mathematics

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Some Certain Fuzzy Aumann Integral Inequalities for Generalized Convexity via Fuzzy Number Valued Mappings

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Abstract: The topic of convex and nonconvex mapping has many applications in engineering and applied mathematics. The Aumann and fuzzy Aumann integrals are the most significant interval and fuzzy operators that allow the classical theory of integrals to be generalized. This paper considers the well-known fuzzy Hermite–Hadamard (HH) type and associated inequalities. With the help of fuzzy Aumann integrals and the newly introduced fuzzy number valued up and down convexity ($\mathcal{UD}$-convexity), we increase this mileage even further. Additionally, with the help of definitions of lower $\mathcal{UD}$-concave (lower $\mathcal{UD}$-concave) and upper $\mathcal{UD}$-convex (concave) fuzzy number valued mappings ($\mathcal{FNVM}$s), we have gathered a sizable collection of both well-known and new extraordinary cases that act as applications of the main conclusions. We also offer a few examples of fuzzy number valued $\mathcal{UD}$-convexity to further demonstrate the validity of the fuzzy inclusion relations presented in this study.

Keywords: fuzzy number valued mapping; fuzzy Aumann integral; up and down convex fuzzy number valued mapping; Hermite–Hadamard inequality; Hermite–Hadamard–Fejér inequality

MSC: 26A33; 26A51; 26D10

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Citation: Khan, M.B.; Othman, H.A.; Voskoglou, M.G.; Abdullah, L.; Alzubaidi, A.M. Some Certain Fuzzy Aumann Integral Inequalities for Generalized Convexity via Fuzzy Number Valued Mappings. *Mathematics* **2023**, *11*, 550. https://doi.org/10.3390/math11030550

Academic Editor: Salvatore Sessa

Received: 9 December 2022
Revised: 30 December 2022
Accepted: 13 January 2023
Published: 19 January 2023

1. Introduction

Many fields make use of the convexity of functions such as game theory, variational science, mathematical programming theory, economics, and optimal control theory. Convex analysis, a brand-new mathematics branch, started taking shape in the 1960s. Many writers have employed related concepts of convexity during the past 20 years and generalized other inequalities, including h-convex functions (see References [1–10]), log convex functions (see References [11–19], and coordinated convex functions (see References [20,21]). Convexity is a fundamental term in optimization theory applied in operations research, economics, control theory, decision-making, and management. Several writers have expanded and generalized integral inequalities using various convex functions; see Refs. [22,23]. For more information, see [24–33] and references therein.

Calculating mistakes in a numerical analysis has always been difficult. The interval analysis has received a lot of attention as a novel method for resolving uncertainty issues because of its capacity to reduce calculation errors and make calculations meaningless. Set-valued analysis, a set-centric approach to mathematics and topology, includes interval analysis. It deals with interval variables rather than point variables, and the computation results are expressed as intervals; therefore, it removes mistakes that lead to incorrect conclusions. Moore [34] first adapted an interval analysis to automatic error analysis to deal with data uncertainty in 1966. The work garnered a lot of attention from academics and led to an improvement in calculation performance. They are helpful in many applications because of their capacity to be expressed as uncertain variables, including computer graphics [35], automatic error analysis [36], decision analysis [37], etc. There are numerous great applications and results for readers interested in interval analysis in other branches of mathematics; see References [38–53].

On the other hand, a generalized convexity mapping has the potential to solve a wide range of issues in both a nonlinear and pure analysis. Recently, well-known inequalities such as Jensen, Simpson, Opial, Ostrowski, Bullen, and the famous Hermite–Hadamard that are extended in the setting of interval-valued functions ($\mathcal{IVM}$) have been constructed using a variety of related classes of convexity. Chalco-Cano [54] established interval-based inequalities for the Ostrowski type using a derivative of the Hukuhara type. Opial-type inequalities for $\mathcal{IVM}$s were developed by Costa in [55]. The Minkowski inequalities for $\mathcal{IVM}$s were one of the inequalities suggested by Beckenbach and Roman-Flores [56]. According to the literature assessment, the majority of authors used an inclusion connection, similarly to in 2018, to evaluate inequality. These inequalities were created by Zhao et al. [57] for the harmonic h-convex $\mathcal{IVM}$s and the h-convex $\mathcal{IVM}$s. The authors who came after used both harmonical (h_1, h_2)-convex functions and (h_1, h_2)-convex functions to create these inequalities; for more information, see Refs. [58–75].

Using the radius and interval midpoint, Bhunia and his co-author defined the center-radius order in 2014; see Ref. [76]. The following findings for the cr-h-convex, harmonically cr-h-convex, and cr-h-GL functions were developed in 2022 by Wei Liu and his co-authors; see References [77–88]. Our examination of the literature showed that inclusion and fuzzy inclusion relations are the main sources of the majority of these discrepancies. The fundamental benefit of the up and down fuzzy relation for up and down functions is that the inequality term generated by employing these conceptions is more exact, and the argument's validity can be supported by intriguing examples of illustrated theorems. For further study related to interval-valued functions and fuzzy mappings, see [89–111].

This study provides an introduced class of convexity based on the fuzzy inclusion order and is known as $\mathcal{UD}$-convex $\mathcal{FNVM}$s, and is inspired by Refs. [56,57]. We create new H.H. inequalities with the aid of these innovative ideas, and eventually, the Jensen inequality is developed. The study includes a variety of examples to help bolster the results reached.

The article is formatted as follows, in order: Section 2 gives some background information. Section 3 each provide an overview of the primary conclusions. A succinct conclusion is explored in Section 4.

2. Preliminaries

We recall a few definitions, which can be found in the literature and that will be relevant in the follow-up.

Let us consider that $\mathbb{X}_o$ is the space of all closed and bounded intervals of $\mathbb{R}$, and that $\mathcal{S} \in \mathbb{X}_o$ is given by

$$\mathcal{S} = [\mathcal{S}_*,\ \mathcal{S}^*] = \{\mathfrak{w} \in \mathbb{R} |\ \mathcal{S}_* \leq \mathfrak{w} \leq \mathcal{S}^*,\ \mathcal{S}_*,\ \mathcal{S}^* \in \mathbb{R}\}, \tag{1}$$

If $\mathcal{S}_* = \mathcal{S}^*$, then $\mathcal{S}$ is degenerate. In the follow-up, all intervals are considered non-degenerate. If $\mathcal{S}_* \geq 0$, then $\mathcal{S}$ is positive. We denote by $\mathbb{X}_o^+ = \{[\mathcal{S}_*,\ \mathcal{S}^*] : [\mathcal{S}_*,\ \mathcal{S}^*] \in \mathbb{X}_o$ and $\mathcal{S}_* \geq 0\}$ the set of all positive intervals.

Let $\varpi \in \mathbb{R}$ and $\varpi \cdot \mathcal{S}$ be given by

$$\varpi \cdot \mathcal{S} = \begin{cases} [\varpi\mathcal{S}_*,\ \varpi\mathcal{S}^*] \text{ if } \varpi > 0, \\ \{0\} \quad \text{ iff } \varpi = 0, \\ [\varpi\mathcal{S}^*, \varpi\mathcal{S}_*] \text{ if } \varpi < 0. \end{cases} \tag{2}$$

We consider the Minkowski sum, $\mathcal{S} + \mathcal{O}$, product, $\mathcal{S} \times \mathcal{O}$, and difference, $\mathcal{O} - \mathcal{S}$, for $\mathcal{S}, \mathcal{O} \in \mathbb{X}_o$, as

$$[\mathcal{O}_*,\ \mathcal{O}^*] + [\mathcal{S}_*,\ \mathcal{S}^*] = [\mathcal{O}_* + \mathcal{S}_*,\ \ \mathcal{O}^* + \mathcal{S}^*], \tag{3}$$

$$[\mathcal{O}_*,\ \mathcal{O}^*] \times [\mathcal{S}_*,\ \mathcal{S}^*] = [\min\{\mathcal{O}_*\mathcal{S}_*,\ \mathcal{O}^*\mathcal{S}_*,\ \mathcal{O}_*\mathcal{S}^*,\ \mathcal{O}^*\mathcal{S}^*\},\ \max\{\mathcal{O}_*\mathcal{S}_*,\ \mathcal{O}^*\mathcal{S}_*,\ \mathcal{O}_*\mathcal{S}^*,\ \mathcal{O}^*\mathcal{S}^*\}] \tag{4}$$

$$[\mathcal{O}_*,\ \mathcal{O}^*] - [\mathcal{S}_*,\ \mathcal{S}^*] = [\mathcal{O}_* - \mathcal{S}^*,\ \ \mathcal{O}^* - \mathcal{S}_*]. \tag{5}$$

Remark 1.
(i) *For given* $[\mathcal{O}_*,\ \mathcal{O}^*],\ [\mathcal{S}_*,\ \mathcal{S}^*] \in \mathbb{R}_I$, *the relation "* $\supseteq_I$ *", defined on* $\mathbb{R}_I$ *by*

$$[\mathcal{S}_*,\ \mathcal{S}^*] \supseteq_I [\mathcal{O}_*,\ \mathcal{O}^*] \text{ if and only if } \mathcal{S}_* \leq \mathcal{O}_*,\ \ \mathcal{O}^* \leq \mathcal{S}^*, \tag{6}$$

for all $[\mathcal{O}_*,\ \mathcal{O}^*],\ [\mathcal{S}_*,\ \mathcal{S}^*] \in \mathbb{R}_I$, *is a partial interval inclusion relation. Moreover,* $[\mathcal{S}_*,\ \mathcal{S}^*] \supseteq_I [\mathcal{O}_*,\ \mathcal{O}^*]$ *coincides with* $[\mathcal{S}_*,\ \mathcal{S}^*] \supseteq [\mathcal{O}_*,\ \mathcal{O}^*]$ *on* $\mathbb{R}_I$. *The relation "* $\supseteq_I$ *" is of UD order [105].*

(ii) *For given* $[\mathcal{O}_*,\ \mathcal{O}^*],\ [\mathcal{S}_*,\ \mathcal{S}^*] \in \mathbb{R}_I$, *the relation "* $\leq_I$ *", defined on* $\mathbb{R}_I$ *by* $[\mathcal{O}_*,\ \mathcal{O}^*] \leq_I [\mathcal{S}_*,\ \mathcal{S}^*]$ *if and only if* $\mathcal{O}_* \leq \mathcal{S}_*,\ \mathcal{O}^* \leq \mathcal{S}^*$ *or* $\mathcal{O}_* \leq \mathcal{S}_*,\ \mathcal{O}^* < \mathcal{S}^*$, *is a partial interval order relation. Plus, we have* $[\mathcal{O}_*,\ \mathcal{O}^*] \leq_I [\mathcal{S}_*,\ \mathcal{S}^*]$ *that coincides with* $[\mathcal{O}_*,\ \mathcal{O}^*] \leq [\mathcal{S}_*,\ \mathcal{S}^*]$ *on* $\mathbb{R}_I$. *The relation "* $\leq_I$ *" is of the left and right (LR) type [104,105].*

Given the intervals $[\mathcal{O}_*,\ \mathcal{O}^*],\ [\mathcal{S}_*,\ \mathcal{S}^*] \in \mathbb{X}_o$, *their Hausdorff–Pompeiu distance is*

$$d_H([\mathcal{O}_*,\ \mathcal{O}^*],\ [\mathcal{S}_*,\ \mathcal{S}^*]) = max\{|\mathcal{O}_* - \mathcal{S}_*|,\ |\mathcal{O}^* - \mathcal{S}^*|\}. \tag{7}$$

We have $(\mathbb{X}_o, d_H)$ *that is a complete metric space [94,102,103].*

Definition 1 ([93,94]). *A fuzzy subset L of* $\mathbb{R}$ *is a mapping* $\widetilde{\mathcal{S}} : \mathbb{R} \to [0, 1]$, *a denoted membership mapping of L. We adopt the symbol to represent the set of all fuzzy subsets of* $\mathbb{R}$.

Let us consider $\widetilde{\mathcal{S}} \in$. If the following properties hold, then $\widetilde{\mathcal{S}}$ is a fuzzy number:

(1) $\widetilde{\mathcal{S}}$ is normal if there exists $\mathfrak{w} \in \mathbb{R}$ and $\widetilde{\mathcal{S}}(\mathfrak{w}) = 1$;
(2) $\widetilde{\mathcal{S}}$ is upper semi-continuous on $\mathbb{R}$ if for a $\mathfrak{w} \in \mathbb{R}$ there exists $\varepsilon > 0$ and $\delta > 0$ yielding $\widetilde{\mathcal{S}}(\mathfrak{w}) - \widetilde{\mathcal{S}}(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|\mathfrak{w} - y| < \delta$;
(3) $\widetilde{\mathcal{S}}$ is a fuzzy convex, meaning that $\widetilde{\mathcal{S}}((1-\varpi)\mathfrak{w} + \varpi y) \geq \min\left(\widetilde{\mathcal{S}}(\mathfrak{w}),\ \widetilde{\mathcal{S}}(y)\right)$, for all $\mathfrak{w}, y \in \mathbb{R}$, and $\varpi \in [0,\ 1]$;
(4) $\widetilde{\mathcal{S}}$ is compactly supported, which means that $\mathrm{cl}\left\{\mathfrak{w} \in \mathbb{R} \middle| \ \widetilde{\mathcal{S}}(\mathfrak{w})\right\rangle 0\}$ is compact.

The symbol $_o$ will be adopted to designate the set of all fuzzy numbers of $\mathbb{R}$.

Definition 2. ([93,94]). *For* $\widetilde{\mathcal{S}} \in {}_o$, *the* $\mathfrak{d}$*-level, or* $\mathfrak{d}$*-cut, sets of* $\widetilde{\mathcal{S}}$ *are* $\left[\widetilde{\mathcal{S}}\right]^{\mathfrak{d}} = \left\{\mathfrak{w} \in \mathbb{R} \middle| \ \widetilde{\mathcal{S}}(\mathfrak{w})\right\rangle \mathfrak{d}\}$ *for all* $\mathfrak{d} \in [0,\ 1]$, *and* $\left[\widetilde{\mathcal{S}}\right]^{0} = \left\{\mathfrak{w} \in \mathbb{R} \middle| \ \widetilde{\mathcal{S}}(\mathfrak{w})\right\rangle 0\}$.

Proposition 1. ([96]). *Let* $\widetilde{\mathcal{S}}, \widetilde{\mathcal{O}} \in {}_o$. *The relation "* $\leq_{\mathbb{F}}$ *", defined on* $_o$ *by*

$$\widetilde{\mathcal{S}} \leq_{\mathbb{F}} \widetilde{\mathcal{O}} \text{ when and only when } \left[\widetilde{\mathcal{S}}\right]^{\mathfrak{d}} \leq_I \left[\widetilde{\mathcal{O}}\right]^{\mathfrak{d}}, \text{ for every } \mathfrak{d} \in [0,\ 1], \tag{8}$$

is a LR order relation.

Proposition 2. ([79]). *Let* $\widetilde{\mathcal{S}}, \widetilde{\mathcal{O}} \in {}_o$. *The relation "* $\supseteq_{\mathbb{F}}$ *", defined on* $_o$ *by*

$$\widetilde{\mathcal{S}} \supseteq_{\mathbb{F}} \widetilde{\mathcal{O}} \text{ when and only when } \left[\widetilde{\mathcal{S}}\right]^{\mathfrak{d}} \supseteq_I \left[\widetilde{\mathcal{O}}\right]^{\mathfrak{d}}, \text{ for every } \mathfrak{d} \in [0,\ 1], \tag{9}$$

is an UD order relation.

