ whenever it serves to simplify the notation. Hence the velocity potential of the subsequent fluid motion after the cylinder has been given a small displacement at time τ is

$$\phi = 2ca^2\delta\tau e^{-\frac{1}{2}\mu(t-\tau)} \int_0^\infty \kappa \mathrm{V} e^{-\kappa(f-y)} \sin \kappa x \sin \kappa \mathrm{V}(t-\tau)\, d\kappa. \tag{12}$$

Finally we obtain the velocity potential for a cylinder in uniform motion by substituting $x+c(t-\tau)$ for x, noting that hereafter x will refer to a moving origin immediately over the centre of the cylinder; we then integrate with respect to τ from the start of the motion up to the instant in question. We could in this way obtain results for any stage of the motion, but we limit the discussion to the final steady state; for this we take $-\infty$ as the lower limit in integrating with respect to τ. Before writing down the result, we must remember to introduce the integrated effect of the original momentary doublet in (1) and its negative image, which were not included in (11); these clearly add up to steady doublets. Hence we find for the steady state

$$\phi = \mathrm{D} - \mathrm{D}_1 + 2ca^2 \int_0^\infty e^{-\kappa(f-y)} (\mathrm{A} \sin \kappa x + \mathrm{B} \cos \kappa x)\, d\kappa, \tag{13}$$

where D represents the doublet ca^2x/r^2 at the point $(0, -f)$, D_1 an equal doublet at the point $(0, f)$, and

$$2\mathrm{A} = \frac{\kappa^2 \mathrm{V}(\mathrm{V}+c)}{\kappa^2(\mathrm{V}+c)^2 + \frac{1}{4}\mu^2} + \frac{\kappa^2 \mathrm{V}(\mathrm{V}-c)}{\kappa^2(\mathrm{V}-c)^2 + \frac{1}{4}\mu^2},$$

$$4\mathrm{B} = \frac{\mu\kappa \mathrm{V}}{\kappa^2(\mathrm{V}-c)^2 + \frac{1}{4}\mu^2} - \frac{\mu\kappa \mathrm{V}}{\kappa^2(\mathrm{V}+c)_2 + \frac{1}{4}\mu^2}. \tag{14}$$

3. Before proceeding further we may obtain the surface elevation from (13) for comparison. The surface condition is now

$$-\partial\phi/\partial y = \partial\eta/\partial t = -c\,\partial\eta/\partial x.$$

Hence we have

$$\eta = 2a^2 f/(x^2+f^2) - 2a^2\kappa_0 \int_0^\infty (\mathrm{A}\cos \kappa x - \mathrm{B} \sin \kappa x)\, e^{-\kappa f}\, d\kappa, \tag{15}$$

in which $\kappa_0 = g/c^2$. Further, since μ is to be small, we may omit irrelevant terms and put

$$\mathrm{A} = -\kappa_0(\kappa-\kappa_0)/\{\kappa-(\kappa_0+i\mu/c)\}\{\kappa-(\kappa_0-i\mu/c)\},$$

$$\mathrm{B} = \kappa_0(\mu/c)/\{\kappa-(\kappa_0+i\mu/c)\}\{\kappa-(\kappa_0-i\mu/c)\}. \tag{16}$$

The integral in (15) can then be written as

$$\int_0^\infty \left\{\frac{e^{-i\kappa x}}{\kappa-\kappa_0-i\mu/c} + \frac{e^{i\kappa x}}{\kappa-\kappa_0+i\mu/c}\right\} e^{-\kappa f}\, d\kappa. \tag{17}$$

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We transform these integrals by contour integration in the plane of a complex variable κ, treating separately the cases of x positive and x negative; after making μ zero in the final results we obtain

$$\eta = \frac{2a^2 f}{x^2+f^2} + 4\pi a^2 \kappa_0 e^{-\kappa_0 f} \sin \kappa_0 x + 2a^2\kappa_0 \int_0^\infty \frac{m\cos mf - \kappa_0 \sin mf}{m^2+\kappa^2} e^{mx}\, dm\,; \quad x<0,$$

$$\eta = \frac{2a^2 f}{x^2+f^2} + 2a^2\kappa_0 \int_0^\infty \frac{m\cos mf - \kappa_0 \sin mf}{m^2+\kappa^2} e^{-mx}\, dm\,; \qquad x>0. \tag{18}$$

These agree with Lamb's results for the circular cylinder in a uniform stream.

