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Some Certain Fuzzy Fractional Inequalities for Up and Down ℏ-Pre-Invex via Fuzzy-Number Valued Mappings

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Abstract: In this study, we apply a recently developed idea of up and down fuzzy-ordered relations between two fuzzy numbers. Here, we consider fuzzy Riemann–Liouville fractional integrals to establish the Hermite–Hadamard-, Fejér-, and Pachpatte-type inequalities. We estimate fuzzy fractional inequalities for a newly introduced class of ℏ-preinvexity over fuzzy-number valued settings. For the first time, such inequalities involving up and down fuzzy-ordered functions are proven using the fuzzy fractional operator. The stated inequalities are supported by a few numerical examples that will be helpful to validate our main results.

Keywords: up and down ℏ-pre-invex fuzzy-interval-valued function; fuzzy Riemann–Liouville fractional integral; Hermite–Hadamard-type inequality; Hermite–Hadamard–Fejér-type inequality

MSC: 26A33; 26A51; 26D10

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1. Introduction

In the subject of inequality theory, researchers have established hundreds of inequality types, which have various applications in mathematical analysis and applied mathematics. Two inequalities that stand out among these types of inequalities in terms of their aesthetic forms, applications, and functioning will be introduced first. A specific function class with applications in statistics, convex programming, numerical analysis, and many other domains are one of the fundamental ideas employed in much of the research on the subject of inequalities. This article provides information on the Hermite–Hadamard inequality, which is produced by employing convex functions and has a very complex structure with inequalities.

In the classical approach, a real-valued mapping $\Psi: K \to \mathbb{R}$ is called convex if

$$\Psi(\mathfrak{s}\mu + (1-\mathfrak{s})z) \le \mathfrak{s}\Psi(\mu) + (1-\mathfrak{s})\Psi(z), \tag{1}$$

for all $\mu,\ z \in K,\ \mathfrak{s} \in [0,\ 1]$, where K is a convex set.

The HH-inequality [1,2] for convex mapping $\Psi : K \to \mathbb{R}$ on an interval $K = [\zeta, z]$ is

$$\Psi\left(\frac{\zeta + z}{2}\right) \le \frac{1}{v - \zeta} \int_{\zeta}^{z} \Psi(\mu) d\mu \le \frac{\Psi(\zeta) + \Psi(z)}{2}, \tag{2}$$

for all $\zeta,\ z \in K$.

Fejér considered the major generalizations of the HH-inequality, which is known as the HH–Fejér inequality. The following is a presentation of the Hermite–Hadamard–Fejér inequality, which has been proven using a weight function and is the generic form of the inequality (2) (see [3]).

Let $\Psi : K \to \mathbb{R}$ be a convex mapping on a convex set K and $\zeta,\ v \in K$ with $\zeta \leq v$. Then,

$$\Psi\left(\frac{\zeta + z}{2}\right) \leq \frac{1}{\int_{\zeta}^{z} \mathfrak{Z}(\mu) d\mu} \int_{\zeta}^{z} \Psi(\mu)\mathfrak{Z}(\mu) d\mu \leq \frac{\Psi(\zeta) + \Psi(z)}{2} \int_{\zeta}^{z} \mathfrak{Z}(\mu)) d\mu \tag{3}$$

If $\mathfrak{Z}(\mu) = 1$, then we obtain (2) from (3). With the support of inequality, a large number of inequalities can be found using the particular symmetric mapping $\mathfrak{Z}(\mu)$ for convex mappings (3). By taking into account various convex function types, various derivative and integral operators, new techniques, and other spaces, researchers working on these two well-known inequalities have generated generalizations, extensions, improvements, and iterations, see [4–13].

The Hermite–Hadamard inequality has been proposed for operator convex and generalized convex functions (see, for example, [14–28]). The Hermite–Hadamard inequalities for the products of two operator preinvex functions were created by Barani [29] in 2015. The Hermite–Hadamard-type inequalities for the operator h-preinvex functions were established by Wang and Sun [28] in 2017. The Hermite–Hadamard-type inequalities for the operator (p, h)-convex functions were proposed in 2022 by Omrani et al. [30]. Research has expanded because of the variety and uses of Hermite–Hadamard inequalities (see, for example, [31–45]).