If $\widetilde{\mathcal{S}}, \widetilde{\mathcal{O}} \in {}_{o}$ and $ə \in \mathbb{R}$, then, for every $ə \in [0,\ 1]$,

$$\left[\widetilde{\mathcal{S}} \oplus \widetilde{\mathcal{O}}\right]^{ə} = \left[\widetilde{\mathcal{S}}\right]^{ə} + \left[\widetilde{\mathcal{O}}\right]^{ə}, \tag{10}$$

$$\left[\widetilde{\mathcal{S}} \otimes \widetilde{\mathcal{O}}\right]^{ə} = \left[\widetilde{\mathcal{S}}\right]^{ə} \times \left[\widetilde{\mathcal{O}}\right]^{ə}, \tag{11}$$

$$\left[\varpi \odot \widetilde{\mathcal{S}}\right]^{ə} = ə.\left[\widetilde{\mathcal{S}}\right]^{ə} \tag{12}$$

result from Equations (4)–(6), respectively.

Theorem 1 ([94]). *For $\widetilde{\mathcal{S}},\ \widetilde{\mathcal{O}} \in {}_{o}$, the supremum metric*

$$d_{\infty}\left(\widetilde{\mathcal{S}},\ \widetilde{\mathcal{O}}\right) = \sup_{0 \le ə \le 1} d_H\left(\left[\widetilde{\mathcal{S}}\right]^{ə},\ \left[\widetilde{\mathcal{O}}\right]^{ə}\right) \tag{13}$$

is a complete metric space, where H stands for the Hausdorff metric on a space of intervals.

Theorem 2 ([94,95]). *If $\mathcal{H} : [\mathfrak{b}, z] \subset \mathbb{R} \to \mathbb{X}_o$ is an $\mathcal{IVM}$ satisfying $\mathcal{H}(\mathfrak{w}) = [\mathcal{H}_*(\mathfrak{w}),\ \mathcal{H}^*(\mathfrak{w})]$, then $\mathcal{H}$ is Aumann integrable (IA-integrable) over $[\mathfrak{b}, z]$ when and only when $\mathcal{H}_*(\mathfrak{w})$ and $\mathcal{H}^*(\mathfrak{w})$ are integrable over $[\mathfrak{b}, z]$, meaning*

$$(IA)\int_{\mathfrak{b}}^{z} \mathcal{H}(\mathfrak{w})d\mathfrak{w} = \left[\int_{\mathfrak{b}}^{z} \mathcal{H}_*(\mathfrak{w})d\mathfrak{w},\ \int_{\mathfrak{b}}^{z} \mathcal{H}^*(\mathfrak{w})d\mathfrak{w}\right] \tag{14}$$

Definition 3 ([104]). *Let $\widetilde{\mathcal{H}} : \mathbb{I} \subset \mathbb{R} \to {}_{o}$ be a $\mathcal{FNVM}$. The family of $\mathcal{IVM}$s, for every $ə \in [0,\ 1]$, is $\mathcal{H}_{ə} : \mathbb{I} \subset \mathbb{R} \to \mathbb{X}_o$ satisfying $\mathcal{H}_{ə}(\mathfrak{w}) = [\mathcal{H}_*(\mathfrak{w}, ə),\ \mathcal{H}^*(\mathfrak{w}, ə)]$ for every $\mathfrak{w} \in \mathbb{I}$. For every $ə \in [0,\ 1]$, the lower and upper mappings of $\mathcal{H}_{ə}$ are the endpoint real-valued mappings $\mathcal{H}_*(\cdot, ə),\ \mathcal{H}^*(\cdot, ə) : \mathbb{I} \to \mathbb{R}$.*

Definition 4 ([104]). *Let $\widetilde{\mathcal{H}} : \mathbb{I} \subset \mathbb{R} \to {}_{o}$ be a $\mathcal{FNVM}$. Then, $\widetilde{\mathcal{H}}(\mathfrak{w})$ is continuous at $\mathfrak{w} \in \mathbb{I}$, if for every $ə \in [0,\ 1]$, $\mathcal{H}_{ə}(\mathfrak{w})$ is continuous when and only when $\mathcal{H}_*(\mathfrak{w}, ə)$ and $\mathcal{H}^*(\mathfrak{w}, ə)$ are continuous at $\mathfrak{w} \in \mathbb{I}$.*

Definition 5 ([95]). *Let $\widetilde{\mathcal{H}} : [\mathfrak{b},\ z] \subset \mathbb{R} \to {}_{o}$ be a $\mathcal{FNVM}$. The fuzzy Aumann integral (FA-integral) of $\widetilde{\mathcal{H}}$ over $[\mathfrak{b},\ z]$ is*

$$\left[(FA)\int_{\mathfrak{b}}^{z} \widetilde{\mathcal{H}}(\mathfrak{w})d\mathfrak{w}\right]^{ə} = (IA)\int_{\mathfrak{b}}^{z} \mathcal{H}_{ə}(\mathfrak{w})d\mathfrak{w} = \left\{\int_{\mathfrak{b}}^{z} \mathcal{H}(\mathfrak{w}, ə)d\mathfrak{w} : \mathcal{H}(\mathfrak{w}, ə) \in S(\mathcal{H}_{ə})\right\}, \tag{15}$$

where $S(\mathcal{H}_{ə}) = \{\mathcal{H}(., ə) \to \mathbb{R} : \mathcal{H}(., ə)$ is integrable, and $\mathcal{H}(\mathfrak{w}, ə) \in \mathcal{H}_{ə}(\mathfrak{w})\}$, for every $ə \in [0,\ 1]$. Moreover, $\widetilde{\mathcal{H}}$ is (FA)-integrable over $[\mathfrak{b},\ z]$ if $(FA)\int_{\mathfrak{b}}^{z} \widetilde{\mathcal{H}}(\mathfrak{w})d\mathfrak{w} \in {}_{o}$.

Theorem 3 [96]. *Let $\widetilde{\mathcal{H}} : [\mathfrak{b},\ z] \subset \mathbb{R} \to {}_{o}$ be a $\mathcal{FNVM}$, whose $ə$-levels define the family of $\mathcal{IVM}$s $\mathcal{H}_{ə} : [\mathfrak{b},\ z] \subset \mathbb{R} \to \mathbb{X}_o$ satisfying $\mathcal{H}_{ə}(\mathfrak{w}) = [\mathcal{H}_*(\mathfrak{w}, ə),\ \mathcal{H}^*(\mathfrak{w}, ə)]$ for every $\mathfrak{w} \in [\mathfrak{b},\ z]$ and $ə \in [0,\ 1]$. $\widetilde{\mathcal{H}}$ is (FA)-integrable over $[\mathfrak{b},\ z]$ when and only when $\mathcal{H}_*(\mathfrak{w}, ə)$ and $\mathcal{H}^*(\mathfrak{w}, ə)$ are integrable over $[\mathfrak{b},\ z]$. Moreover, if $\widetilde{\mathcal{H}}$ is (FA)-integrable over $[\mathfrak{b},\ z]$, then we have*

$$\left[(FA)\int_{\mathfrak{b}}^{z} \widetilde{\mathcal{H}}(\mathfrak{w})d\mathfrak{w}\right]^{ə} = \left[\int_{\mathfrak{b}}^{z} \mathcal{H}_*(\mathfrak{w}, ə)d\mathfrak{w},\ \int_{\mathfrak{b}}^{z} \mathcal{H}^*(\mathfrak{w}, ə)d\mathfrak{w}\right] = (IA)\int_{\mathfrak{b}}^{z} \mathcal{H}_{ə}(\mathfrak{w})d\mathfrak{w} \tag{16}$$

for every $ə \in [0,\ 1]$.

Breckner discussed the coming emerging idea of interval-valued convexity in [97].

An I·V·M $\mathcal{H} : \mathbb{I} = [\mathfrak{b},\ z] \to \mathcal{X}_o$ is called convex I·V·M if

$$\mathcal{H}(\varpi\mathfrak{w} + (1-\varpi)s) \supseteq \varpi\mathcal{H}(\mathfrak{w}) + (1-\varpi)\mathcal{H}(s), \tag{17}$$

for all $\mathfrak{w},\ y \in [\mathfrak{b},\ z]$, $\varpi \in [0,\ 1]$, where $\mathcal{X}_o$ is the collection of real-valued intervals. If (17) is reversed, then $\mathcal{H}$ is called concave.

Definition 6 ([89])**.** *The $\mathcal{FNVM}$ $\widetilde{\mathcal{H}} : [\mathfrak{b},\ z] \to {}_o$ is called convex $\mathcal{FNVM}$ on $[\mathfrak{b},\ z]$ if*

$$\widetilde{\mathcal{H}}(\varpi\mathfrak{w} + (1-\varpi)s\ ) \leq_{\mathbb{F}} \varpi \odot \widetilde{\mathcal{H}}(\mathfrak{w}) \oplus (1-\varpi) \odot \widetilde{\mathcal{H}}(s), \tag{18}$$

for all $\mathfrak{w},\ s \in [\mathfrak{b},\ z]$, $\varpi \in [0,\ 1]$, where $\widetilde{\mathcal{H}}(\mathfrak{w}) \geq_{\mathbb{F}} \widetilde{0}$ for all $\mathfrak{w} \in [\mathfrak{b},\ z]$. If (18) is reversed, then $\widetilde{\mathcal{H}}$ is called concave $\mathcal{FNVM}$ on $[\mathfrak{b},\ z]$. $\widetilde{\mathcal{H}}$ is affine if and only if it is both convex and concave $\mathcal{FNVM}$.

Definition 7 ([105])**.** *The $\mathcal{FNVM}$ $\widetilde{\mathcal{H}} : [\mathfrak{b},\ z] \to {}_o$ is called $\mathcal{UD}$-convex $\mathcal{FNVM}$ on $[\mathfrak{b},\ z]$ if*

$$\widetilde{\mathcal{H}}(\varpi\mathfrak{w} + (1-\varpi)s) \supseteq_{\mathbb{F}} \varpi \odot \widetilde{\mathcal{H}}(\mathfrak{w}) \oplus (1-\varpi) \odot \widetilde{\mathcal{H}}(s), \tag{19}$$

for all $\mathfrak{w},\ s \in [\mathfrak{b},\ z]$, $\varpi \in [0,\ 1]$, where $\widetilde{\mathcal{H}}(\mathfrak{w}) \geq_{\mathbb{F}} \widetilde{0}$ for all $\mathfrak{w} \in [\mathfrak{b},\ z]$. If (19) is reversed then, $\widetilde{\mathcal{H}}$ is called $\mathcal{UD}$-concave $\mathcal{FNVM}$ on $[\mathfrak{b},\ z]$. $\widetilde{\mathcal{H}}$ is $\mathcal{UD}$-affine $\mathcal{FNVM}$ if and only if it is both $\mathcal{UD}$-convex and $\mathcal{UD}$-concave $\mathcal{FNVM}$.

Theorem 4 ([105])**.** *Let $\widetilde{\mathcal{H}} : [\mathfrak{b}, z] \to {}_o$ be a $\mathcal{FNVM}$, whose $\mathfrak{s}$-cuts define the family of inteval-valued mappings $\mathcal{H}_{\mathfrak{s}} : [\mathfrak{b}, z] \to \mathcal{X}_o^+ \subset \mathcal{X}_o$ are given by*

$$\mathcal{H}_{\mathfrak{s}}(\mathfrak{w}) = [\mathcal{H}_*(\mathfrak{w}, \mathfrak{s}),\ \mathcal{H}^*(\mathfrak{w}, \mathfrak{s})], \tag{20}$$

for all $\mathfrak{w} \in [\mathfrak{b}, z]$ and for all $\mathfrak{s} \in [0,\ 1]$. Then, $\widetilde{\mathcal{H}}$ is $\mathcal{UD}$-convex $\mathcal{FNVM}$ on $[\mathfrak{b}, z]$, if and only if, for all $\mathfrak{s} \in [0,\ 1]$, $\mathcal{H}_*(\mathfrak{w},\ \mathfrak{s})$ is a convex mapping and $\mathcal{H}^*(\mathfrak{w},\ \mathfrak{s})$ is a concave mapping.

Remark 2. *If $\mathcal{H}_*(\mathfrak{w}, \mathfrak{s}) \neq \mathcal{H}^*(\mathfrak{w}, \mathfrak{s})$ and $\mathfrak{s} = 1$, then we obtain the inequality (17).*

If $\mathcal{H}_*(\mathfrak{w}, \mathfrak{s}) = \mathcal{H}^*(\mathfrak{w}, \mathfrak{s})$ and $\mathfrak{s} = 1$, then we obtain the classical definition of convex mappings.

Now we have obtained some new definitions from the literature which will be helpful to investigate some classical and new results as special cases of main results.

Definition 8. ([79])**.** *Let $\widetilde{\mathcal{H}} : [\mathfrak{b}, z] \to {}_o$ be a $\mathcal{FNVM}$, whose $\mathfrak{s}$-cuts define the family of $\mathcal{IVM}$s $\mathcal{H}_{\mathfrak{s}} : [\mathfrak{b}, z] \to \mathcal{X}_o^+ \subset \mathcal{X}_o$ are given by*

$$\mathcal{H}_{\mathfrak{s}}(\mathfrak{w}) = [\mathcal{H}_*(\mathfrak{w}, \mathfrak{s}),\ \mathcal{H}^*(\mathfrak{w}, \mathfrak{s})], \tag{21}$$

for all $\mathfrak{w} \in [\mathfrak{b}, z]$ and for all $\mathfrak{s} \in [0,\ 1]$. Then, $\widetilde{\mathcal{H}}$ is lower $\mathcal{UD}$-convex (concave) $\mathcal{FNVM}$ on $[\mathfrak{b}, z]$, if and only if, for all $\mathfrak{s} \in [0,\ 1]$, $\mathcal{H}_*(\mathfrak{w},\ \mathfrak{s})$ is a convex (concave) mapping and $\mathcal{H}^*(\mathfrak{w},\ \mathfrak{s})$ is an affine mapping.

Definition 9. ([79])**.** *Let $\widetilde{\mathcal{H}} : [\mathfrak{b}, z] \to {}_o$ be a $\mathcal{FNVM}$, whose $\mathfrak{s}$-cuts define the family of $\mathcal{IVM}$s $\mathcal{H}_{\mathfrak{s}} : [\mathfrak{b}, z] \to \mathcal{X}_o^+ \subset \mathcal{X}_o$ are given by*

$$\mathcal{H}_{\mathfrak{s}}(\mathfrak{w}) = [\mathcal{H}_*(\mathfrak{w}, \mathfrak{s}),\ \mathcal{H}^*(\mathfrak{w}, \mathfrak{s})], \tag{22}$$

for all $\mathfrak{w} \in [\mathfrak{b}, z]$ and for all $\mathfrak{s} \in [0,\ 1]$. Then, $\mathcal{H}$ is upper $\mathcal{UD}$-convex (concave) $\mathcal{FNVM}$ on $[\mathfrak{b}, z]$, if and only if, for all $\mathfrak{s} \in [0,\ 1]$, $\mathcal{H}_*(\mathfrak{w},\ \mathfrak{s})$ is an affine mapping and $\mathcal{H}^*(\mathfrak{w},\ \mathfrak{s})$ is a convex (concave) mapping.

Remark 3. *Both concepts "$\mathcal{UD}$-convex $\mathcal{FNVM}$" and classical "convex $\mathcal{FNVM}$, see [41]" behave alike when $\widetilde{\mathcal{H}}$ is lower $\mathcal{UD}$-convex $\mathcal{FNVM}$.*

3. Fuzzy Number Hermite–Hadamard Inequalities

In this section, we propose Hermite–Hadamard and Hermite–Hadamard–Fejér inequalities for $\mathcal{UD}$-convex $\mathcal{FNVM}$s, and verify with the help of nontrivial examples.