The wave resistance R is derived from the regular waves in the rear, by considering the rate of increase of energy and taking into account the propagation of energy in a regular train; we have

$$\mathrm{R} = \tfrac{1}{4} g\rho\, (\text{amplitude})^2 = 4\pi^2 g\rho a^4 \kappa_0^{\,2} e^{-2\kappa_0 f}. \tag{19}$$

4. We have now to obtain the resistance R by direct summation of the horizontal component of fluid pressure on the cylinder. It is clearly necessary to proceed to a further stage with the velocity potential, since we have assumed so far that the surface effect is negligible in the neighbourhood of the cylinder. If we write (13) as

$$\phi = \mathrm{D} + \mathrm{X}_1, \tag{20}$$

the doublet D is the first approximation, satisfying the boundary conditions on the cylinder; X_1 is the image of the doublet in the free surface, found by satisfying the conditions there. The next step is to find X_2, the image of X_1 in the cylinder, ignoring then the effect of X_2 at the free surface. It follows that X_2 is the image of X_1 in the cylinder, found as if the cylinder were at rest in a field defined by X_1. Taking polar co-ordinates with the origin at the centre of the circular section of the cylinder, we have

$$x = r\cos\theta\,; \qquad y+f = r\sin\theta\,; \tag{21}$$

also the conditions for X_2 are that it should be a potential function, the components of velocity must vanish as r becomes infinite, and

$$\partial(\mathrm{X}_1+\mathrm{X}_2)/\partial r = 0, \qquad \text{for } r = a. \tag{22}$$

But from (13), X_1 consists of a summation of terms of the form

$$e^{\kappa y} \begin{matrix}\cos\\ \sin\end{matrix}\, \kappa x.$$

We obtain X_2 by replacing each term by the expressions

$$e^{-\kappa f} e^{\kappa a^2 \sin\theta/r} \begin{matrix}\cos\\ \sin\end{matrix} (\kappa a^2 \cos\theta/r),$$

and the above conditions for X_2 are then satisfied. This process amounts simply to inversion; we may think of X_1 as due to a line distribution of sources and X_2 is then a circle of sources on the inverse of this line with respect to the cylinder. We have now for the velocity potential to this stage

$$\phi = D + 2ca^2 \int_0^\infty e^{-\kappa(f-y)} \{ (A-\tfrac{1}{2}) \sin \kappa x + B \cos \kappa x \} \, d\kappa$$

$$+ 2ca^2 \int_0^\infty e^{-\kappa f + \kappa a^2 y/r^2} \{ (A-\tfrac{1}{2}) \sin (\kappa a^2 x/r^2) + B \cos (\kappa a^2 x/r^2) \} \, d\kappa. \quad (23)$$

We have put $A-\frac{1}{2}$ for A so as to include under the integral sign the doublet previously denoted by D_1.

The method could theoretically be carried on step by step; however, we stop at this stage because it is sufficient for obtaining the wave resistance R from the pressure equation to the same approximation as by the energy method.