A fundamental idea in applied sciences and mathematics is fractional calculus. Fractional calculus is actively used by researchers to address a wide range of real-world problems. In the modern era, fractional analysis and inequality theory have coevolved. Over the years, much attention has been paid to various fractional versions of inequalities of the Hermite–Hadamard, Fejer, Ostrowski, and Pachpatte types, [46,47]. In addition to the aforementioned inequalities, several researchers have employed the Riemann–Liouville fractional integral operators to examine the Ostrowski inequality (see [48]), Simpson-type inequality (see [49]), and Hermite–Hadamard–Mercer inequalities (see [50]). The Hermite–Hadamard inequality and its Fejér analog were investigated by Katugampola et al. using fractional integral operators of the Katugampola type (see [51]). In order to illustrate alternate versions of the Hermite–Hadamard inequality, Fernandez and Mohammed (see [52]) used Atangana–Baleanu fractional operators, and Noor et al. (see [53]) demonstrated Hermite–Hadamard-type inequalities. The Caputo–Fabrizio fractional integrals were also used to study the Hermite–Hadamard inequality (see [54,55]). Butt et al. introduced new iterations of fractal-based Jensen- and Jensen–Mercer-type inequalities (see [56]). For log-preinvex [57] and harmonically convex functions [58], new fractional forms of Hermite–Hadamard–Mercer- and Pachpatte–Mercer-type inclusions have been created. Hermite–Hadamard inequalities have been further generalized for convex interval-valued functions [59] and convex fuzzy interval-valued functions [60]. For more information, see [61–67].

Each described notion and its definitions, despite initially seeming to be comparable, are wholly different. In the context of interval-valued analysis, numerous academics have coupled a variety of convex functions with integral inequalities, leading to a number of notable findings. Román-Flores established Minkowski-type inequalities (see [68]), Chalco-Cano researched Ostrowski-type inequalities (see [69]), and Opial investigated Opial-type inequalities (see [70]). Zhao et al. (see [71]) also established a refinement of the Hermite–Hadamard inequality and suggested the interval-valued h-convex function. Zhang et al. (see [72]) and Costa et al. (see [73]) presented left–right interval-valued and fuzzy interval-valued functions, respectively, to demonstrate Jensen's inequalities. Recently, Khan et al. introduced the novel versions of inequalities that are known as fuzzy fractional Hermite–Hadamard, Fejér-, and Pachpatte-type inequalities for $\mathcal{UD}$-convex $\mathcal{F}\mathfrak{I}\mathcal{VM}$ via fuzzy left and right Riemann–Liouville fractional integrals. Many more inequalities have been introduced related to real-valued, interval-valued, and fuzzy-number valued mappings, (see [74–82]).

The article is structured as follows: We cover the necessary prerequisites and relevant details for the accompanying integral inequalities and interval-valued analysis in Section 1. Preinvexity and fuzzy UD-order functions are concepts that are explained in Section 2. We derive the Hermite–Hadamard and any applicable inequalities for the h-preinvex functions in fuzzy-number valued settings in Section 3. We offer a brief conclusion in Section 4 and go over a number of unanswered research problems that are relevant to the results of this work.

2. Preliminaries

We recall a few definitions that can be found in the literature and will be relevant in the follow-up.

Let us consider that $\mathbb{X}_o$ is the space of all closed and bounded intervals of $\mathbb{R}$, and that $\aleph \in \mathbb{X}_o$ is given by

$$\aleph = [\aleph_*,\ \aleph^*] = \{\mathfrak{w} \in \mathbb{R} |\ \aleph_* \leq \mathfrak{w} \leq \aleph^*\},\ (\aleph_*,\ \aleph^* \in \mathbb{R}) \tag{4}$$

If $\aleph_* = \aleph^*$, then $\aleph$ is degenerate. In the follow-up, all intervals are considered non-degenerate. If $\aleph_* \geq 0$, then $\aleph$ is positive. We denote by $\mathbb{X}_o^+ = \{[\aleph_*,\ \aleph^*] : [\aleph_*,\ \aleph^*] \in \mathbb{X}_o$ and $\aleph_* \geq 0\}$, the set of all positive intervals.