Theorem 5. *Let $\widetilde{\mathcal{H}} : [\mathfrak{b}, z] \to _0$ be a $\mathcal{UD}$-convex $\mathcal{FNVM}$ on $[\mathfrak{b}, z]$, whose $\mathfrak{s}$-cuts define the family of $\mathcal{IVM}$s $\mathcal{H}_{\mathfrak{s}} : [\mathfrak{b}, z] \subset \mathbb{R} \to \mathcal{X}_o^+$ are given by $\mathcal{H}_{\mathfrak{s}}(\mathfrak{w}) = [\mathcal{H}_*(\mathfrak{w}, \mathfrak{s}),\ \mathcal{H}^*(\mathfrak{w}, \mathfrak{s})]$ for all $\mathfrak{w} \in [\mathfrak{b}, z]$ and for all $\mathfrak{s} \in [0, 1]$. If $\widetilde{\mathcal{H}} \in \mathcal{FA}_{([\mathfrak{b}, z],\ \mathfrak{s})}$, then*

$$\widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+z}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{z-\mathfrak{b}} \odot (FA) \int_{\mathfrak{b}}^{z} \widetilde{\mathcal{H}}(\mathfrak{w})d\mathfrak{w} \supseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{H}}(\mathfrak{b}) \ \oplus\ \widetilde{\mathcal{H}}(z)}{2}. \tag{23}$$

If $\widetilde{\mathcal{H}}(\mathfrak{w})$ concave $\mathcal{FNVM}$, then (23) is reversed.

Proof. Let $\widetilde{\mathcal{H}} : [\mathfrak{b}, z] \to _0$ be a $\mathcal{UD}$-convex $\mathcal{FNVM}$. Then, by hypothesis, we have

$$2\widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+z}{2}\right) \supseteq_{\mathbb{F}} \widetilde{\mathcal{H}}(\varpi\mathfrak{b} + (1-\varpi)z) \oplus \widetilde{\mathcal{H}}((1-\varpi)\mathfrak{b} + \varpi z).$$

Therefore, for every $\mathfrak{s} \in [0, 1]$, we have

$$\begin{aligned} 2\mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right) &\leq \mathcal{H}_*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) + \mathcal{H}_*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}), \\ 2\mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right) &\geq \mathcal{H}^*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) + \mathcal{H}^*((1-\varpi)\mathfrak{b} + \varpi z, \mathfrak{s}). \end{aligned}$$

Then

$$\begin{aligned} 2\int_0^1 \mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right)d\varpi &\leq \int_0^1 \mathcal{H}_*(\varpi\mathfrak{b} + (1-\varpi)z, \mathfrak{s})d\varpi + \int_0^1 \mathcal{H}_*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s})d\varpi, \\ 2\int_0^1 \mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2}, \mathfrak{s}\right)d\varpi &\geq \int_0^1 \mathcal{H}^*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s})d\varpi + \int_0^1 \mathcal{H}^*((1-\varpi)\mathfrak{b} + \varpi z, \mathfrak{s})d\varpi. \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right) &\leq \frac{1}{z-\mathfrak{b}} \ \int_{\mathfrak{b}}^{z} \mathcal{H}_*(\mathfrak{w},\ \mathfrak{s})d\mathfrak{w}, \\ \mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right) &\geq \frac{1}{z-\mathfrak{b}} \ \int_{\mathfrak{b}}^{z} \mathcal{H}^*(\mathfrak{w},\ \mathfrak{s})d\mathfrak{w}. \end{aligned}$$

That is

$$\left[\mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right),\ \mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right)\right] \supseteq_I \frac{1}{z-\mathfrak{b}}\left[\int_{\mathfrak{b}}^{z} \mathcal{H}_*(\mathfrak{w},\ \mathfrak{s})d\mathfrak{w},\ \int_{\mathfrak{b}}^{z} \mathcal{H}^*(\mathfrak{w},\ \mathfrak{s})d\mathfrak{w}\right].$$

Thus,

$$\widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+z}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{z-\mathfrak{b}} \ \odot (FA) \int_{\mathfrak{b}}^{z} \widetilde{\mathcal{H}}(\mathfrak{w})d\mathfrak{w}. \tag{24}$$

In a similar way as above, we have

$$\frac{1}{z-\mathfrak{b}} \odot (FA) \int_{\mathfrak{b}}^{z} \widetilde{\mathcal{H}}(\mathfrak{w})d\mathfrak{w} \supseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{H}}(\mathfrak{b}) \ \oplus\ \widetilde{\mathcal{H}}(z)}{2}. \tag{25}$$

Combining (24) and (25), we have

$$\widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+z}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{z-\mathfrak{b}} \odot (FA) \int_{\mathfrak{b}}^{z} \widetilde{\mathcal{H}}(\mathfrak{w})d\mathfrak{w} \supseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{H}}(\mathfrak{b}) \ \oplus\ \widetilde{\mathcal{H}}(z)}{2}.$$

Hence, the required result. □

Remark 4. *The following are some exceptional cases which can be obtained from inequality (23):*

If one lays $\mathcal{H}$ is lower $\mathcal{UD}$-convex $\mathcal{FNVM}$ on $[\mathfrak{b}, z]$, then one acquires the following coming inequality, see [90]:

$$\mathcal{H}\left(\frac{\mathfrak{b}+z}{2}\right) \leq_{\mathbb{F}} \frac{1}{z-\mathfrak{b}} \odot (FA) \int_{\mathfrak{b}}^{z} \mathcal{H}(\mathfrak{w}) d\mathfrak{w} \leq_{\mathbb{F}} \frac{\mathcal{H}(\mathfrak{b}) \oplus \mathcal{H}(z)}{2} \tag{26}$$

If one takes $\mathcal{H}$ is lower $\mathcal{UD}$-convex $\mathcal{FNVM}$ on $[\mathfrak{b}, z]$ and $\partial =$, then one achieves the following coming inequality, see [98]:

$$\mathcal{H}\left(\frac{\mathfrak{b}+z}{2}\right) \leq_{I} \frac{1}{z-\mathfrak{b}} (IA) \int_{\mathfrak{b}}^{z} \mathcal{H}(\mathfrak{w}) d\mathfrak{w} \leq_{I} \frac{\mathcal{H}(\mathfrak{b}) + \mathcal{H}(z)}{2} \tag{27}$$

Let $\partial = 1$. Then, from Theorem 5, we acquire the following inequality, see [99]:

$$\mathcal{H}\left(\frac{\mathfrak{b}+z}{2}\right) \supseteq \frac{1}{z-\mathfrak{b}} (IA) \int_{\mathfrak{b}}^{z} \mathcal{H}(\mathfrak{w}) d\mathfrak{w} \supseteq \frac{\mathcal{H}(\mathfrak{b}) + \mathcal{H}(z)}{2}. \tag{28}$$

Let $\partial =$ and $\mathcal{H}_*(\mathfrak{w}, \partial) = \mathcal{H}^*(\mathfrak{w}, \partial)$. Then, from Theorem 5, we achieve the classical Hermite–Hadamard inequality.

Example 1. *Let* $\mathfrak{w} \in [2, 3]$, and the $\mathcal{FNVM}$ $\widetilde{\mathcal{H}} : [\mathfrak{b}, z] = [2, 3] \to {}_{o}$, defined by

$$\widetilde{\mathcal{H}}(\mathfrak{w})(\theta) = \begin{cases} \dfrac{\theta - 2 + \mathfrak{w}^{\frac{1}{2}}}{1 - \mathfrak{w}^{\frac{1}{2}}} & \theta \in \left[2 - \mathfrak{w}^{\frac{1}{2}}, 3\right], \\ \dfrac{2 + \mathfrak{w}^{\frac{1}{2}} - \theta}{\mathfrak{w}^{\frac{1}{2}} - 1} & \theta \in \left(3, 2 + \mathfrak{w}^{\frac{1}{2}}\right], \\ 0 & \text{otherwise}, \end{cases} \tag{29}$$

Then, for each $\partial \in [0, 1]$, *we have* $\mathcal{H}_{\partial}(\mathfrak{w}) = \left[(1-\partial)\left(2 - \mathfrak{w}^{\frac{1}{2}}\right) + 3\partial, (1-\partial)\left(2 + \mathfrak{w}^{\frac{1}{2}}\right) + 3\partial\right]$. *Since left and right end point mappings* $\mathcal{H}_*(\mathfrak{w}, \partial) = (1-\partial)\left(2 - \mathfrak{w}^{\frac{1}{2}}\right) + 3\partial$, *and* $\mathcal{H}^*(\mathfrak{w}, \partial) = (1-\partial)\left(2 + \mathfrak{w}^{\frac{1}{2}}\right) + 3\partial$, *are convex and concave mappings, respectively, for each* $\partial \in [0, 1]$, *then* $\widetilde{\mathcal{H}}(\mathfrak{w})$ *is* $\mathcal{UD}$*-convex* $\mathcal{FNVM}$. *We clearly see that* $\widetilde{\mathcal{H}} \in L([\mathfrak{b}, z], {}_{o})$ *and*

$$\mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2}, \partial\right) = \mathcal{H}_*\left(\frac{5}{2}, \partial\right) = (1-\partial)\frac{4-\sqrt{10}}{2} + 3\partial,$$

$$\mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2}, \partial\right) = \mathcal{H}^*\left(\frac{5}{2}, \partial\right) = (1-\partial)\frac{4+\sqrt{10}}{2} + 3\partial.$$

Note that

$$\frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}_*(\mathfrak{w}, \partial) d\mathfrak{w} = \int_{2}^{3} \left((1-\partial)\left(2 - \mathfrak{w}^{\frac{1}{2}}\right) + 3\partial\right) d\mathfrak{w} \approx 0.4215(1-\partial) + 3\partial,$$

$$\frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}^*(\mathfrak{w}, \partial) d\mathfrak{w} = \int_{2}^{3} \left((1+\partial)\left(2 + \mathfrak{w}^{\frac{1}{2}}\right) + 3\partial\right) d\mathfrak{w} \approx 3.58(1-\partial) + 3\partial,$$

and

$$\frac{\mathcal{H}_*(\mathfrak{b},\mathfrak{s})+\mathcal{H}_*(z,\ \mathfrak{s})}{2}=(1-\mathfrak{s})\left(\frac{4-\sqrt{2}-\sqrt{3}}{2}\right)+3\mathfrak{s},$$

$$\frac{\mathcal{H}^*(\mathfrak{b},\mathfrak{s})+\mathcal{H}^*(z,\ \mathfrak{s})}{2}=(1-\mathfrak{s})\left(\frac{4+\sqrt{2}+\sqrt{3}}{2}\right)+3\mathfrak{s}.$$

Therefore,

$$\left[(1-\mathfrak{s})\frac{4-\sqrt{10}}{2}+3\mathfrak{s},(1-\mathfrak{s})\frac{4+\sqrt{10}}{2}+3\mathfrak{s}\right]\supseteq_I\left[\frac{843}{2000}(1-\mathfrak{s})+3\mathfrak{s},\frac{179}{50}(1-\mathfrak{s})+3\mathfrak{s}\right]$$
$$\supseteq_I\left[(1-\mathfrak{s})\left(\frac{4-\sqrt{2}-\sqrt{3}}{2}\right)+3\mathfrak{s},(1-\mathfrak{s})\left(\frac{4+\sqrt{2}+\sqrt{3}}{2}\right)+3\mathfrak{s}\right],$$

Hence,

$$\widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+z}{2}\right)\supseteq_{\mathbb{F}}\frac{1}{z-\mathfrak{b}}\odot(FA)\int_{\mathfrak{b}}^{z}\widetilde{\mathcal{H}}(\mathfrak{w})d\mathfrak{w}\supseteq_{\mathbb{F}}\frac{\widetilde{\mathcal{H}}(\mathfrak{b})\oplus\widetilde{\mathcal{H}}(z)}{2},$$

and Theorem 5 is verified.

Theorem 6. *Let $\widetilde{\mathcal{H}}:[\mathfrak{b},\ z]\to{}_{o}$ be a $\mathcal{UD}$-convex $\mathcal{FNVM}$ on $[\mathfrak{b},\ z]$, whose $\mathfrak{s}$-cuts define the family of $\mathcal{IVM}$s $\mathcal{H}_{\mathfrak{s}}:[\mathfrak{b},\ z]\subset\mathbb{R}\to\mathcal{X}_o^+$ are given by $\mathcal{H}_{\mathfrak{s}}(\mathfrak{w})=[\mathcal{H}_*(\mathfrak{w},\mathfrak{s}),\ \mathcal{H}^*(\mathfrak{w},\mathfrak{s})]$ for all $\mathfrak{w}\in[\mathfrak{b},\ z]$ and for all $\mathfrak{s}\in[0,\ 1]$. If $\widetilde{\mathcal{H}}\in\ \mathcal{FA}_{([\mathfrak{b},\ z],\ \mathfrak{s})}$, then*

$$\widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+z}{2}\right)\supseteq_{\mathbb{F}}\ \mathfrak{T}_2\supseteq_{\mathbb{F}}\frac{1}{z-\mathfrak{b}}\odot(FA)\int_{\mathfrak{b}}^{z}\widetilde{\mathcal{H}}(\mathfrak{w})d\mathfrak{w}\supseteq_{\mathbb{F}}\mathfrak{T}_1\supseteq_{\mathbb{F}}\frac{\widetilde{\mathcal{H}}(\mathfrak{b})\ \oplus\ \widetilde{\mathcal{H}}(z)}{2},\tag{30}$$

where

$$\mathfrak{T}_1=\frac{\frac{\widetilde{\mathcal{H}}(\mathfrak{b})\ \oplus\ \widetilde{\mathcal{H}}(z)}{2}\ \oplus\ \widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+z}{2}\right)}{2},\ \mathfrak{T}_2=\frac{\widetilde{\mathcal{H}}\left(\frac{3\mathfrak{b}+z}{4}\right)\ \oplus\ \widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+3z}{4}\right)}{2}$$

and $\mathfrak{T}_1=[\mathfrak{T}_{1*},\ \mathfrak{T}_1{}^*]$, $\mathfrak{T}_2=[\mathfrak{T}_{2*},\ \mathfrak{T}_2{}^*]$.