We have
$$R = \int_0^{2\pi} ap \cos \theta \, d\theta; \quad (24)$$

$$p/\rho = -c\partial\phi/\partial x - gy + \mu\phi - \tfrac{1}{2} q^2. \quad (25)$$

If we write (23) as $\phi = D + X_1 + X_2$, and omit terms which obviously contribute nothing to the value of R, we have, when $r = a$,

$$\frac{p}{\rho} = -c \frac{\partial}{\partial x}(X_1+X_2) + \mu (X_1+X_2) - \frac{1}{a^2} \frac{\partial D}{\partial \theta} \frac{\partial (X_1+X_2)}{\partial \theta}$$

$$= (2c/a) \sin \theta \, \partial (X_1+X_2)/\partial \theta + \mu (X_1+X_2), \quad (26)$$

where we have used (22) and the value of D. From (23), omitting the doublets D and D_1, which will from symmetry give no contribution to R when μ is zero, we have

$$p = 4ca^2 \int_0^\infty e^{-2\kappa f + \kappa a \sin \theta} \{ 2\kappa c A \sin \theta \sin (\phi - \theta) + \mu A \sin \phi$$

$$+ 2\kappa c B \sin \theta \cos (\phi - \theta) + \mu B \cos \phi \} \, d\kappa, \quad (27)$$

where $\phi = \kappa a \cos \theta$. Substituting in (24) we have an expression for R. We may now change the order of integration and take first that with respect to θ; we can carry this out, after some transformation, by means of the integrals

$$\int_0^\pi e^{h \cos \theta} \cos (h \sin \theta - n\theta) \, d\theta = \pi h^n / \Gamma(n+1),$$

$$\int_0^\pi e^{h \cos \theta} \cos (h \sin \theta + n\theta) \, d\theta = 0, \quad (28)$$

where n is a positive, odd integer. In fact the integration with respect to θ gives simply $\pi \kappa a (\kappa c B + \mu A)$; hence we have

$$R = 4\pi \rho c a^4 \int_0^\infty \kappa (\kappa c B + \mu A) e^{-2\kappa f} \, d\kappa, \quad (29)$$

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where A and B are given by (14), or by (16) since we suppose μ small. Thus we have

$$R = 4\pi\rho c a^4 \operatorname*{Lim}_{\mu\to 0} \int_0^\infty \frac{\mu\kappa_0^2\kappa e^{-2\kappa f}\, d\kappa}{\{\kappa-(\kappa_0+i\mu/c)\}\{\kappa-(\kappa_0-i\mu/c)\}}$$

$$= 4\pi\rho c a^4 \operatorname{Lim} \mu\{2\pi i\kappa_0^3 e^{-2\kappa_0 f}/2i(\mu/c) + \text{finite quantity}\}$$

$$= 4\pi^2 g^3 \rho a^4 c^{-4} e^{-2gf/c^2}, \tag{30}$$

which is the same as the previous expression (19).

Sphere.

5. A sphere of radius a is at depth f below the surface and is moving with uniform velocity c parallel to the axis of x. The origin is in the free surface, the axis of z being drawn vertically upwards. As before, the first approximation is a doublet D given by

$$\phi = ca^3 x/2r^3; \qquad r^2 = x^2+y^2+(z+f)^2. \tag{31}$$

For the purpose of satisfying the conditions at the free surface we have

$$\phi = D = -\tfrac{1}{2}ca^3 \frac{\partial}{\partial x}\int_0^\infty e^{-\kappa(z+f)} J_0\{\kappa\sqrt{(x^2+y^2)}\}\, d\kappa; \qquad z+f>0. \tag{32}$$

This suggests at once suitable forms for the next approximation and for the free surface; the equations are similar to (6) of the previous case, and we obtain in the same way

$$\phi = D - D_1 + X_1, \tag{33}$$

where D_1 is a doublet at the image point $(0, 0, f)$ and

$$X_1 = ca^3 \frac{\partial}{\partial x}\int_0^\infty \sqrt{(g\kappa)}\, e^{-\kappa(f-z)}\, d\kappa \int_0^\infty e^{-\frac{1}{2}\mu u} J_0[\kappa\sqrt{\{(x+cu)^2+y^2\}}] \sin(\kappa V u)\, du. \tag{34}$$