Let $\mathfrak{u} \in \mathbb{R}$ and $\mathfrak{u}{\cdot}\aleph$ be given by

$$\mathfrak{u}{\cdot}\aleph = \begin{cases} [\mathfrak{u}\aleph_*,\ \mathfrak{u}\aleph^*] & \text{if } \mathfrak{u} > 0, \\ \{0\} & \text{if } \mathfrak{u} = 0, \\ [\mathfrak{u}\aleph^*, \mathfrak{u}\aleph_*] & \text{if } \mathfrak{u} < 0. \end{cases} \tag{5}$$

We consider the Minkowski sum, $\aleph + \omega$, product, $\aleph \times \omega$, and difference, $\omega - \aleph$, for $\aleph, \omega \in \mathbb{X}_o$, as

$$[\omega_*,\ \omega^*] + [\aleph_*,\ \aleph^*] = [\omega_* + \aleph_*,\ \ \omega^* + \aleph^*], \tag{6}$$

$$[\omega_*,\ \omega^*] \times [\aleph_*,\ \aleph^*] = [\min\{\omega_*\aleph_*,\ \omega^*\aleph_*,\ \omega_*\aleph^*,\ \omega^*\aleph^*\},\ \max\{\omega_*\aleph_*,\ \omega^*\aleph_*,\ \omega_*\aleph^*,\ \omega^*\aleph^*\}] \tag{7}$$

$$[\omega_*,\ \omega^*] - [\aleph_*,\ \aleph^*] = [\omega_* - \aleph^*,\ \ \omega^* - \aleph_*]. \tag{8}$$

Remark 1. *(i) For the given* $[\omega_*,\ \omega^*],\ [\aleph_*,\ \aleph^*] \in \mathbb{R}_I$*, the relation "* $\supseteq_I$ *" is defined on* $\mathbb{R}_I$ *by*

$$[\aleph_*,\ \aleph^*] \supseteq_I [\omega_*,\ \omega^*] \text{ if and only if } \aleph_* \leq \omega_*,\ \ \omega^* \leq \aleph^*, \tag{9}$$

for all $[\omega_*,\ \omega^*],\ [\aleph_*,\ \aleph^*] \in \mathbb{R}_I$*, is a partial interval inclusion relation.*

Moreover, $[\aleph_*,\ \aleph^*] \supseteq_I [\omega_*,\ \omega^*]$ coincides with $[\aleph_*,\ \aleph^*] \supseteq [\omega_*,\ \omega^*]$ on $\mathbb{R}_I$. The relation " $\supseteq_I$ " is of UD order [72].

(ii) For the given $[\omega_*,\ \omega^*],\ [\aleph_*,\ \aleph^*] \in \mathbb{R}_I$, the relation " $\leq_I$ " is defined on $\mathbb{R}_I$ by $[\omega_*,\ \omega^*] \leq_I [\aleph_*,\ \aleph^*]$ if and only if $\omega_* \leq \aleph_*,\ \omega^* \leq \aleph^*$ or $\omega_* \leq \aleph_*,\ \omega^* < \aleph^*$, is a partial interval order relation. Plus, we have that $[\omega_*,\ \omega^*] \leq_I [\aleph_*,\ \aleph^*]$ coincides with $[\omega_*,\ \omega^*] \leq [\aleph_*,\ \aleph^*]$ on $\mathbb{R}_I$. The relation " $\leq_I$ " is of left and right (LR) type [72,73].

Given the intervals $[\omega_*,\ \omega^*],\ [\aleph_*,\ \aleph^*] \in \mathbb{X}_o$, their Hausdorff–Pompeiu distance is

$$d_H([\omega_*,\ \omega^*],\ [\aleph_*,\ \aleph^*]) = \max\{|\omega_* - \aleph_*|,\ |\omega^* - \aleph^*|\}. \tag{10}$$

We have that $(\mathbb{X}_o, d_H)$ is a complete metric space [77,79,82].

Definition 1. *[76] A fuzzy subset L of* $\mathbb{R}$ *is a mapping* $\widetilde{\aleph} : \mathbb{R} \to [0,1]$*, denoting membership mapping of L. We adopt the symbol £ to represent the set of all fuzzy subsets of* $\mathbb{R}$*.*

Let us consider $\widetilde{\aleph} \in £$. If the following properties hold, then $\widetilde{\aleph}$ is a fuzzy number:

1. $\widetilde{\aleph}$ *is normal if there exists* $\mathfrak{w} \in \mathbb{R}$ *and* $\widetilde{\aleph}(\mathfrak{w}) = 1$;
2. $\widetilde{\aleph}$ *is upper semi-continuous on* $\mathbb{R}$ *if for a* $\mathfrak{w} \in \mathbb{R}$ *there exist* $\varepsilon > 0$ *and* $\delta > 0$ *yielding* $\widetilde{\aleph}(\mathfrak{w}) - \widetilde{\aleph}(y) < \varepsilon$ *for all* $y \in \mathbb{R}$ *with* $|\mathfrak{w} - y| < \delta$;
3. $\widetilde{\aleph}$ *is fuzzy convex, meaning that* $\widetilde{\aleph}((1-\mathfrak{u})\mathfrak{w} + \mathfrak{u}y) \geq min\left(\widetilde{\aleph}(\mathfrak{w}),\ \widetilde{\aleph}(y)\right)$, *for all* $\mathfrak{w}, y \in \mathbb{R}$, *and* $\mathfrak{u} \in [0,\ 1]$;
4. $\widetilde{\aleph}$ *is compactly supported, which means that* $cl\left\{\mathfrak{w} \in \mathbb{R} \middle| \ \widetilde{\aleph}(\mathfrak{w})\right\rangle 0\Big\}$ *is compact.*

The symbol $£_o$ *will be adopted to designate the set of all fuzzy numbers of* $\mathbb{R}$.

Definition 2. *[76,77] For* $\widetilde{\aleph} \in £_o$ *, the* $\mho$ *-level, or* $\mho$*-cut, sets of* $\widetilde{\aleph}$ *are* $\left[\widetilde{\aleph}\right]^{\mho} = \left\{\mathfrak{w} \in \mathbb{R} \middle| \ \widetilde{\aleph}(\mathfrak{w})\right\rangle \mho\Big\}$ *for all* $\mho \in [0,\ 1]$, *and* $\left[\widetilde{\aleph}\right]^{0} = \left\{\mathfrak{w} \in \mathbb{R} \middle| \ \widetilde{\aleph}(\mathfrak{w})\right\rangle 0\Big\}$.

Proposition 1. *[78] Let* $\widetilde{\aleph}, \widetilde{\omega} \in £_o$ *. The relation "* $\leq_{\mathbb{F}}$ *", defined on* $£_o$ *by*

$$\widetilde{\aleph} \leq_{\mathbb{F}} \widetilde{\omega} \text{ when and only when } \left[\widetilde{\aleph}\right]^{\mho} \leq_I [\widetilde{\omega}]^{\mho}, \text{ for every } \mho \in [0,\ 1], \tag{11}$$

is an LR order relation.

Proposition 2. *[70] Let* $\widetilde{\aleph}, \widetilde{\omega} \in £_o$ *. The relation "* $\supseteq_{\mathbb{F}}$ *", defined on* $£_o$ *by*

$$\widetilde{\aleph} \supseteq_{\mathbb{F}} \widetilde{\omega} \text{ when and only when } \left[\widetilde{\aleph}\right]^{\mho} \supseteq_I [\widetilde{\omega}]^{\mho} \text{ for every } \mho \in [0,\ 1], \tag{12}$$

is an UD order relation.

Proof. The proof relies on the UD relation $\supseteq_I$ on $\mathbb{X}_o$. If $\widetilde{\aleph}, \widetilde{\omega} \in £_o$ and $\mho \in \mathbb{R}$, then, for every $\mho \in [0,\ 1]$,

$$\left[\widetilde{\aleph} \oplus \widetilde{\omega}\right]^{\mho} = \left[\widetilde{\aleph}\right]^{\mho} + [\widetilde{\omega}]^{\mho}, \tag{13}$$

$$\left[\widetilde{\aleph} \otimes \widetilde{\omega}\right]^{\mho} = \left[\widetilde{\aleph}\right]^{\mho} \times [\widetilde{\omega}]^{\mho}, \tag{14}$$

$$\left[\mathfrak{u} \odot \widetilde{\aleph}\right]^{\mho} = \mathfrak{u}.\left[\widetilde{\aleph}\right]^{\mho}. \tag{15}$$

result from Equations (5)–(7), respectively. □

Theorem 1. *[77] For* $\widetilde{\aleph},\ \widetilde{\omega} \in £_o$, *the supremum metric*

$$d_\infty\left(\widetilde{\aleph},\ \widetilde{\omega}\right) = \sup_{0 \leq \mho \leq 1} d_H\left(\left[\widetilde{\aleph}\right]^{\mho},\ [\widetilde{\omega}]^{\mho}\right). \tag{16}$$

is a complete metric space, where H stands for the Hausdorff metric on a space of intervals.