Proof. Take $\left[\mathfrak{b},\ \frac{\mathfrak{b}+z}{2}\right]$, we have

$$2\widetilde{\mathcal{H}}\left(\frac{\varpi\mathfrak{b}+(1-\varpi)\frac{\mathfrak{b}+z}{2}}{2}+\frac{(1-\varpi)\mathfrak{b}+\varpi\frac{\mathfrak{b}+z}{2}}{2}\right)\supseteq_{\mathbb{F}}\widetilde{\mathcal{H}}\left(\varpi\mathfrak{b}+(1-\varpi)\frac{\mathfrak{b}+z}{2}\right)\oplus\widetilde{\mathcal{H}}\left((1-\varpi)\mathfrak{b}+\varpi\frac{\mathfrak{b}+z}{2}\right).$$

Therefore, for every $\mathfrak{s}\in[0,\ 1]$, we have

$$2\mathcal{H}_*\left(\frac{\varpi\mathfrak{b}+(1-\varpi)\frac{\mathfrak{b}+z}{2}}{2}+\frac{(1-\varpi)\mathfrak{b}+\varpi\frac{\mathfrak{b}+z}{2}}{2},\ \mathfrak{s}\right)\le\mathcal{H}_*\left(\varpi\mathfrak{b}+(1-\varpi)\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right)+\mathcal{H}_*\left((1-\varpi)\mathfrak{b}+\varpi\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right),$$
$$2\mathcal{H}^*\left(\frac{\varpi\mathfrak{b}+(1-\varpi)\frac{\mathfrak{b}+z}{2}}{2}+\frac{(1-\varpi)\mathfrak{b}+\varpi\frac{\mathfrak{b}+z}{2}}{2},\ \mathfrak{s}\right)\ge\mathcal{H}^*\left(\varpi\mathfrak{b}+(1-\varpi)\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right)+\mathcal{H}^*\left((1-\varpi)\mathfrak{b}+\varpi\frac{\mathfrak{b}+z}{2},\mathfrak{s}\right).$$

In consequence, we obtain

$$\frac{\mathcal{H}_*\left(\frac{3\mathfrak{b}+z}{4},\ \mathfrak{s}\right)}{2} \leq \frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{\frac{\mathfrak{b}+z}{2}} \mathcal{H}_*(\mathfrak{w},\ \mathfrak{s})d\mathfrak{w},$$
$$\frac{\mathcal{H}^*\left(\frac{3\mathfrak{b}+z}{4},\ \mathfrak{s}\right)}{2} \geq \frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{\frac{\mathfrak{b}+z}{2}} \mathcal{H}^*(\mathfrak{w},\ \mathfrak{s})d\mathfrak{w}.$$

That is

$$\frac{\left[\mathcal{H}_*\left(\frac{3\mathfrak{b}+z}{4},\ \mathfrak{s}\right),\ \mathcal{H}^*\left(\frac{3\mathfrak{b}+z}{4},\ \mathfrak{s}\right)\right]}{2} \leq \frac{1}{z-\mathfrak{b}}\left[\int_{\mathfrak{b}}^{\frac{\mathfrak{b}+z}{2}} \mathcal{H}_*(\mathfrak{w},\ \mathfrak{s})d\mathfrak{w},\ \int_{\mathfrak{b}}^{\frac{\mathfrak{b}+z}{2}} \mathcal{H}^*(\mathfrak{w},\ \mathfrak{s})d\mathfrak{w}\right].$$

It follows that

$$\frac{\widetilde{\mathcal{H}}\left(\frac{3\mathfrak{b}+z}{4}\right)}{2} \supseteq_{\mathbb{F}} \frac{1}{z-\mathfrak{b}} \odot (FA)\int_{\mathfrak{b}}^{\frac{\mathfrak{b}+z}{2}} \widetilde{\mathcal{H}}(\mathfrak{w})d\mathfrak{w}. \tag{31}$$

In a similar way as above, we have

$$\frac{\widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+3z}{4}\right)}{2} \supseteq_{\mathbb{F}} \frac{1}{z-\mathfrak{b}} \odot (FA)\int_{\frac{\mathfrak{b}+z}{2}}^{z} \widetilde{\mathcal{H}}(\mathfrak{w})d\mathfrak{w}. \tag{32}$$

Combining (31) and (32), we have

$$\frac{\left[\widetilde{\mathcal{H}}\left(\frac{3\mathfrak{b}+z}{4}\right) \oplus \widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+3z}{4}\right)\right]}{2} \supseteq_{\mathbb{F}} \frac{1}{z-\mathfrak{b}} \odot (FA)\int_{\mathfrak{b}}^{z} \widetilde{\mathcal{H}}(\mathfrak{w})d\mathfrak{w}.$$

By using Theorem 5, we have

$$\widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+z}{2}\right) = \widetilde{\mathcal{H}}\left(\frac{1}{2}.\frac{3\mathfrak{b}+z}{4} + \frac{1}{2}.\frac{\mathfrak{b}+3z}{4}\right).$$

Therefore, for every $\mathfrak{s} \in [0,\ 1]$, we have

$$\mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right) = \mathcal{H}_*\left(\frac{1}{2}.\frac{3\mathfrak{b}+z}{4} + \frac{1}{2}.\frac{\mathfrak{b}+3z}{4},\ \mathfrak{s}\right)$$
$$\mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right) = \mathcal{H}^*\left(\frac{1}{2}.\frac{3\mathfrak{b}+z}{4} + \frac{1}{2}.\frac{\mathfrak{b}+3z}{4},\ \mathfrak{s}\right),$$

$$\leq \left[\frac{1}{2}\mathcal{H}_*\left(\frac{3\mathfrak{b}+z}{4},\ \mathfrak{s}\right) + \frac{1}{2}\mathcal{H}_*\left(\frac{\mathfrak{b}+3z}{4},\ \mathfrak{s}\right)\right]$$
$$\geq \left[\frac{1}{2}\mathcal{H}^*\left(\frac{3\mathfrak{b}+z}{4},\ \mathfrak{s}\right) + \frac{1}{2}\mathcal{H}^*\left(\frac{\mathfrak{b}+3z}{4},\ \mathfrak{s}\right)\right],$$

$$\leq \frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}_*(\mathfrak{w},\ \mathfrak{s})d\mathfrak{w}$$
$$\geq \frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}^*(\mathfrak{w},\ \mathfrak{s})d\mathfrak{w},$$

$$= \mathfrak{T}_{2*}$$
$$= \mathfrak{T}_2{}^*,$$

$$\leq \frac{1}{2}\left[\frac{\mathcal{H}_*(\mathfrak{b},\ \mathfrak{s}) + \mathcal{H}_*(z,\ \mathfrak{s})}{2} + \mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right)\right]$$
$$\geq \frac{1}{2}\left[\frac{\mathcal{H}^*(\mathfrak{b},\ \mathfrak{s}) + \mathcal{H}^*(z,\ \mathfrak{s})}{2} + \mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right)\right],$$

$$= \mathfrak{T}_{1*}$$
$$= \mathfrak{T}_1{}^*,$$

$$\leq \frac{1}{2}\left[\frac{\mathcal{H}_*(\mathfrak{b},\ \vartheta)+\mathcal{H}_*(z,\ \vartheta)}{2}+\frac{\mathcal{H}_*(\mathfrak{b},\ \vartheta)+\mathcal{H}_*(z,\ \vartheta)}{2}\right]$$
$$\geq \frac{1}{2}\left[\frac{\mathcal{H}^*(\mathfrak{b},\ \vartheta)+\mathcal{H}^*(z,\ \vartheta)}{2}+\frac{\mathcal{H}^*(\mathfrak{b},\ \vartheta)+\mathcal{H}^*(z,\ \vartheta)}{2}\right],$$
$$=\frac{\mathcal{H}_*(\mathfrak{b},\ \vartheta)+\mathcal{H}_*(z,\ \vartheta)}{2}$$
$$=\frac{\mathcal{H}^*(\mathfrak{b},\ \vartheta)+\mathcal{H}^*(z,\ \vartheta)}{2},$$

that is

$$\widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+z}{2}\right) \supseteq_{\mathbb{F}} \mathfrak{T}_2 \supseteq_{\mathbb{F}} \frac{1}{z-\mathfrak{b}} \odot (FA)\int_{\mathfrak{b}}^{z} \widetilde{\mathcal{H}}(\mathfrak{w})d\mathfrak{w} \supseteq_{\mathbb{F}} \mathfrak{T}_1 \supseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{H}}(\mathfrak{b})\ \oplus\ \widetilde{\mathcal{H}}(z)}{2},$$

hence, the result follows. □

Example 2. *We consider the $\mathcal{FNVM}$ $\widetilde{\mathcal{H}} : [\mathfrak{b},\ z] = [2,\ 3] \to {}_0$ defined by, $\mathcal{H}_{\vartheta}(\mathfrak{w}) = \left[(1-\vartheta)\left(2-\mathfrak{w}^{\frac{1}{2}}\right)+3\vartheta, (1+\vartheta)\left(2+\mathfrak{w}^{\frac{1}{2}}\right)+3\vartheta\right]$, as in Example 1, then $\widetilde{\mathcal{H}}(\mathfrak{w})$ is $\mathcal{UD}$-convex $\mathcal{FNVM}$ and satisfying (10). We have $\mathcal{H}_*(\mathfrak{w},\ \vartheta) = (1-\vartheta)\left(2-\mathfrak{w}^{\frac{1}{2}}\right)+3\vartheta$ and $\mathcal{H}^*(\mathfrak{w},\ \vartheta) = (1+\vartheta)\left(2+\mathfrak{w}^{\frac{1}{2}}\right)+3\vartheta$. We now compute the following*

$$\frac{\mathcal{H}_*(\mathfrak{b},\ \vartheta)+\mathcal{H}_*(z,\ \vartheta)}{2}=\frac{4+2\vartheta-(1-\vartheta)\left(\sqrt{2}+\sqrt{3}\right)}{2}$$
$$\frac{\mathcal{H}^*(\mathfrak{b},\vartheta)\ \widetilde{+}\ \mathcal{H}^*(z,\vartheta)}{2}=\frac{4+10\vartheta+(1+\vartheta)\left(\sqrt{2}+\sqrt{3}\right)}{2},$$

$$\mathfrak{T}_{1*}=\frac{\frac{\mathcal{H}_*(\mathfrak{b},\ \vartheta)+\mathcal{H}_*(z,\ \vartheta)}{2}+\mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2},\ \vartheta\right)}{2}=\frac{8+4\vartheta-(1-\vartheta)\left(\sqrt{2}+\sqrt{3}+\sqrt{2}\times\sqrt{5}\right)}{4}$$
$$\mathfrak{T}_1{}^*=\frac{\frac{\mathcal{H}^*(\mathfrak{b},\ \vartheta)+\mathcal{H}^*(z,\ \vartheta)}{2}+\mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2},\ \vartheta\right)}{2}=\frac{8+20\vartheta+(1+\vartheta)\left(\sqrt{2}+\sqrt{3}+\sqrt{2}\times\sqrt{5}\right)}{4},$$

$$\mathfrak{T}_{2*}=\frac{\mathcal{H}_*\left(\frac{3\mathfrak{b}+z}{4},\ \vartheta\right)+\mathcal{H}_*\left(\frac{\mathfrak{b}+3z}{4},\ \vartheta\right)}{2}=\frac{5+7\vartheta-\sqrt{11}(1-\vartheta)}{4}$$
$$\mathfrak{T}_2{}^*=\frac{\mathcal{H}^*\left(\frac{3\mathfrak{b}+z}{4},\ \vartheta\right)+\mathcal{H}^*\left(\frac{\mathfrak{b}+3z}{4},\ \vartheta\right)}{2}=\frac{11+23\vartheta+\sqrt{11}(1+\vartheta)}{4},$$

Then we obtain that

$$(1-\vartheta)\frac{4-\sqrt{10}}{2}+3\vartheta \leq \frac{5+7\vartheta-\sqrt{11}(1-\vartheta)}{4} \leq \frac{843}{2000}(1-\vartheta)+3\vartheta$$
$$\leq \frac{8+4\vartheta-(1-\vartheta)\left(\sqrt{2}+\sqrt{3}+\sqrt{2}\times\sqrt{5}\right)}{4} \leq (1-\vartheta)\left(\frac{4-\sqrt{2}-\sqrt{3}}{2}\right)+3\vartheta$$
$$(1+\vartheta)\frac{4+\sqrt{10}}{2}+3\vartheta \geq \frac{11+23\vartheta+\sqrt{11}(1+\vartheta)}{4} \geq \frac{179}{50}(1+\vartheta)+3\vartheta$$
$$\geq \frac{8+20\vartheta+(1+\vartheta)\left(\sqrt{2}+\sqrt{3}+\sqrt{2}\times\sqrt{5}\right)}{4} \geq (1+\vartheta)\left(\frac{4+\sqrt{2}+\sqrt{3}}{2}\right)+3\vartheta.$$

Hence, Theorem 6 is verified.

We now obtain some HH-inequalities for the product of $\mathcal{UD}$-convex $\mathcal{FNVM}$s. These inequalities are refinements of some known inequalities, see [57].

Theorem 7. *Let $\widetilde{\mathcal{H}}, \widetilde{T} : [\mathfrak{b}, z] \to {}_o$ be two $\mathcal{UD}$-convex $\mathcal{FNVM}s$ on $[\mathfrak{b}, z]$, whose $\mathfrak{o}$-cuts $\mathcal{H}_{\mathfrak{o}}$, $T_{\mathfrak{o}} : [\mathfrak{b}, z] \subset \mathbb{R} \to \mathcal{X}_o^+$ are defined by $\mathcal{H}_{\mathfrak{o}}(\mathfrak{w}) = [\mathcal{H}_*(\mathfrak{w}, \mathfrak{o}), \mathcal{H}^*(\mathfrak{w}, \mathfrak{o})]$ and $T_{\mathfrak{o}}(\mathfrak{w}) = [T_*(\mathfrak{w}, \mathfrak{o}), T^*(\mathfrak{w}, \mathfrak{o})]$ for all $\mathfrak{w} \in [\mathfrak{b}, z]$ and for all $\mathfrak{o} \in [0, 1]$. If $\widetilde{\mathcal{H}} \otimes \widetilde{T} \in \mathcal{FA}_{([\mathfrak{b}, z], \mathfrak{o})}$, then*

$$\frac{1}{z-\mathfrak{b}} \odot (FA) \int_{\mathfrak{b}}^{z} \widetilde{\mathcal{H}}(\mathfrak{w}) \otimes \widetilde{T}(\mathfrak{w}) d\mathfrak{w} \supseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{M}}(\mathfrak{b}, z)}{3} \oplus \frac{\widetilde{\mathcal{N}}(\mathfrak{b}, z)}{6}. \quad (33)$$

where $\widetilde{\mathcal{M}}(\mathfrak{b}, z) = \widetilde{\mathcal{H}}(\mathfrak{b}) \otimes \widetilde{T}(\mathfrak{b}) \oplus \widetilde{\mathcal{H}}(z) \otimes \widetilde{T}(z)$, $\widetilde{\mathcal{N}}(\mathfrak{b}, z) = \widetilde{\mathcal{H}}(\mathfrak{b}) \otimes \widetilde{T}(z) \oplus \widetilde{\mathcal{H}}(z) \otimes \widetilde{T}(\mathfrak{b})$, and $\mathcal{M}_{\mathfrak{o}}(\mathfrak{b}, z) = [\mathcal{M}_*((\mathfrak{b}, z), \mathfrak{o}), \mathcal{M}^*((\mathfrak{b}, z), \mathfrak{o})]$ and $\mathcal{N}_{\mathfrak{o}}(\mathfrak{b}, z) = [\mathcal{N}_*((\mathfrak{b}, z), \mathfrak{o}), \mathcal{N}^*((\mathfrak{b}, z), \mathfrak{o})]$.