The corresponding surface elevation is

$$\eta = a^3\int_0^\infty e^{-\kappa f} J_0\{\kappa\sqrt{(x^2+y^2)}\}\kappa\, d\kappa$$

$$+ a^3\int_0^\infty \sqrt{(g\kappa)}\, e^{-\kappa f}\kappa\, d\kappa \int_0^\infty e^{-\frac{1}{2}\mu u} J_0[\kappa\sqrt{\{(x+cu)^2+y^2\}}] \sin(\kappa V u)\, du. \tag{35}$$

The first term represents the effect of the doublets D and D_1. It can be verified by approximate methods that the second term includes a main part like the well-known wave pattern for ship waves. Since the expression in (35) gives finite and continuous values for the surface elevation, it might be of interest to examine some points in detail; for instance, the elevation near the lines corresponding to the lines of cusps for a moving point disturbance. However, we pass now to the calculation of the resultant horizontal pressure

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on the sphere. We have to find X_2 the image of X_1 in the sphere; for this we first put X_1 into a different form by using

$$\pi J_0[\kappa\{(x+cu)^2+y^2\}^{\frac{1}{2}}] = \int_0^\pi \cos\{\kappa(x+cu)\cos\phi\}\cos(\kappa y\sin\phi)\,d\phi. \quad (36)$$

From (36) and (34), after carrying out the integration with respect to u, we obtain

$$\pi X_1 = ca^3\int_0^\infty e^{-\kappa(f-z)}\,\kappa d\kappa\int_0^\pi\{A\sin(\kappa x\cos\phi)+B\cos(\kappa x\cos\phi)\}$$
$$\times\cos(\kappa y\sin\phi)\cos\phi\,d\phi, \quad (37)$$

where A and B are given by (14) after writing $c\cos\phi$ for c.

For convenience in the following analysis, we transfer the origin to the centre of the sphere, noting that in (37) we shall have $\exp.(-2\kappa f+\kappa z)$ in place of $\exp.(-\kappa f+\kappa z)$. Also we use polar co-ordinates

$$x = r\cos\alpha; \qquad y = r\sin\alpha\cos\beta; \qquad z = r\sin\alpha\sin\beta.$$

The conditions for X_2 are that it must be a potential function, the disturbance due to it must ultimately vanish as we recede from the sphere, and on the sphere

$$\partial(X_1+X_2)/\partial r = 0. \quad (38)$$

To avoid repetition of expressions like (37), we take out of it a typical term and write

$$X_1 = e^{\kappa z}\sin(\kappa x\cos\phi)\cos(\kappa y\sin\phi). \quad (39)$$

We know that the function

$$r^{-1}e^{\kappa a^2 z/r^2}\sin(\kappa a^2 x\cos\phi/r^2)\cos(\kappa a^2 y\sin\phi/r^2) \quad (40)$$

satisfies the first two conditions for X_2, but we find it does not fulfil (38). An additional term is required, and it can be found in the following way. Suppose that on the sphere we have

$$e^{\kappa z}\sin(\kappa x\cos\phi)\cos(\kappa y\sin\phi) = \Sigma A_m Y_m(\alpha,\beta), \quad (41)$$

where the right-hand side is an expansion in surface spherical harmonics. Then for the term (39), all the conditions for X_2 are satisfied by

$$ar^{-1}e^{\kappa a^2 z/r^2}\sin(\kappa a^2 x\cos\phi/r^2)\cos(\kappa a^2 y\sin\phi/r^2)-\Sigma a^{m+1}A_m Y_m/(m+1)r^{m+1}. \quad (42)$$

Suppose, similarly, that on the sphere we have

$$e^{\kappa z}\cos(\kappa x\cos\phi)\cos(\kappa y\sin\phi) = \Sigma B_m Y_m(\alpha,\beta). \quad (43)$$