Theorem 2. *[77,78] If* $\Psi : [\zeta, \mathrm{z}] \subset \mathbb{R} \to \mathbb{X}_o$ *is an I-V·M satisfying* $\Psi(\mathfrak{w}) = [\Psi_*(\mathfrak{w}),\ \Psi^*(\mathfrak{w})]$, *then* Ψ *is Aumann integrable (IA-integrable) over* $[\zeta, \mathrm{z}]$ *when and only when* $\Psi_*(\mathfrak{w})$ *and* $\Psi^*(\mathfrak{w})$ *are integrable over* $[\zeta, \mathrm{z}]$, *meaning*

$$(IA)\int_{\zeta}^{\mathrm{z}} \Psi(\mathfrak{w})d\mathfrak{w} = \left[\int_{\zeta}^{\mathrm{v}} \Psi_*(\mathfrak{w})d\mathfrak{w},\ \int_{\zeta}^{\mathrm{z}} \Psi^*(\mathfrak{w})d\mathfrak{w}\right]. \tag{17}$$

Definition 3. *[73] Let $\widetilde{\Psi} : \mathbb{I} \subset \mathbb{R} \to \pounds_o$ be a F-N·V·M. The family of I-V·Ms, for every $\mho \in [0,\ 1]$, is $\Psi_\mho : \mathbb{I} \subset \mathbb{R} \to \mathbb{X}_o$ satisfying $\Psi_\mho(\mathfrak{w}) = [\Psi_*(\mathfrak{w}, \mho),\ \Psi^*(\mathfrak{w}, \mho)]$ for every $\mathfrak{w} \in \mathbb{I}$. For every $\mho \in [0,\ 1]$, the lower and upper mappings of $\Psi_\mho$ are the endpoint real-valued mappings $\Psi_*(\cdot, \mho),\ \Psi^*(\cdot, \mho) : \mathbb{I} \to \mathbb{R}$.*

Definition 4. *[73] Let $\widetilde{\Psi} : \mathbb{I} \subset \mathbb{R} \to \pounds_o$ be a F-N·V·M. Then, $\widetilde{\Psi}(\mathfrak{w})$ is continuous at $\mathfrak{w} \in \mathbb{I}$, if for every $\mho \in [0,\ 1]$, $\Psi_\mho(\mathfrak{w})$ is continuous when and only when $\Psi_*(\mathfrak{w}, \mho)$ and $\Psi^*(\mathfrak{w}, \mho)$ are continuous at $\mathfrak{w} \in \mathbb{I}$.*

Definition 5. *[77] Let $\widetilde{\Psi} : [\zeta,\ \mathfrak{z}] \subset \mathbb{R} \to \pounds_o$ be a F-N·V·M. The fuzzy Aumann integral (FA-integral) of $\widetilde{\Psi}$ over $[\zeta,\ \mathfrak{z}]$ is*

$$\left[(FA)\int_\zeta^{\mathfrak{z}} \widetilde{\Psi}(\mathfrak{w})d\mathfrak{w}\right]^\mho = (IA)\int_\zeta^{\mathfrak{z}} \Psi_\mho(\mathfrak{w})d\mathfrak{w} = \left\{\int_\zeta^{\mathfrak{z}} \Psi(\mathfrak{w}, \mho)d\mathfrak{w} : \Psi(\mathfrak{w}, \mho) \in S(\Psi_\mho)\right\}, \tag{18}$$

where $S(\Psi_\mho) = \{\Psi(., \mho) \to \mathbb{R} : \Psi(., \mho)$ is integrable, and $\Psi(\mathfrak{w}, \mho) \in \Psi_\mho(\mathfrak{w})\}$, for every $\mho \in [0,\ 1]$. Moreover, $\widetilde{\Psi}$ is (FA)-integrable over $[\zeta,\ \mathfrak{z}]$ if $(FA)\int_\zeta^{\mathfrak{z}} \widetilde{\Psi}(\mathfrak{w})d\mathfrak{w} \in \pounds_o$