Proof. Since $\widetilde{\mathcal{H}}, \widetilde{T} \in \mathcal{FA}_{([\mathfrak{b}, z])}$, then we have

$$\begin{gathered} \mathcal{H}_*(\varsigma\mathfrak{b} + (1-\varsigma)z, \mathfrak{o}) \le \varsigma\mathcal{H}_*(\mathfrak{b}, \mathfrak{o}) + (1-\varsigma)\mathcal{H}_*(z, \mathfrak{o}), \\ \mathcal{H}^*(\varsigma\mathfrak{b} + (1-\varsigma)z, \mathfrak{o}) \ge \varsigma\mathcal{H}^*(\mathfrak{b}, \mathfrak{o}) + (1-\varsigma)\mathcal{H}^*(z, \mathfrak{o}). \end{gathered}$$

And

$$\begin{gathered} T_*(\varsigma\mathfrak{b} + (1-\varsigma)z, \mathfrak{o}) \le \varsigma T_*(\mathfrak{b}, \mathfrak{o}) + (1-\varsigma)T_*(z, \mathfrak{o}), \\ T^*(\varsigma\mathfrak{b} + (1-\varsigma)z, \mathfrak{o}) \ge \varsigma T^*(\mathfrak{b}, \mathfrak{o}) + (1-\varsigma)T^*(z, \mathfrak{o}). \end{gathered}$$

From the definition of $\mathcal{UD}$-convex $\mathcal{FNVM}s$, it follows that $\widetilde{0} \le_{\mathbb{F}} \widetilde{\mathcal{H}}(\mathfrak{w})$ and $\widetilde{0} \le_{\mathbb{F}} \widetilde{T}(\mathfrak{w})$, so

$$\begin{gathered} \mathcal{H}_*(\varsigma\mathfrak{b} + (1-\varsigma)z, \mathfrak{o}) \times T_*(\varsigma\mathfrak{b} + (1-\varsigma)z, \mathfrak{o}) \\ \le (\varsigma\mathcal{H}_*(\mathfrak{b}, \mathfrak{o}) + (1-\varsigma)\mathcal{H}_*(z, \mathfrak{o})) \times (\varsigma T_*(\mathfrak{b}, \mathfrak{o}) + (1-\varsigma)T_*(z, \mathfrak{o})) \\ = \mathcal{H}_*(\mathfrak{b}, \mathfrak{o}\times)T_*(\mathfrak{b}, \mathfrak{o})\varsigma^2 + \mathcal{H}_*(z, \mathfrak{o}) \times T_*(z, \mathfrak{o})\varsigma^2 \\ +\mathcal{H}_*(\mathfrak{b}, \mathfrak{o}) \times T_*(z, \mathfrak{o})\varsigma(1-\varsigma) + \mathcal{H}_*(z, \mathfrak{o}) \times T_*(\mathfrak{b}, \mathfrak{o})\varsigma(1-\varsigma) \\ \mathcal{H}^*(\varsigma\mathfrak{b} + (1-\varsigma)z, \mathfrak{o}) \times T^*(\varsigma\mathfrak{b} + (1-\varsigma)z, \mathfrak{o}) \\ \ge (\varsigma\mathcal{H}^*(\mathfrak{b}, \mathfrak{o}) + (1-\varsigma)\mathcal{H}^*(z, \mathfrak{o})) \times (\varsigma T^*(\mathfrak{b}, \mathfrak{o}) + (1-\varsigma)T^*(z, \mathfrak{o})) \\ = \mathcal{H}^*(\mathfrak{b}, \mathfrak{o}) \times T^*(\mathfrak{b}, \mathfrak{o})\varsigma^2 + \mathcal{H}^*(z, \mathfrak{o}) \times T^*(z, \mathfrak{o})\varsigma^2 \\ +\mathcal{H}^*(\mathfrak{b}, \mathfrak{o})T^* \times (z, \mathfrak{o})\varsigma(1-\varsigma) + \mathcal{H}^*(z, \mathfrak{o}) \times T^*(\mathfrak{b}, \mathfrak{o})\varsigma(1-\varsigma), \end{gathered}$$

Integrating both sides of the above inequality over [0, 1], we get

$$\begin{gathered} \int_0^1 \mathcal{H}_*(\varsigma\mathfrak{b} + (1-\varsigma)z, \mathfrak{o}) \times T_*(\varsigma\mathfrak{b} + (1-\varsigma)z, \mathfrak{o})d\varsigma \\ = \frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}_*(\mathfrak{w}, \mathfrak{o}) \times T_*(\mathfrak{w}, \mathfrak{o})d\mathfrak{w} \\ \le (\mathcal{H}_*(\mathfrak{b}, \mathfrak{o}) \times T_*(\mathfrak{b}, \mathfrak{o}) + \mathcal{H}_*(z, \mathfrak{o}) \times T_*(z, \mathfrak{o})) \int_0^1 \varsigma^2 d\varsigma \\ +(\mathcal{H}_*(\mathfrak{b}, \mathfrak{o}) \times T_*(z, \mathfrak{o}) + \mathcal{H}_*(z, \mathfrak{o}) \times T_*(\mathfrak{b}, \mathfrak{o})) \int_0^1 \varsigma(1-\varsigma)d\varsigma, \\ \int_0^1 \mathcal{H}^*(\varsigma\mathfrak{b} + (1-\varsigma)z, \mathfrak{o}) \times T^*(\varsigma\mathfrak{b} + (1-\varsigma)z, \mathfrak{o})d\varsigma \\ = \frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}^*(\mathfrak{w}, \mathfrak{o}) \times T^*(\mathfrak{w}, \mathfrak{o})d\mathfrak{w} \\ \ge (\mathcal{H}^*(\mathfrak{b}, \mathfrak{o}) \times T^*(\mathfrak{b}, \mathfrak{o}) + \mathcal{H}^*(z, \mathfrak{o}) \times T^*(z, \mathfrak{o})) \int_0^1 \varsigma^2 d\varsigma \\ +(\mathcal{H}^*(\mathfrak{b}, \mathfrak{o}) \times T^*(z, \mathfrak{o}) + \mathcal{H}^*(z, \mathfrak{o}) \times T^*(\mathfrak{b}, \mathfrak{o})) \int_0^1 \varsigma(1-\varsigma)d\varsigma. \end{gathered}$$

It follows that,

$$\begin{gathered} \tfrac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}_*(\mathfrak{w}, \mathfrak{o}) \times T_*(\mathfrak{w}, \mathfrak{o})dw \le \mathfrak{B}_*((\mathfrak{b}, z), \mathfrak{o}) \int_0^1 \varsigma^2 d\varsigma + \mathfrak{C}_*((\mathfrak{b}, z), \mathfrak{o}) \int_0^1 \varsigma(1-\varsigma)d\varsigma, \\ \tfrac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}^*(\mathfrak{w}, \mathfrak{o}) \times T^*(\mathfrak{w}, \mathfrak{o})d\mathfrak{w} \ge \mathfrak{B}^*((\mathfrak{b}, z), \mathfrak{o}) \int_0^1 \varsigma^2 d\varsigma + \mathfrak{C}^*((\mathfrak{b}, z), \mathfrak{o}) \int_0^1 \varsigma(1-\varsigma)d\varsigma, \end{gathered}$$

that is

$$\begin{gathered} \frac{1}{z-\mathfrak{b}} \left[\int_{\mathfrak{b}}^{z} \mathcal{H}_*(\mathfrak{w}, \mathfrak{o}) \times T_*(\mathfrak{w}, \mathfrak{o})d\mathfrak{w}, \int_{\mathfrak{b}}^{z} \mathcal{H}^*(\mathfrak{w}, \mathfrak{o}) \times T^*(\mathfrak{w}, \mathfrak{o})d\mathfrak{w} \right] \\ \supseteq_I \left[\frac{\mathfrak{B}_*((\mathfrak{b}, z), \mathfrak{o})}{3}, \frac{\mathfrak{B}^*((\mathfrak{b}, z), \mathfrak{o})}{3} \right] + \left[\frac{\mathfrak{C}_*((\mathfrak{b}, z), \mathfrak{o})}{6}, \frac{\mathfrak{C}^*((\mathfrak{b}, z), \mathfrak{o})}{6} \right]. \end{gathered}$$

Thus,

$$\frac{1}{z-\mathfrak{b}} \odot (FA)\int_{\mathfrak{b}}^{z} \widetilde{\mathcal{H}}(\mathfrak{w}) \otimes \widetilde{T}(\mathfrak{w}) d\mathfrak{w} \supseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{M}}(\mathfrak{b},z)}{3} \oplus \frac{\widetilde{\mathcal{N}}(\mathfrak{b},z)}{6}.$$

And the theorem has been established. □

Example 3. *Let* $[\mathfrak{b},z]=[0,2]$, *and the* $\mathcal{FNVM}s$ $\mathcal{H},T:[\mathfrak{b},\ z]=[0,2]\to{}_{o}$, *defined by*

$$\mathcal{H}(\mathfrak{w})(\theta)=\begin{cases}\dfrac{\theta}{\mathfrak{w}} & \theta\in[0,\ \mathfrak{w}],\\ \dfrac{2\mathfrak{w}-\theta}{\mathfrak{w}} & \theta\in(\mathfrak{w},\ 2\mathfrak{w}],\\ 0 & otherwise,\end{cases}$$

$$T(\mathfrak{w})(\theta)=\begin{cases}\dfrac{\theta-\mathfrak{w}}{2-\mathfrak{w}} & \theta\in[\mathfrak{w},\ 2],\\ \dfrac{8-e^{\mathfrak{w}}-\theta}{8-e^{\mathfrak{w}}-2} & \theta\in(2,\ 8-e^{\mathfrak{w}}],\\ 0 & otherwise.\end{cases}$$

Then, for each $\partial\in[0,\ 1]$, *we have* $\mathcal{H}_{\partial}(\mathfrak{w})=[\partial\mathfrak{w},(2-\partial)\mathfrak{w}]$ *and* $T_{\partial}(\mathfrak{w})=[(1-\partial)\mathfrak{w}+2\partial,(1-\partial)(8-e^{\mathfrak{w}})+2\partial]$ *Since left and right end point mappings* $\mathcal{H}_{*}(\mathfrak{w},\partial)=\partial\mathfrak{w}$, *and* $\mathcal{H}^{*}(\mathfrak{w},\ \partial)=(2-\partial)\mathfrak{w}$, *are convex and concave mappings, respectively, and* $T_{*}(\mathfrak{w},\partial)=(1-\partial)\mathfrak{w}+2\partial$ *and* $T^{*}(\mathfrak{w},\ \partial)=(1-\partial)(8-e^{\mathfrak{w}})+2\partial$ *are convex and concave mappings, respectively, for each* $\partial\in[0,\ 1]$, *then* $\widetilde{\mathcal{H}}(\mathfrak{w})$ *and* $\widetilde{T}(\mathfrak{w})$ *both are* $\mathcal{UD}$*-convex* $\mathcal{FNVM}s$. *We clearly see that* $\widetilde{\mathcal{H}}\otimes\widetilde{T}\in L([\mathfrak{b},\ z],{}_{o})$ *and*

$$\frac{1}{z-\mathfrak{b}}\ \int_{\mathfrak{b}}^{z}\mathcal{H}_{*}(\mathfrak{w},\partial)\times T_{*}(\mathfrak{w},\partial)d\mathfrak{w}=\frac{1}{2}\int_{0}^{2}\Big(\partial(1-\partial)\mathfrak{w}^{2}+2\partial^{2}\mathfrak{w}\Big)d\mathfrak{w}=\frac{2}{3}\partial(2+\partial),$$

$$\frac{1}{z-\mathfrak{b}}\int_{\mathfrak{b}}^{z}\mathcal{H}^{*}(\mathfrak{w},\partial)\times T^{*}(\mathfrak{w},\partial)d\mathfrak{w}=\frac{1}{2}\int_{0}^{2}((1-\partial)(2-\partial)\mathfrak{w}(8-e^{\mathfrak{w}})+2\partial(2-\partial)\mathfrak{w})d\mathfrak{w}$$
$$\approx\frac{(2-\partial)}{2}\bigg(\frac{1903}{250}-\frac{903}{250}\partial\bigg).$$

Note that

$$\Delta_{*}(\mathfrak{b},z)=[\mathcal{H}_{*}(\mathfrak{b})\times T_{*}(\mathfrak{b})+\mathcal{H}_{*}(z)\times T_{*}(z)]=4\partial,$$
$$\Delta^{*}(\mathfrak{b},z)=[\mathcal{H}^{*}(\mathfrak{b})\times T^{*}(\mathfrak{b})+\mathcal{H}^{*}(z)\times T^{*}(z)]=2(2-\partial)\Big[(1-\partial)\Big(8-e^{2}\Big)+2\partial\Big],$$
$$\nabla_{*}(\mathfrak{b},z)=[\mathcal{H}_{*}(\mathfrak{b})\times T_{*}(z)+\mathcal{H}_{*}(z)\times T_{*}(\mathfrak{b})]=4\partial^{2},$$
$$\nabla_{*}(\mathfrak{b},z)=[\mathcal{H}^{*}(\mathfrak{b})\times T^{*}(z)+\mathcal{H}^{*}(z)\times T^{*}(\mathfrak{b})]=2(2-\partial)(7-5\partial).$$

Therefore, we have

$$\frac{1}{3}\Delta_{\partial}(\mathfrak{b},z)+\frac{1}{6}\nabla_{\partial}(\mathfrak{b},z)$$
$$=\frac{1}{3}\Big[4\partial,2(2-\partial)\Big[(1-\partial)\Big(8-e^{2}\Big)+2\partial\Big]\Big]+\frac{1}{3}\Big[2\partial^{2},(2-\partial)(7-5\partial)\Big]$$
$$=\frac{1}{3}\Big[2\partial(2+\partial),(2-\partial)\Big[2(1-\partial)\Big(8-e^{2}\Big)-\partial+7\Big]\Big].$$

It follows that

$$[\frac{2}{3}\partial(1+2\partial),\frac{(2-\partial)}{2}\bigg(\frac{1903}{250}-\frac{903}{250}\partial\bigg)]\supseteq_{I}\frac{1}{3}\big[2\partial(2+\partial),(2-\partial)\big[2(1-\partial)\big(8-e^{2}\big)-\partial+7\big]\big],$$

and Theorem 7 has been demonstrated.

Theorem 8. *Let* $\widetilde{\mathcal{H}}, \widetilde{T} : [\mathfrak{b}, z] \to {}_{o}$ *be two* $\mathcal{UD}$*-convex* $\mathcal{FNVM}$*s, whose* $\mathfrak{s}$*-cuts define the family of* $\mathcal{IVM}$*s* $\mathcal{H}_{\mathfrak{s}}$, $T_{\mathfrak{s}} : [\mathfrak{b}, z] \subset \mathbb{R} \to \mathcal{X}_o^+$ *are given by* $\mathcal{H}_{\mathfrak{s}}(\mathfrak{w}) = [\mathcal{H}_*(\mathfrak{w}, \mathfrak{s}),\ \mathcal{H}^*(\mathfrak{w}, \mathfrak{s})]$ *and* $T_{\mathfrak{s}}(\mathfrak{w}) = [T_*(\mathfrak{w}, \mathfrak{s}),\ T^*(\mathfrak{w}, \mathfrak{s})]$ *for all* $\mathfrak{w} \in [\mathfrak{b}, z]$ *and for all* $\mathfrak{s} \in [0, 1]$*, respectively. If* $\widetilde{\mathcal{H}} \otimes \widetilde{T} \in \mathcal{FA}_{([\mathfrak{b}, z], \mathfrak{s})}$*, then*

$$2\,\widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+z}{2}\right) \otimes \widetilde{T}\left(\frac{\mathfrak{b}+z}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{z-\mathfrak{b}} \odot (FA) \int_{\mathfrak{b}}^{z} \widetilde{\mathcal{H}}(\mathfrak{w}) \otimes \widetilde{T}(\mathfrak{w}) d\mathfrak{w} \oplus \frac{\widetilde{\mathcal{M}}(\mathfrak{b}, z)}{6} \oplus \frac{\widetilde{\mathcal{N}}(\mathfrak{b}, z)}{3}. \tag{34}$$

where $\widetilde{\mathcal{M}}(\mathfrak{b}, z) = \widetilde{\mathcal{H}}(\mathfrak{b}) \otimes \widetilde{T}(\mathfrak{b}) \oplus \widetilde{\mathcal{H}}(z) \otimes \widetilde{T}(z)$, $\widetilde{\mathcal{N}}(\mathfrak{b}, z) = \widetilde{\mathcal{H}}(\mathfrak{b}) \otimes \widetilde{T}(z) \oplus \widetilde{\mathcal{H}}(z) \otimes \widetilde{T}(\mathfrak{b})$*, and* $\mathcal{M}_{\mathfrak{s}}(\mathfrak{b}, z) = [\mathcal{M}_*((\mathfrak{b}, z),\ \mathfrak{s}),\ \mathcal{M}^*((\mathfrak{b}, z),\ \mathfrak{s})]$ *and* $\mathcal{N}_{\mathfrak{s}}(\mathfrak{b}, z) = [\mathcal{N}_*((\mathfrak{b}, z),\ \mathfrak{s}),\ \mathcal{N}^*((\mathfrak{b}, z),\ \mathfrak{s})]$.