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Then the complete expression for X_2 is

$$\pi X_2 = ca^3 \int_0^\infty e^{-2\kappa f}\,\kappa d\kappa \int_0^\pi ar^{-1} e^{\kappa a^2 z/r^2} \cos(\kappa a^2 y \sin\phi/r^2)\cos\phi$$
$$\times\{A\sin(\kappa a^2 x\cos\phi/r^2) + B\cos(\kappa a^2 x\cos\phi/r^2)\}\,d\phi$$
$$-ca^3 \int_0^\infty e^{-2\kappa f}\,\kappa d\kappa \int_0^\pi \sum_m (AA_m + BB_m)(m+1)^{-1}(a/r)^{m+1} Y_m \cos\phi\,d\phi. \tag{44}$$

We have now

$$\phi = D - D_1 + X_1 + X_2 = D + X, \tag{45}$$

and the pressure equation is

$$p/\rho = -c\,\partial\phi/\partial x - gz + \mu\phi - \tfrac{1}{2}q^2. \tag{46}$$

The wave resistance, or the resultant horizontal pressure on the sphere, is

$$R = \int_0^\pi d\alpha \int_0^{2\pi} a^2 p \sin\alpha\cos\alpha\,d\beta. \tag{47}$$

Omitting terms which, from symmetry, will give no contribution to R, we have

$$\frac{p}{\rho} = -c\frac{\partial X}{\partial x} + \mu X - \frac{\partial D}{\partial r}\frac{\partial X}{\partial r} - \frac{1}{r^2}\frac{\partial D}{\partial \alpha}\frac{\partial X}{\partial \alpha} - \frac{1}{r^2\sin^2\alpha}\frac{\partial D}{\partial \beta}\frac{\partial X}{\partial \beta}. \tag{48}$$

But when $r = a$, we have

$$\partial D/\partial\beta = 0; \qquad \partial D/\partial\alpha = -\tfrac{1}{2}ca\sin\alpha; \qquad \partial X/\partial r = 0,$$

hence

$$p/\rho = (3c/2a)\sin\alpha\,\partial X/\partial\alpha + \mu X. \tag{49}$$

We must now substitute (49) in (47) and use the value of X given by the sum of (37) and (44) on the sphere; it is clear that we may omit the doublet D_1 as it will not affect the limiting value of R when μ is zero.

6. Consider, in the first place, the contribution of the first term in the value of p given in (49). In the repeated integrals which are obtained, we may change the order of integration, and we shall carry out first the summation over the surface of the sphere. We notice that, when $r = a$, the first term in the value of X_2 in (44) is equal to the value of X_1; the additional part of X_2 is the term involving the expansions in spherical surface harmonics. Choose a typical term from the latter part, and we find we have to evaluate

$$\int \tfrac{3}{2}\sin\alpha\cos\alpha\,(\partial Y_m/\partial\alpha)\,dS, \tag{50}$$

taken over the surface of unit sphere.

But this integral is equal to

$$-3\int P_2(\cos\alpha)\,Y_m(\alpha,\beta)\,dS. \tag{51}$$

Hence, the only term which has a value different from zero is the term in Y_2, the surface harmonic of the second order. From the manner in which

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the expansions were introduced, in (41) and (43), it follows that the contribution of the second term in (44) is one-third of that of the first term; hence, summing up the result so far as the first term of (49) is concerned, we have

$$\pi R' = -5c^2a^4\rho \int_0^\infty e^{-2\kappa f}\kappa d\kappa \int_0^\pi \cos\phi\, d\phi \int_0^{2\pi} d\beta \int_0^\pi \sin\alpha\, P_2(\cos\alpha)\, e^{\kappa a \sin\alpha \sin\beta}$$

$$\times \cos(\kappa a \sin\alpha \cos\beta \sin\phi)$$

$$\times \{A \sin(\kappa a \cos\alpha \cos\phi) + B\cos(\kappa a \cos\alpha\cos\phi)\}\, d\alpha. \quad (52)$$

Taking the integration with respect to β, we find it is equal to

$$2\int_0^\pi e^{\kappa a \sin\alpha\cos\beta} \cos(\kappa a \sin\alpha \sin\phi \sin\beta)\, d\beta = 2\pi I_0(\kappa a \sin\alpha\cos\phi), \quad (53)$$

where $I_0(x)$ is the Bessel function $J_0(ix)$, a result which may be obtained by direct expansion and integration term by term. For the integration with respect to α the term in A in (52) obviously gives zero, and we are left with

$$2\pi \int_0^\pi I_0(\kappa a \cos\phi \sin\alpha) \cos(\kappa a \cos\phi \cos\alpha)\, P_2(\cos\alpha) \sin\alpha\, d\alpha. \quad (54)$$

Here also we may expand in powers of κa and integrate term by term; it can be shown that the integral of the coefficient of $(\kappa a)^n$ vanishes except for the single term κ^2a^2; thus we find that (54) reduces to

$$-(2\pi/5)\kappa^2a^2\cos^2\phi.$$

7. We have now to consider the term μX in the value for p in (49). We might omit this term, on general grounds, as giving no contribution to R ultimately when μ vanishes; for X is the velocity potential for a sphere at rest in a given field X_1. However, it may be left in, and we have a similar calculation. Taking the second integral in (44), we find it is now only the term in Y_1 which counts; hence the contribution of this part is one-half of that of the first integral in (44). Further, it is the term involving A which gives a value different from zero when integrating with respect to α, and instead of (54) we have

$$2\pi \int_0^\pi I_0(\kappa a \cos\phi \sin\alpha) \sin(\kappa a \cos\phi \cos\alpha)\, P_1(\cos\alpha) \sin\alpha\, d\alpha,$$

which reduces to $(4\pi/3)\kappa a \cos\phi$.

8. Collecting the various results, we have now

$$R = -2ca^6\rho \int_0^\infty e^{-2\kappa f}\kappa^2 d\kappa \int_0^\pi (\kappa c B\cos\phi + \mu A)\cos^2\phi\, d\phi, \quad (55)$$

a form which may be compared with the corresponding expression for the cylinder in (29).

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A and B are given by (14) when we replace c by $c\cos\phi$; putting these values in (55), we see that we may change the order of integration. Further, as we make μ vanish ultimately, we may use simplified forms of A and B corresponding to (16). These give

$$\mathrm{R} = 4\kappa_0^2 ca^6\rho\mu\int_0^{\pi/2}\sec^2\phi\, d\phi\int_0^\infty \frac{\kappa^2 e^{-2\kappa f}\, d\kappa}{(\kappa-\kappa_0\sec^2\phi)^2+(\mu/c)^2\sec^2\phi}.$$

To obtain the limiting value for μ zero we may treat this like the similar expressions in (30); or, alternatively, we may put $(\mu/c)\sec\phi = 1/n$, and use the general result

$$\operatorname*{Lim}_{n\to\infty}\int_a^b \frac{f(x)\,dx}{1+n^2(x-\alpha)^2} = \frac{\pi}{2}\ \{f(\alpha-0)+f(\alpha+0)\}.$$

The apparent difficulty with regard to values of ϕ near $\pi/2$ is overcome by noticing that with the particular functions involved in R no extra contribution arises from such terms near the upper limits of the variables. Carrying out the integration in κ in this way, and changing the remaining variable by putting $\tan\phi = t$, we obtain

$$\mathrm{R} = 4\pi g^4\rho a^6 c^{-6} e^{-2gf/c^2}\int_0^\infty (1+t^2)^{3/2} e^{-2gft^2/c^2}. \tag{56}$$