Proof. By hypothesis, for each $\mathfrak{s} \in [0, 1]$, we have

$$\begin{aligned}
&\mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2}, \mathfrak{s}\right) \times T_*\left(\frac{\mathfrak{b}+z}{2}, \mathfrak{s}\right) \\
&\mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2}, \mathfrak{s}\right) \times T^*\left(\frac{\mathfrak{b}+z}{2}, \mathfrak{s}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4}\begin{bmatrix} \mathcal{H}_*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \times T_*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \\ +\mathcal{H}_*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \times T_*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \end{bmatrix} \\
&+\frac{1}{4}\begin{bmatrix} \mathcal{H}_*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \times T_*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \\ +\mathcal{H}_*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \times T_*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \end{bmatrix}, \\
&\geq \frac{1}{4}\begin{bmatrix} \mathcal{H}^*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \times T^*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \\ +\mathcal{H}^*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \times T^*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \end{bmatrix} \\
&+\frac{1}{4}\begin{bmatrix} \mathcal{H}^*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \times T^*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \\ +\mathcal{H}^*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \times T^*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4}\begin{bmatrix} \mathcal{H}_*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \times T_*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \\ +\mathcal{H}_*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \times T_*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \end{bmatrix} \\
&\qquad +\frac{1}{4}\begin{bmatrix} (\varpi\mathcal{H}_*(\mathfrak{b},\ \mathfrak{s}) + (1-\varpi)\mathcal{H}_*(z,\ \mathfrak{s})) \\ \times((1-\varpi)T_*(\mathfrak{b},\ \mathfrak{s}) + \varpi T_*(z,\ \mathfrak{s})) \\ +((1-\varpi)\mathcal{H}_*(\mathfrak{b},\ \mathfrak{s}) + \varpi\mathcal{H}_*(z,\ \mathfrak{s})) \\ \times(\varpi T_*(\mathfrak{b},\ \mathfrak{s}) + (1-\varpi)T_*(z,\ \mathfrak{s})) \end{bmatrix}, \\
&\geq \frac{1}{4}\begin{bmatrix} \mathcal{H}^*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \times T^*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \\ +\mathcal{H}^*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \times T^*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \end{bmatrix} \\
&\qquad +\frac{1}{4}\begin{bmatrix} (\varpi\mathcal{H}^*(\mathfrak{b},\ \mathfrak{s}) + (1-\varpi)\mathcal{H}^*(z,\ \mathfrak{s})) \\ \times((1-\varpi)T^*(\mathfrak{b},\ \mathfrak{s}) + \mathfrak{s}T^*(z,\ \mathfrak{s})) \\ +((1-\varpi)\mathcal{H}^*(\mathfrak{b},\ \mathfrak{s}) + \mathfrak{s}\mathcal{H}^*(z,\ \mathfrak{s})) \\ \times(\varpi T^*(\mathfrak{b},\ \mathfrak{s}) + (1-\varpi)T^*(z,\ \mathfrak{s})) \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}\begin{bmatrix} \mathcal{H}_*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \times T_*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \\ +\mathcal{H}_*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \times T_*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \end{bmatrix} \\
&\qquad +\frac{1}{2}\begin{bmatrix} \left\{\varpi^2 + (1-\varpi)^2\right\}\mathcal{N}_*((\mathfrak{b}, z),\ \mathfrak{s}) \\ +\{\varpi(1-\varpi) + (1-\varpi)\varpi\}\mathcal{M}_*((\mathfrak{b}, z),\ \mathfrak{s}) \end{bmatrix}, \\
&= \frac{1}{4}\begin{bmatrix} \mathcal{H}^*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \times T^*(\varpi\mathfrak{b} + (1-\varpi)z,\ \mathfrak{s}) \\ +\mathcal{H}^*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \times T^*((1-\varpi)\mathfrak{b} + \varpi z,\ \mathfrak{s}) \end{bmatrix} \\
&\qquad +\frac{1}{2}\begin{bmatrix} \left\{\varpi^2 + (1-\varpi)^2\right\}\mathcal{N}^*((\mathfrak{b}, z),\ \mathfrak{s}) \\ +\{\varpi(1-\varpi) + (1-\varpi)\varpi\}\mathcal{M}^*((\mathfrak{b}, z),\ \mathfrak{s}) \end{bmatrix}.
\end{aligned}$$

Taking integration over $[0, 1]$, we have

$$\begin{aligned}
2\,\mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2}, \mathfrak{s}\right) \times T_*\left(\frac{\mathfrak{b}+z}{2}, \mathfrak{s}\right) &\leq \frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}_*(\mathfrak{w}, \mathfrak{s}) \times T_*(\mathfrak{w}, \mathfrak{s}) d\mathfrak{w} + \frac{\mathcal{M}_*((\mathfrak{b}, z),\ \mathfrak{s})}{6} + \frac{\mathcal{N}_*((\mathfrak{b}, z),\ \mathfrak{s})}{3}, \\
2\,\mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2}, \mathfrak{s}\right) \times T^*\left(\frac{\mathfrak{b}+z}{2}, \mathfrak{s}\right) &\geq \frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}^*(\mathfrak{w}, \mathfrak{s}) \times T^*(\mathfrak{w}, \mathfrak{s}) d\mathfrak{w} + \frac{\mathcal{M}^*((\mathfrak{b}, z),\ \mathfrak{s})}{6} + \frac{\mathcal{N}^*((\mathfrak{b}, z),\ \mathfrak{s})}{3},
\end{aligned}$$

that is

$$2\,\widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+z}{2}\right)\otimes\widetilde{T}\left(\frac{\mathfrak{b}+z}{2}\right)\supseteq_{\mathbb{F}}\frac{1}{z-\mathfrak{b}}\odot(FA)\int_{\mathfrak{b}}^{z}\widetilde{\mathcal{H}}(\mathfrak{w})\otimes\widetilde{T}(\mathfrak{w})d\mathfrak{w}\ \oplus\frac{\widetilde{\mathcal{M}}(\mathfrak{b},z)}{6}\oplus\frac{\widetilde{\mathcal{N}}(\mathfrak{b},z)}{3}.$$

Hence, the required result. □

Example 4. *We consider the $\mathcal{FNVM}$s $\widetilde{\mathcal{H}}$, $\widetilde{T}:[\mathfrak{b},\ z]=[0,\ 2]\to{}_{o}$. Then, for each $\mathfrak{s}\in[0,\ 1]$, we have $\mathcal{H}_{\mathfrak{s}}(\mathfrak{w})=[\mathfrak{s}\mathfrak{w},(2-\mathfrak{s})\mathfrak{w}]$ and $T_{\mathfrak{s}}(\mathfrak{w})=[(1-\mathfrak{s})\mathfrak{w}+2\mathfrak{s},(1-\mathfrak{s})(8-e^{\mathfrak{w}})+2\mathfrak{s}]$, as in Example 3, then $\widetilde{\mathcal{H}}$ and $\widetilde{T}$ both are $\mathcal{UD}$-convex mappings. We have $\mathcal{H}_{*}(\mathfrak{w},\mathfrak{s})=\mathfrak{s}\mathfrak{w}$, $\mathcal{H}^{*}(\mathfrak{w},\ \mathfrak{s})=(2-\mathfrak{s})\mathfrak{w}$ and $T_{*}(\mathfrak{w},\mathfrak{s})=(1-\mathfrak{s})\mathfrak{w}+2\mathfrak{s}$, $T^{*}(\mathfrak{w},\ \mathfrak{s})=(1-\mathfrak{s})(8-e^{\mathfrak{w}})+2\mathfrak{s}$, then*

$$2\mathcal{H}_{*}\left(\frac{\mathfrak{b}+z}{2},\mathfrak{s}\right)\times T_{*}\left(\frac{\mathfrak{b}+z}{2},\mathfrak{s}\right)=2\mathfrak{s}(1+\mathfrak{s}),$$

$$2\mathcal{H}^{*}\left(\frac{\mathfrak{b}+z}{2},\mathfrak{s}\right)\times T^{*}\left(\frac{\mathfrak{b}+z}{2},\mathfrak{s}\right)=2\left[16-20\mathfrak{s}+6\mathfrak{s}^{2}+\left(2-3\mathfrak{s}+\mathfrak{s}^{2}\right)e\right],$$

$$\frac{1}{z-\mathfrak{b}}\int_{\mathfrak{b}}^{z}\mathcal{H}_{*}(\mathfrak{w},\mathfrak{s})\times T_{*}(\mathfrak{w},\mathfrak{s})d\mathfrak{w}=\frac{1}{2}\int_{0}^{2}\left(\mathfrak{s}(1-\mathfrak{s})\mathfrak{w}^{2}+2\mathfrak{s}^{2}\mathfrak{w}\right)d\mathfrak{w}=\frac{4}{3}\mathfrak{s}(3-\mathfrak{s}),$$

$$\frac{1}{z-\mathfrak{b}}\int_{\mathfrak{b}}^{z}\mathcal{H}^{*}(\mathfrak{w},\mathfrak{s})\times T^{*}(\mathfrak{w},\mathfrak{s})d\mathfrak{w}=\frac{1}{2}\int_{0}^{2}((1-\mathfrak{s})(2-\mathfrak{s})\mathfrak{w}(8-e^{\mathfrak{w}})+2\mathfrak{s}(2-\mathfrak{s})\mathfrak{w})d\mathfrak{w}$$

$$\approx\frac{(2-\mathfrak{s})}{2}\left(\frac{1903}{250}-\frac{903}{250}\mathfrak{s}\right).$$

$$\Delta_{*}(\mathfrak{b},z)=[\mathcal{H}_{*}(\mathfrak{b})\times T_{*}(\mathfrak{b})+\mathcal{H}_{*}(z)\times T_{*}(z)]=4\mathfrak{s},$$

$$\Delta^{*}(\mathfrak{b},z)=[\mathcal{H}^{*}(\mathfrak{b})\times T^{*}(\mathfrak{b})+\mathcal{H}^{*}(z)\times T^{*}(z)]=2(2-\mathfrak{s})\left[(1-\mathfrak{s})\left(8-e^{2}\right)+2\mathfrak{s}\right],$$

$$\nabla_{*}(\mathfrak{b},z)=[\mathcal{H}_{*}(\mathfrak{b})\times T_{*}(z)+\mathcal{H}_{*}(z)\times T_{*}(\mathfrak{b})]=4\mathfrak{s}^{2},$$

$$\nabla_{*}(\mathfrak{b},z)=[\mathcal{H}^{*}(\mathfrak{b})\times T^{*}(z)+\mathcal{H}^{*}(z)\times T^{*}(\mathfrak{b})]=2(2-\mathfrak{s})(7-5\mathfrak{s}).$$

Therefore, we have

$$\frac{1}{6}\Delta_{\mathfrak{s}}((\mathfrak{b},z),\ \mathfrak{s})+\frac{1}{3}\nabla_{\mathfrak{s}}((\mathfrak{b},z),\ \mathfrak{s})$$

$$=\frac{1}{3}\left[2\mathfrak{s},(2-\mathfrak{s})\left[(1-\mathfrak{s})\left(8-e^{2}\right)+2\mathfrak{s}\right]\right]+\frac{2}{3}\left[2\mathfrak{s}^{2},(2-\mathfrak{s})(7-5\mathfrak{s})\right]$$

$$=\frac{1}{3}\left[2\mathfrak{s}(1+2\mathfrak{s}),(2-\mathfrak{s})\left[(1-\mathfrak{s})\left(8-e^{2}\right)-8\mathfrak{s}+14\right]\right].$$

It follows that

$$2\left[\mathfrak{s}(1+\mathfrak{s}),\left[16-20\mathfrak{s}+6\mathfrak{s}^{2}+\left(2-3\mathfrak{s}+\mathfrak{s}^{2}\right)e\right]\right]\supseteq_{I}\left[\frac{2}{3}\mathfrak{s}(2+\mathfrak{s}),\frac{(2-\mathfrak{s})}{2}\left(\frac{1903}{250}-\frac{903}{250}\mathfrak{s}\right)\right]$$

$$+\frac{1}{3}\left[2\mathfrak{s}(1+2\mathfrak{s}),(2-\mathfrak{s})\left[(1-\mathfrak{s})\left(8-e^{2}\right)-8\mathfrak{s}+14\right]\right],$$

and Theorem 8 has been demonstrated.

We now give *HH*-Fejér inequalities for $\mathcal{UD}$-convex $\mathcal{FNVM}$s. Firstly, we obtain the second *HH*-Fejér inequality for $\mathcal{UD}$-convex $\mathcal{FNVM}$.

Theorem 9. *Let $\widetilde{\mathcal{H}}:[\mathfrak{b},\ z]\to{}_{o}$ be a $\mathcal{UD}$-convex $\mathcal{FNVM}$ with $\mathfrak{b}<z$, whose $\mathfrak{s}$-cuts define the family of $\mathcal{IVM}$s $\mathcal{H}_{\mathfrak{s}}:[\mathfrak{b},\ z]\subset\mathbb{R}\to\mathcal{X}_{o}^{+}$ are given by $\mathcal{H}_{\mathfrak{s}}(\mathfrak{w})=[\mathcal{H}_{*}(\mathfrak{w},\mathfrak{s}),\ \mathcal{H}^{*}(\mathfrak{w},\mathfrak{s})]$ for*

all $\mathfrak{w} \in [\mathfrak{b}, z]$ *and for all* $\mathfrak{s} \in [0, 1]$. *If* $\widetilde{\mathcal{H}} \in \mathcal{FA}_{([\mathfrak{b}, z], \mathfrak{s})}$ *and* $\mathfrak{B} : [\mathfrak{b}, z] \to \mathbb{R}$, $\mathfrak{B}(\mathfrak{w}) \geq 0$, *symmetric with respect to* $\frac{\mathfrak{b}+z}{2}$, *then*

$$\frac{1}{z-\mathfrak{b}} \odot (FA) \int_{\mathfrak{b}}^{z} \mathcal{H}(\mathfrak{w}) \odot \mathfrak{B}(\mathfrak{w}) d\mathfrak{w} \supseteq_{\mathbb{F}} [\mathcal{H}(\mathfrak{b}) \oplus \mathcal{H}(z)] \odot \int_0^1 \varpi \mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z) d\varpi. \quad (35)$$

Proof. Let $\widetilde{\mathcal{H}}$ be a $\mathcal{UD}$-convex $\mathcal{FNVM}$. Then, for each $\mathfrak{s} \in [0, 1]$, we have

$$\begin{aligned} &\mathcal{H}_*(\varpi\mathfrak{b} + (1-\varpi)z, \mathfrak{s}) B(\varpi\mathfrak{b} + (1-\varpi)z) \\ &\leq (\varpi\mathcal{H}_*(\mathfrak{b}, \mathfrak{s}) + (1-\varpi)\mathcal{H}_*(z, \mathfrak{s}))\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z), \\ &\mathcal{H}^*(\varpi\mathfrak{b} + (1-\varpi)z, \mathfrak{s})\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z) \\ &\geq (\varpi\mathcal{H}^*(\mathfrak{b}, \mathfrak{s}) + (1-\varpi)\mathcal{H}^*(z, \mathfrak{s}))\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z). \end{aligned} \quad (36)$$