The remaining integral can be expressed in terms of known functions. Possibly the simplest method is to use the confluent hypergeometric function* defined, for real positive values of α and for real values of k and m for which $k-m-\frac{1}{2}\leqslant 0$, by

$$\mathrm{W}_{k,m}(\alpha) = \frac{e^{-\alpha/2}\alpha^k}{\Gamma(\frac{1}{2}-k+m)}\int_0^\infty u^{-k-\frac{1}{2}+m}(1+u/\alpha)^{k-\frac{1}{2}+m}e^{-u}\,du. \tag{57}$$

We have now the wave resistance of the sphere given by

$$\mathrm{R} = \tfrac{1}{4}\pi^{3/2} g\rho a^6 f^{-3}\alpha^{3/2}e^{-\alpha/2}\mathrm{W}_{1,1}(\alpha); \qquad \alpha = 2gf/c^2. \tag{58}$$

8. For purposes of calculation, we require expansions of $\mathrm{W}_{1,1}(\alpha)$. This function belongs to the logarithmic type of confluent hypergeometric function, and general expansions are not available in this case; however, they can be obtained without difficulty for $\mathrm{W}_{1,1}$. In the first place, the differential equation satisfied by $\mathrm{W}_{1,1}$ is

$$\frac{d^2y}{d\alpha^2}+\left(-\frac{1}{4}+\frac{1}{\alpha}-\frac{3}{4\alpha^2}\right)y = 0. \tag{59}$$

We use the ordinary methods for solving by means of power series. The roots of the indicial equation are $\frac{3}{2}$ and $-\frac{1}{2}$; hence one of the fundamental

* E. T. Whittaker, 'Bull. Amer. Math. Soc.,' vol. 10, p. 125; also Whittaker and Watson, 'Modern Analysis,' Chap. XVI.

solutions will contain logarithms. Calculating the coefficients step by step, we obtain as a fundamental system

$$\left.\begin{aligned} y_1 &= \alpha^{3/2}\left(1-\tfrac{1}{3}\alpha+\tfrac{7}{96}\alpha^2-\tfrac{1}{96}\alpha^3+\tfrac{11}{9216}\alpha^4-\ldots\right) \\ y_2 &= y_1\log\alpha+\alpha^{-3/2}\left(-\tfrac{8}{3}-\tfrac{8}{3}\alpha+\tfrac{2}{9}\alpha^3-\tfrac{95}{1152}\alpha^4+\ldots\right) \end{aligned}\right\}. \tag{60}$$

We know that $W_{1,1}$ is a linear function of y_1 and y_2; however, it is simpler to obtain an expansion directly and use (60) to verify it. For this purpose we use the equivalent contour integral for the confluent hypergeometric function,

$$W_{k,m} = \frac{\alpha^k e^{-\alpha/2}}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(s)\,\Gamma(-s-k-m+\frac{1}{2})\,\Gamma(-s-k+m+\frac{1}{2})}{\Gamma(-k-m+\frac{1}{2})\,\Gamma(-k+m+\frac{1}{2})}\,\alpha^s\,ds, \tag{61}$$

where the contour has loops if necessary, so that the poles of $\Gamma(s)$ and those of $\Gamma(-s-k-m+\frac{1}{2})\,\Gamma(-s-k+m+\frac{1}{2})$ are on opposite sides of it. The integral can be evaluated by the method of residues. When $k=m=1$, the poles at which the residues have to be found are simple poles at $s=-\frac{1}{2}, -\frac{3}{2}$, together with double poles at $s=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. The latter series gives rise to logarithmic residues. Carrying out the calculation, we obtain

$$W_{1,1} = \pi^{-1/2}\alpha e^{-\alpha/2}\left(\alpha^{-3/2}+\frac{3}{2}\alpha^{-1/2}\right)-\frac{3}{4\pi}\,\alpha^{3/2}e^{-\alpha/2}\left\{\log\alpha\sum_0^\infty\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)\Gamma(p+3)}\,\alpha^p\right.$$
$$\left.+\tfrac{1}{2}\pi^{\frac{1}{2}}\left(\gamma-2\log 2-\frac{3}{2}\right)+\sum_1^\infty\alpha^p\,\frac{d}{dp}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)\,\Gamma(p+3)}\right\}, \tag{62}$$

where γ is Euler's constant 0·5772.... The coefficients may be put into alternative forms more suited for calculation; for instance

$$\frac{d}{dp}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)\,\Gamma(p+3)}$$
$$=\frac{1.3.5\ldots(2p-1)\,\pi^{\frac{1}{2}}}{2^p.p!(p+2)!}\left\{\gamma-2\log 2+\sum_1^p\frac{1}{n(2n-1)}-\sum_1^{p+2}\frac{1}{n}\right\}.$$

For numerical calculation we have

$$W_{1,1} = \frac{3}{8}\pi^{-1/2}\alpha^{3/2}e^{-\alpha/2}\ \left\{\frac{8}{3\alpha^2}+\frac{4}{\alpha}+\frac{3}{2}+\frac{5}{36}\alpha+\frac{11}{384}\alpha^2+\frac{7}{1280}\alpha^3+\ldots\right.$$
$$\left.-\left(\gamma+\log\frac{1}{4}\alpha\right)\left(1+\frac{1}{6}\alpha+\frac{1}{32}\alpha^2+\frac{1}{192}\alpha^3+\ldots\right)\right. \tag{63}$$

The expansion may be confirmed by comparison with the fundamental solutions of the differential equation given in (60); we find that

$$(8/3)\,\pi^{\frac{1}{2}}\,W_{1,1} = (2\log 2-\gamma-\tfrac{1}{6})\,y_1-y_2.$$

For large values of α the general asymptotic expansion of $W_{k,m}$ is available; and in this case we have

$$W_{1,1}\sim\alpha e^{-\alpha/2}\left(1+\frac{3}{4}\frac{1}{\alpha}+\frac{9}{32}\frac{1}{\alpha^2}-\frac{15}{128}\frac{1}{\alpha^3}+\ldots\right). \tag{64}$$

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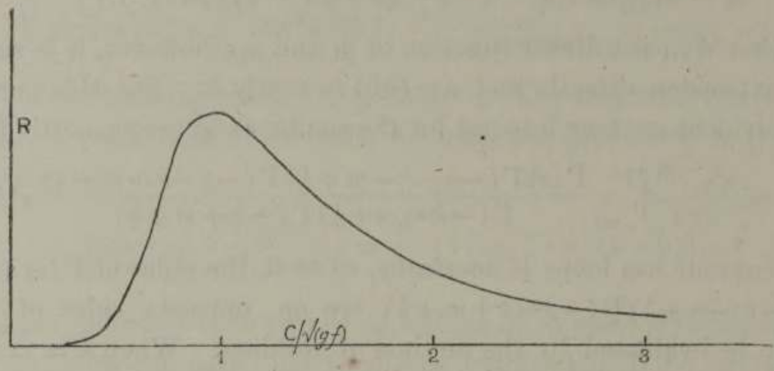
9. With (63) and (64) we can now calculate the resistance R from (58). For a given depth f, the variation of the resistance with the velocity is shown in the following curve, for which R has been calculated for various values of $c/\sqrt{(gf)}$.



The curve is very similar in form to the two-dimensional case of a circular cylinder. For small velocities, that is α large, if we take the first term of the asymptotic expansion (64), we have

$$R = \sqrt{(2\pi^3 g^7/f)} \cdot \rho a^6 c^{-5} e^{-2gf/c^2},$$

which may be compared with (30) for the cylinder. It is of interest to notice the similar law of variation of wave resistance with speed for the few cases of rigid bodies which have been worked out. The method adopted here can be applied to bodies of different forms, and it is hoped to illustrate later some interference effects.

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