And

$$\begin{aligned} &\mathcal{H}_*((1-\varpi)\mathfrak{b} + \varpi z, \mathfrak{s}) B((1-\varpi)\mathfrak{b} + \varpi z) \\ &\quad \leq ((1-\varpi)\mathcal{H}_*(\mathfrak{b}, \mathfrak{s}) + \varpi\mathcal{H}_*(z, \mathfrak{s}))\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z), \\ &\mathcal{H}^*((1-\varpi)\mathfrak{b} + \varpi z, \mathfrak{s})\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z) \\ &\quad \geq ((1-\varpi)\mathcal{H}^*(\mathfrak{b}, \mathfrak{s}) + \varpi\mathcal{H}^*(z, \mathfrak{s}))\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z). \end{aligned} \quad (37)$$

After adding (36) and (37), and integrating over $[0, 1]$, we get

$$\begin{aligned} &\int_0^1 \mathcal{H}_*(\varpi\mathfrak{b} + (1-\varpi)z, \mathfrak{s})\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z) d\varpi \\ &\quad + \int_0^1 \mathcal{H}_*((1-\varpi)\mathfrak{b} + \varpi z, \mathfrak{s})\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z) d\varpi \\ &\leq \int_0^1 \begin{bmatrix} \mathcal{H}_*(\mathfrak{b}, \mathfrak{s})\{\varpi\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z) + (1-\varpi)\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z)\} \\ +\mathcal{H}_*(z, \mathfrak{s})\{(1-\varpi)\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z) + \varpi\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z)\} \end{bmatrix} d\varpi, \\ &\int_0^1 \mathcal{H}^*((1-\varpi)\mathfrak{b} + \varpi z, \mathfrak{s})\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z) d\varpi \\ &\quad + \int_0^1 \mathcal{H}^*(\varpi\mathfrak{b} + (1-\varpi)z, \mathfrak{s})\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z) d\varpi \\ &\geq \int_0^1 \begin{bmatrix} \mathcal{H}^*(\mathfrak{b}, \mathfrak{s})\{\varpi\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z) + (1-\varpi)\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z)\} \\ +\mathcal{H}^*(z, \mathfrak{s})\{(1-\varpi)\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z) + \varpi\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z)\} \end{bmatrix} d\varpi, \end{aligned}$$

$$\begin{aligned} &= 2\mathcal{H}_*(\mathfrak{b}, \mathfrak{s}) \int_0^1 \varpi B(\varpi\mathfrak{b} + (1-\varpi)z) d\varpi + 2\mathcal{H}_*(z, \mathfrak{s}) \int_0^1 \varpi B((1-\varpi)\mathfrak{b} + \varpi z) d\varpi, \\ &= 2\mathcal{H}^*(\mathfrak{b}, \mathfrak{s}) \int_0^1 \varpi B(\varpi\mathfrak{b} + (1-\varpi)z) d\varpi + 2\mathcal{H}^*(z, \mathfrak{s}) \int_0^1 \varpi B((1-\varpi)\mathfrak{b} + \varpi z) d\varpi. \end{aligned}$$

Since $\mathfrak{B}$ is symmetric, then

$$\begin{aligned} &\int_0^1 \mathcal{H}_*(\varpi\mathfrak{b} + (1-\varpi)z, \mathfrak{s})\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z) d\varpi \\ &+ \int_0^1 \mathcal{H}_*(\varpi\mathfrak{b} + (1-\varpi)z, \mathfrak{s})\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z) d\varpi \\ &\leq 2[\mathcal{H}_*(\mathfrak{b}, \mathfrak{s}) + \mathcal{H}_*(z, \mathfrak{s})] \int_0^1 \varpi B((1-\varpi)\mathfrak{b} + \varpi z) d\varpi, \\ &\int_0^1 \mathcal{H}^*((1-\varpi)\mathfrak{b} + \varpi z, \mathfrak{s})\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z) d\varpi \\ &+ \int_0^1 \mathcal{H}^*(\varpi\mathfrak{b} + (1-\varpi)z, \mathfrak{s})\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z) d\varpi \\ &\geq 2[\mathcal{H}^*(\mathfrak{b}, \mathfrak{s}) + \mathcal{H}^*(z, \mathfrak{s})] \int_0^1 \varpi B((1-\varpi)\mathfrak{b} + \varpi z) d\varpi. \end{aligned} \quad (38)$$

Since

$$\begin{aligned} &\int_0^1 \mathcal{H}_*(\varpi\mathfrak{b} + (1-\varpi)z, \mathfrak{s})\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z) d\varpi \\ &= \int_0^1 \mathcal{H}_*((1-\varpi)\mathfrak{b} + \varpi z, \mathfrak{s})\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z) d\varpi = \frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}_*(\mathfrak{w}, \mathfrak{s})\mathfrak{B}(\mathfrak{w}) d\mathfrak{w} \\ &\int_0^1 \mathcal{H}^*((1-\varpi)\mathfrak{b} + \varpi z, \mathfrak{s})\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z) d\varpi \\ &= \int_0^1 \mathcal{H}^*(\varpi\mathfrak{b} + (1-\varpi)z, \mathfrak{s})\mathfrak{B}(\varpi\mathfrak{b} + (1-\varpi)z) d\varpi = \frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}^*(\mathfrak{w}, \mathfrak{s})\mathfrak{B}(\mathfrak{w}) d\mathfrak{w} \end{aligned} \quad (39)$$

Then from (38), we have

$$\begin{aligned} &\frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}_*(\mathfrak{w}, \mathfrak{s})\mathfrak{B}(\mathfrak{w}) d\mathfrak{w} \leq [\mathcal{H}_*(\mathfrak{b}, \mathfrak{s}) + \mathcal{H}_*(z, \mathfrak{s})] \int_0^1 \varpi\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z) d\varpi, \\ &\frac{1}{z-\mathfrak{b}} \int_{\mathfrak{b}}^{z} \mathcal{H}^*(\mathfrak{w}, \mathfrak{s})\mathfrak{B}(\mathfrak{w}) d\mathfrak{w} \geq [\mathcal{H}^*(\mathfrak{b}, \mathfrak{s}) + \mathcal{H}^*(z, \mathfrak{s})] \int_0^1 \varpi\mathfrak{B}((1-\varpi)\mathfrak{b} + \varpi z) d\varpi, \end{aligned}$$

that is

$$\left[\frac{1}{z-\mathfrak{b}}\int_{\mathfrak{b}}^{z}\mathcal{H}_*(\mathfrak{w},\mathfrak{s})\mathfrak{B}(\mathfrak{w})d\mathfrak{w},\ \frac{1}{z-\mathfrak{b}}\int_{\mathfrak{b}}^{z}\mathcal{H}^*(\mathfrak{w},\mathfrak{s})\mathfrak{B}(\mathfrak{w})d\mathfrak{w}\right]$$
$$\supseteq_I [\mathcal{H}_*(\mathfrak{b},\ \mathfrak{s})+\mathcal{H}_*(z,\ \mathfrak{s}),\ \mathcal{H}^*(\mathfrak{b},\ \mathfrak{s})+$$
$$\mathcal{H}^*(z,\ \mathfrak{s})]\int_0^1 \varpi\mathfrak{B}((1-\varpi)\mathfrak{b}+\varpi z)d\varpi,$$

hence

$$\frac{1}{z-\mathfrak{b}}\odot(FA)\int_{\mathfrak{b}}^{z}\widetilde{\mathcal{H}}(\mathfrak{w})\odot\mathfrak{B}(\mathfrak{w})d\mathfrak{w}\supseteq_{\mathbb{F}}\left[\widetilde{\mathcal{H}}(\mathfrak{b})\oplus\widetilde{\mathcal{H}}(z)\right]\odot\int_0^1\varpi\mathfrak{B}((1-\varpi)\mathfrak{b}+\varpi z)d\varpi.$$

Next, we construct first *HH*-Fejér inequality for $\mathcal{UD}$-convex $\mathcal{FNVM}$, which generalizes first *HH*-Fejér inequalities for classical convex mapping. □

Theorem 10. *Let $\widetilde{\mathcal{H}}:[\mathfrak{b},\ z]\to{}_0$ be a $\mathcal{UD}$-convex $\mathcal{FNVM}$ with $\mathfrak{b}<z$, whose $\mathfrak{s}$-cuts define the family of $I\mathcal{VM}s$ $\mathcal{H}_{\mathfrak{s}}:[\mathfrak{b},\ z]\subset\mathbb{R}\to\mathcal{X}_{\varrho}^{+}$ are given by $\mathcal{H}_{\mathfrak{s}}(\mathfrak{w})=[\mathcal{H}_*(\mathfrak{w},\mathfrak{s}),\ \mathcal{H}^*(\mathfrak{w},\mathfrak{s})]$ for all $\mathfrak{w}\in[\mathfrak{b},\ z]$ and for all $\mathfrak{s}\in[0,\ 1]$. If $\widetilde{\mathcal{H}}\in\mathcal{FA}_{([\mathfrak{b},\ z],\ \mathfrak{s})}$ and $\mathfrak{B}:[\mathfrak{b},\ z]\to\mathbb{R}$, $\mathfrak{B}(\mathfrak{w})\geq 0$, symmetric with respect to $\frac{\mathfrak{b}+z}{2}$, and $\int_{\mathfrak{b}}^{z}\mathfrak{B}(\mathfrak{w})d\mathfrak{w}>0$, then*

$$\widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+z}{2}\right)\supseteq_{\mathbb{F}}\frac{1}{\int_{\mathfrak{b}}^{z}\mathfrak{B}(\mathfrak{w})d\mathfrak{w}}\odot(FA)\int_{\mathfrak{b}}^{z}\widetilde{\mathcal{H}}(\mathfrak{w})\odot\mathfrak{B}(\mathfrak{w})d\mathfrak{w}. \tag{40}$$

Proof. Since $\widetilde{\mathcal{H}}$ is a $\mathcal{UD}$-convex, then for $\mathfrak{s}\in[0,\ 1]$, we have

$$\begin{aligned}\mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right)&\leq\frac{1}{2}(\mathcal{H}_*(\varpi\mathfrak{b}+(1-\varpi)z,\ \mathfrak{s})+\mathcal{H}_*((1-\varpi)\mathfrak{b}+\varpi z,\ \mathfrak{s})),\\ \mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right)&\geq\frac{1}{2}(\mathcal{H}^*(\varpi\mathfrak{b}+(1-\varpi)z,\ \mathfrak{s})+\mathcal{H}^*((1-\varpi)\mathfrak{b}+\varpi z,\ \mathfrak{s})),\end{aligned}\tag{41}$$

Since $\mathfrak{B}(\varpi\mathfrak{b}+(1-\varpi)z)=\mathfrak{B}((1-\varpi)\mathfrak{b}+\varpi z)$, then by multiplying (41) by $\mathfrak{B}((1-\varpi)\mathfrak{b}+\varpi z)$ and integrating it with respect to ϖ over $[0,\ 1]$, we obtain

$$\begin{aligned}&\mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right)\int_0^1\mathfrak{B}((1-\varpi)\mathfrak{b}+\varpi z)d\varpi\\ &\leq\frac{1}{2}\left(\begin{array}{c}\int_0^1\mathcal{H}_*(\varpi\mathfrak{b}+(1-\varpi)z,\ \mathfrak{s})\mathfrak{B}(\varpi\mathfrak{b}+(1-\varpi)z)d\varpi\\ +\int_0^1\mathcal{H}_*((1-\varpi)\mathfrak{b}+\varpi z,\ \mathfrak{s})\mathfrak{B}((1-\varpi)\mathfrak{b}+\varpi z)d\varpi\end{array}\right),\\ &\mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right)\int_0^1\mathfrak{B}((1-\varpi)\mathfrak{b}+\varpi z)d\varpi\\ &\geq\frac{1}{2}\left(\begin{array}{c}\int_0^1\mathcal{H}^*(\varpi\mathfrak{b}+(1-\varpi)z,\ \mathfrak{s})\mathfrak{B}(\varpi\mathfrak{b}+(1-\varpi)z)d\varpi\\ +\int_0^1\mathcal{H}^*((1-\varpi)\mathfrak{b}+\varpi z,\ \mathfrak{s})\mathfrak{B}((1-\varpi)\mathfrak{b}+\varpi z)d\varpi\end{array}\right).\end{aligned}\tag{42}$$

Since

$$\begin{aligned}&\int_0^1\mathcal{H}_*(\varpi\mathfrak{b}+(1-\varpi)z,\ \mathfrak{s})\mathfrak{B}(\varpi\mathfrak{b}+(1-\varpi;)z)d\varpi\\ &=\int_0^1\mathcal{H}_*((1-\varpi)\mathfrak{b}+\varpi z,\ \mathfrak{s})\mathfrak{B}((1-\varpi)\mathfrak{b}+\varpi z)d\varpi=\tfrac{1}{z-\mathfrak{b}}\int_{\mathfrak{b}}^{z}\mathcal{H}_*(\mathfrak{w},\mathfrak{s})\mathfrak{B}(\mathfrak{w})d\mathfrak{w}\\ &\int_0^1\mathcal{H}^*((1-\varpi)\mathfrak{b}+\varpi z,\ \mathfrak{s})\mathfrak{B}((1-\varpi)\mathfrak{b}+\varpi z)d\varpi\\ &=\int_0^1\mathcal{H}^*(\varpi\mathfrak{b}+(1-\varpi)z,\ \mathfrak{s})\mathfrak{B}(\varpi\mathfrak{b}+(1-\varpi)z)d\varpi=\tfrac{1}{z-\mathfrak{b}}\int_{\mathfrak{b}}^{z}\mathcal{H}^*(\mathfrak{w},\mathfrak{s})\mathfrak{B}(\mathfrak{w})d\mathfrak{w}.\end{aligned}\tag{43}$$

Then from (43), we have

$$\begin{aligned}\mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right)&\leq\frac{1}{\int_{\mathfrak{b}}^{z}\mathfrak{B}(\mathfrak{w})d\mathfrak{w}}\int_{\mathfrak{b}}^{z}\mathcal{H}_*(\mathfrak{w},\mathfrak{s})\mathfrak{B}(\mathfrak{w})d\mathfrak{w},\\ \mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{s}\right)&\geq\frac{1}{\int_{\mathfrak{b}}^{z}\mathfrak{B}(\mathfrak{w})d\mathfrak{w}}\int_{\mathfrak{b}}^{z}\mathcal{H}^*(\mathfrak{w},\mathfrak{s})\mathfrak{B}(\mathfrak{w})d\mathfrak{w},\end{aligned}$$

from which, we have

$$\left[\mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{d}\right),\ \mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2},\ \mathfrak{d}\right)\right]$$
$$\supseteq_I \frac{1}{\int_{\mathfrak{b}}^{z}\mathfrak{B}(\mathfrak{w})d\mathfrak{w}}\left[\int_{\mathfrak{b}}^{z}\mathcal{H}_*(\mathfrak{w},\mathfrak{d})\mathfrak{B}(\mathfrak{w})d\mathfrak{w},\ \int_{\mathfrak{b}}^{z}\mathcal{H}^*(\mathfrak{w},\mathfrak{d})\mathfrak{B}(\mathfrak{w})d\mathfrak{w}\right],$$

that is

$$\widetilde{\mathcal{H}}\left(\frac{\mathfrak{b}+z}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\int_{\mathfrak{b}}^{z}\mathfrak{B}(\mathfrak{w})d\mathfrak{w}} \odot (FA)\int_{\mathfrak{b}}^{z}\widetilde{\mathcal{H}}(\mathfrak{w})\odot\mathfrak{B}(\mathfrak{w})d\mathfrak{w}.$$

This completes the proof. □

Remark 5. *From Theorem 9 and Theorem 10, we clearly see that:*

If $\mathcal{W}(\mathfrak{w}) = 1$, then we acquire the inequality (23).

If $\mathcal{H}$ is lower $\mathcal{UD}$-convex $\mathcal{FNVM}$ on $[\mathfrak{b}, z]$, then we acquire the following coming inequality, see [90]*:*

$$\mathcal{H}\left(\frac{\mathfrak{b}+z}{2}\right) \leq_{\mathbb{F}} \frac{1}{\int_{\mathfrak{b}}^{z}\mathcal{W}(\mathfrak{w})d\mathfrak{w}} \odot (FA)\int_{\mathfrak{b}}^{z}\mathcal{H}(\mathfrak{w})\odot\mathcal{W}(\mathfrak{w})d\mathfrak{w} \leq_{\mathbb{F}} \frac{\mathcal{H}(\mathfrak{b})\oplus\mathcal{H}(z)}{2}. \tag{44}$$

If $\mathcal{H}$ is lower $\mathcal{UD}$-convex $\mathcal{FNVM}$ on $[\mathfrak{b}, z]$ with $\mathfrak{d}\ =$, then from (35) and (40) we acquire the following coming inequality, see [99]*:*

$$\mathcal{H}\left(\frac{\mathfrak{b}+z}{2}\right) \leq_I \frac{1}{z-\mathfrak{b}}\ (IA)\int_{\mathfrak{b}}^{z}\mathcal{H}(\mathfrak{w})d\mathfrak{w} \leq_I \frac{\mathcal{H}(\mathfrak{b})+\mathcal{H}(z)}{2}. \tag{45}$$

If $\mathcal{H}$ is lower $\mathcal{UD}$-convex $\mathcal{FNVM}$ on $[\mathfrak{b}, z]$ with $\mathfrak{d}\ =$, then from (35) and (40) we acquire the following coming inequality, see [99]*:*

$$\mathcal{H}\left(\frac{\mathfrak{b}+z}{2}\right) \leq_I \frac{1}{\int_{\mathfrak{b}}^{z}\mathcal{W}(\mathfrak{w})d\mathfrak{w}}\ (IA)\int_{\mathfrak{b}}^{z}\mathcal{H}(\mathfrak{w})\mathcal{W}(\mathfrak{w})d\mathfrak{w} \leq_I \frac{\mathcal{H}(\mathfrak{b})+\mathcal{H}(z)}{2}. \tag{46}$$

Let $\mathfrak{d}\ =$. Then from (35) and (40), we acquire the following inequality, see [56]*:*

$$\mathcal{H}\left(\frac{\mathfrak{b}+z}{2}\right) \supseteq \frac{1}{\int_{\mathfrak{b}}^{z}\mathcal{W}(\mathfrak{w})d\mathfrak{w}}\ (IA)\int_{\mathfrak{b}}^{z}\mathcal{H}(\mathfrak{w})\mathcal{W}(\mathfrak{w})d\mathfrak{w} \supseteq \frac{\mathcal{H}(\mathfrak{b})+\mathcal{H}(z)}{2}. \tag{47}$$

Let $\mathfrak{d} = 1$ and $\mathcal{H}_*(\mathfrak{w},\ \mathfrak{d}) = \mathcal{H}^*(\mathfrak{w},\ \mathfrak{d})$. Then, from(35) and (40), we obtain the following classical Fejér inequality:

$$\mathcal{H}\left(\frac{\mathfrak{b}+z}{2}\right) \leq \frac{1}{\int_{\mathfrak{b}}^{z}\mathcal{W}(\mathfrak{w})d\mathfrak{w}}\int_{\mathfrak{b}}^{z}\mathcal{H}(\mathfrak{w})\mathcal{W}(\mathfrak{w})d\mathfrak{w} \leq \frac{\mathcal{H}(\mathfrak{b})+\mathcal{H}(z)}{2}. \tag{48}$$

Example 5. *We consider the $\mathcal{FNVM}$ $\mathcal{H} : [0,\ 2] \to {}_I$ defined by*

$$\mathcal{H}(\mathfrak{w})(\theta) = \begin{cases} \dfrac{\theta - 2 + \mathfrak{w}^{\frac{1}{2}}}{\frac{3}{2} - 2 - \mathfrak{w}^{\frac{1}{2}}} & \theta \in \left[2 - \mathfrak{w}^{\frac{1}{2}},\ \frac{3}{2}\right], \\ \dfrac{2 + \mathfrak{w}^{\frac{1}{2}} - \theta}{2 + \mathfrak{w}^{\frac{1}{2}} - \frac{3}{2}} & \theta \in \left(\frac{3}{2},\ 2 + \mathfrak{w}^{\frac{1}{2}}\right], \\ 0 & \text{otherwise}, \end{cases}$$

Then, for each $\vartheta \in [0,\ 1]$, *we have* $\mathcal{H}_{\vartheta}(\mathfrak{w}) = \left[(1-\vartheta)\left(2-\mathfrak{w}^{\frac{1}{2}}\right)+\frac{3}{2}\vartheta, (1+\vartheta)\left(2+\mathfrak{w}^{\frac{1}{2}}\right)+\frac{3}{2}\vartheta\right]$. *Since end point mappings* $\mathcal{H}_*(\mathfrak{w},\vartheta)$, *and* $\mathcal{H}^*(\mathfrak{w},\vartheta)$ *are convex and concave mappings, respectively, for each* $\vartheta \in [0,\ 1]$, *then* $\mathcal{H}(\mathfrak{w})$ *is* $\mathcal{UD}$*-convex* $\mathcal{FNVM}$. *If*

$$\mathfrak{B}(\mathfrak{w}) = \begin{cases} \sqrt{\mathfrak{w}}, & \sigma \in [0,1], \\ \sqrt{2-\mathfrak{w}}, & \sigma \in (1,\ 2], \end{cases}$$

then $\mathfrak{B}(2-\mathfrak{w}) = \mathfrak{B}(\mathfrak{w}) \geq 0$, *for all* $\mathfrak{w} \in [0,\ 2]$.

Since $\mathcal{H}_*(\mathfrak{w},\vartheta) = (1-\vartheta)\left(2-\mathfrak{w}^{\frac{1}{2}}\right)+\frac{3}{2}\vartheta$ *and* $\mathcal{H}^*(\mathfrak{w},\vartheta) = (1+\vartheta)\left(2+\mathfrak{w}^{\frac{1}{2}}\right)+\frac{3}{2}\vartheta$. *Now we compute the following*:

$$\begin{aligned} \frac{1}{z-\mathfrak{b}} \textstyle\int_{\mathfrak{b}}^{z}[\mathcal{H}_*(\mathfrak{w},\vartheta)]\mathfrak{B}(\mathfrak{w})d\mathfrak{w} &= \frac{1}{2}\textstyle\int_0^2[\mathcal{H}_*(\mathfrak{w},\vartheta)]\mathfrak{B}(\mathfrak{w})d\mathfrak{w} \\ &= \frac{1}{2}\textstyle\int_0^1[\mathcal{H}_*(\mathfrak{w},\vartheta)]\mathfrak{B}(\mathfrak{w})d\mathfrak{w} + \frac{1}{2}\int_1^2 \mathcal{H}_*(\mathfrak{w},\vartheta)\mathfrak{B}(\mathfrak{w})d\mathfrak{w}, \\ \frac{1}{z-\mathfrak{b}} \textstyle\int_{\mathfrak{b}}^{z}[\mathcal{H}^*(\mathfrak{w},\vartheta)]\mathfrak{B}(\mathfrak{w})d\mathfrak{w} &= \frac{1}{2}\textstyle\int_0^2[\mathcal{H}^*(\mathfrak{w},\vartheta)]\mathfrak{B}(\mathfrak{w})d\mathfrak{w} \\ &= \frac{1}{2}\textstyle\int_0^1[\mathcal{H}^*(\mathfrak{w},\vartheta)]\mathfrak{B}(\mathfrak{w})d\mathfrak{w} + \frac{1}{2}\int_1^2 \mathcal{H}^*(\mathfrak{w},\vartheta)\mathfrak{B}(\mathfrak{w})d\mathfrak{w}, \end{aligned}$$

$$\begin{aligned} &= \tfrac{1}{2}\textstyle\int_0^1\left[(1-\vartheta)\left(2-\mathfrak{w}^{\frac{1}{2}}\right)+\frac{3}{2}\vartheta\right](\sqrt{\mathfrak{w}})d\mathfrak{w} + \frac{1}{2}\int_1^2\left[(1-\vartheta)\left(2-\mathfrak{w}^{\frac{1}{2}}\right)+\frac{3}{2}\vartheta\right](\sqrt{2-\mathfrak{w}})d\mathfrak{w} \\ &= \tfrac{1}{4}\left[\tfrac{13}{3}-\tfrac{\pi}{2}\right]+\vartheta\left[\tfrac{\pi}{8}-\tfrac{1}{12}\right], \\ &= \tfrac{1}{2}\textstyle\int_0^1\left[(1+\vartheta)\left(2+\mathfrak{w}^{\frac{1}{2}}\right)+\frac{3}{2}\vartheta\right](\sqrt{\mathfrak{w}})d\mathfrak{w} + \frac{1}{2}\int_1^2\left[(1+\vartheta)\left(2+\mathfrak{w}^{\frac{1}{2}}\right)+\frac{3}{2}\vartheta\right](\sqrt{2-\mathfrak{w}})d\mathfrak{w} \\ &= \tfrac{1}{4}\left[\tfrac{19}{3}+\tfrac{\pi}{2}\right]+\vartheta\left[\tfrac{\pi}{8}+\tfrac{31}{12}\right]. \end{aligned} \tag{49}$$

And

$$\begin{aligned} &[\mathcal{H}_*(\mathfrak{b},\ \vartheta)+\mathcal{H}_*(z,\ \vartheta)]\textstyle\int_0^1 \varpi B((1-\varpi)\mathfrak{b}+\varpi z)d\varpi \\ &= \left[4(1-\vartheta)-\sqrt{2}(1-\vartheta)+3\vartheta\right]\left[\textstyle\int_0^{\frac{1}{2}}\varpi\sqrt{2\varpi}d\varpi+\int_{\frac{1}{2}}^1\varpi\sqrt{2(1-\varpi)}d\varpi\right] \\ &= \tfrac{1}{3}\left(4(1-\vartheta)-\sqrt{2}(1-\vartheta)+3\vartheta\right), \\ &[\mathcal{H}^*(\mathfrak{b},\ \vartheta)+\mathcal{H}^*(z,\ \vartheta)]\textstyle\int_0^1 \varpi\mathfrak{B}((1-\varpi)\mathfrak{b}+\varpi z)d\varpi \\ &= \left[4(1+\vartheta)+\sqrt{2}(1+\vartheta)+3\vartheta\right]\left[\textstyle\int_0^{\frac{1}{2}}\varpi\sqrt{2\varpi}d\varpi+\int_{\frac{1}{2}}^1\varpi\sqrt{2(1-\varpi)}d\varpi\right] \\ &= \tfrac{1}{3}\left(4(1+\vartheta)+\sqrt{2}(1+\vartheta)+3\vartheta\right). \end{aligned} \tag{50}$$

From (49) and (50), we have

$$\left[\frac{1}{4}\left[\frac{13}{3}-\frac{\pi}{2}\right]+\vartheta\left[\frac{\pi}{4}-\frac{7}{6}\right],\ \frac{1}{4}\left[\frac{19}{3}+\frac{\pi}{2}\right]+\vartheta\left[\frac{\pi}{4}+\frac{25}{6}\right]\right]$$

$$\supseteq_I \left[\frac{1}{3}\left(4(1-\vartheta)-\sqrt{2}(1-\vartheta)+3\vartheta\right),\ \frac{1}{3}\left(4(1+\vartheta)+\sqrt{2}(1+\vartheta)+3\vartheta\right)\right], \text{for all } \vartheta \in [0,\ 1].$$

Hence, Theorem 9 is verified.

For Theorem 10, we have

$$\mathcal{H}_*\left(\frac{\mathfrak{b}+z}{2},\ \vartheta\right) = \mathcal{H}_*(1,\ \vartheta) = \frac{2+\vartheta}{2},$$
$$\mathcal{H}^*\left(\frac{\mathfrak{b}+z}{2},\ \vartheta\right) = \mathcal{H}^*(1,\ \vartheta) = \frac{3(2+3\vartheta)}{2},$$

$$\int_{\mathfrak{b}}^{z}\mathfrak{B}(\mathfrak{w})d\mathfrak{w} = \int_0^1\sqrt{\mathfrak{w}}d\mathfrak{w}+\int_1^2\sqrt{2-\mathfrak{w}}d\mathfrak{w} = \frac{4}{3}, \tag{51}$$

$$\begin{aligned} \frac{1}{\int_{\mathfrak{b}}^{z}\mathfrak{B}(\mathfrak{w})d\mathfrak{w}}\int_{\mathfrak{b}}^{z}\mathcal{H}_*(\mathfrak{w},\vartheta)\mathfrak{B}(\mathfrak{w})d\mathfrak{w} &= \frac{3}{8}\left[\frac{13}{3}-\frac{\pi}{2}\right]+\frac{3\vartheta}{2}\left[\frac{\pi}{8}-\frac{1}{12}\right], \\ \frac{1}{\int_{\mathfrak{b}}^{z}\mathfrak{B}(\mathfrak{w})d\mathfrak{w}}\int_{\mathfrak{b}}^{z}\mathcal{H}^*(\mathfrak{w},\vartheta)\mathfrak{B}(\mathfrak{w})d\mathfrak{w} &= \frac{3}{8}\left[\frac{19}{3}+\frac{\pi}{2}\right]+\frac{3\vartheta}{2}\left[\frac{\pi}{8}+\frac{31}{12}\right]. \end{aligned} \tag{52}$$

From (51) and (52), we have

$$\left[\frac{2+\vartheta}{2}, \frac{3(2+3\vartheta)}{2}\right] \supseteq_I \left[\frac{3}{8}\left[\frac{13}{3}-\frac{\pi}{2}\right]+\frac{3\vartheta}{2}\left[\frac{\pi}{8}-\frac{1}{12}\right], \frac{3}{8}\left[\frac{19}{3}+\frac{\pi}{2}\right]+\frac{3\vartheta}{2}\left[\frac{\pi}{8}+\frac{31}{12}\right]\right].$$

Hence, Theorem 10 has been verified.

4. Conclusions

This paper provides the introduced class $\mathcal{UD}$-convex concept for $\mathcal{FNVM}$s. The H.H. and Jensen-type inequalities were developed utilizing this idea and a fuzzy-inclusion relation. This study expands on several recent findings made by Zhao et al. [56,57] and the writers who came after them, Refs. [61,62]. Furthermore, some nontrivial cases are provided to verify our primary conclusions' accuracy. In the future, it will be fascinating to look into how analogous inequalities are established for other convexity types and by employing various integral operators. Our study of interval integral operator-type integral inequalities will broaden their practical applications because integral operators are widely used in engineering technology, such as various forms of mathematical modeling, and because different integral operators are suitable for different forms of practical problems. Convex optimization theory may take a new turn as a result of this idea. Other researchers working on a range of scientific subjects may probably find the idea useful.

Author Contributions: Conceptualization, M.B.K.; methodology, M.B.K.; validation, A.M.A.; formal analysis, H.A.O. and A.M.A.; investigation, M.B.K.; resources, H.A.O. and A.M.A.; data curation, M.G.V.; writing—original draft preparation, M.B.K.; writing—review and editing, M.B.K. and A.M.A.; visualization, M.G.V.; supervision, M.B.K. and L.A.; project administration, M.B.K. and L.A.; funding acquisition, A.M.A. and M.G.V. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the Rector, COMSATS University Islamabad, Islamabad, Pakistan. The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant Code: 22UQU4330052DSR11. This research was also supported by Office of Research Management, Universiti Malaysia Terengganu, Malaysia.

Conflicts of Interest: The authors declare no conflict of interest.

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