Theorem 3. *[78] Let $\widetilde{\Psi} : [\zeta,\ \mathfrak{z}] \subset \mathbb{R} \to \pounds_o$ be a F-N·V·M, for which the $\mho$-levels define the family of I-V·Ms $\Psi_\mho : [\zeta,\ \mathfrak{z}] \subset \mathbb{R} \to \mathbb{X}_o$ satisfying $\Psi_\mho(\mathfrak{w}) = [\Psi_*(\mathfrak{w}, \mho),\ \Psi^*(\mathfrak{w}, \mho)]$ for every $\mathfrak{w} \in [\zeta,\ \mathfrak{z}]$ and $\mho \in [0,\ 1]$. $\widetilde{\Psi}$ is (FA)-integrable over $[\zeta,\ \mathfrak{z}]$ when and only when $\Psi_*(\mathfrak{w}, \mho)$ and $\Psi^*(\mathfrak{w}, \mho)$ are integrable over $[\zeta,\ \mathfrak{z}]$. Moreover, if $\widetilde{\Psi}$ is (FA)-integrable over $[\zeta,\ \mathfrak{z}]$, then we have*

$$\left[(FA)\int_\zeta^{\mathfrak{z}} \widetilde{\Psi}(\mathfrak{w})d\mathfrak{w}\right]^\mho = \left[\int_\zeta^{\mathfrak{z}} \Psi_*(\mathfrak{w}, \mho)d\mathfrak{w},\ \int_\zeta^{\mathfrak{z}} \Psi^*(\mathfrak{w}, \mho)d\mathfrak{w}\right] = (IA)\int_\zeta^{\mathfrak{z}} \Psi_\mho(\mathfrak{w})d\mathfrak{w} \tag{19}$$

for every $\mho \in [0,\ 1]$.

Definition 6. *[82] Let $\beta > 0$ and $L([\zeta, \mathfrak{z}], \pounds_o)$ be the collection of all Lebesgue measurable fuzzy-number valued mappings on $[\zeta, \mathfrak{z}]$. Then, the fuzzy left and right RL fractional integrals of order $\beta > 0$ of $\Psi \in L(\ [\zeta,\ \mathfrak{z}], \pounds_o)$ are*

$$\mathcal{I}_{\zeta^+}^\beta\ \widetilde{\Psi}(\mathfrak{w}) = \frac{1}{\Gamma(\beta)}\int_\zeta^{\mathfrak{w}} (\mathfrak{w} - m)^{\beta-1}\widetilde{\Psi}(m)dm, \quad (\mathfrak{w} > \zeta), \tag{20}$$

and

$$\mathcal{I}_{\mathfrak{z}^-}^\beta\ \widetilde{\Psi}(\mathfrak{w}) = \frac{1}{\Gamma(\beta)}\int_{\mathfrak{w}}^{\mathfrak{z}} (m - \mathfrak{w})^{\beta-1}\widetilde{\Psi}(m)dm, \quad (\mathfrak{w} < \mathfrak{z}) \tag{21}$$

respectively, where $\Gamma(\mathfrak{w}) = \int_0^\infty m^{\mathfrak{w}-1}e^{-m}dm$ is the Euler gamma function. The fuzzy left and right RL fractional integrals $\mathfrak{w}$ based on left and right end point mappings are

$$\begin{gathered}\left[\mathcal{I}_{\zeta^+}^\beta\ \widetilde{\Psi}(\mathfrak{w})\right]^\mho = \tfrac{1}{\Gamma(\beta)}\int_\zeta^{\mathfrak{w}} (\mathfrak{w} - m)^{\beta-1}\Psi_\mho(m)dm \\ = \tfrac{1}{\Gamma(\beta)}\int_\zeta^{\mathfrak{w}} (\mathfrak{w} - m)^{\beta-1}[\Psi_*(m,\ \mho), \Psi^*(m,\ \mho)]dm,\ (\mathfrak{w} > \zeta)\end{gathered} \tag{22}$$

where

$$\mathcal{I}_{\zeta^+}^\beta\ \Psi_*(\mathfrak{w},\ \mho) = \frac{1}{\Gamma(\beta)}\int_\zeta^{\mathfrak{w}} (\mathfrak{w} - m)^{\beta-1}\Psi_*(m,\ \mho)dm, \quad (\mathfrak{w} > \zeta), \tag{23}$$

and

$$\mathcal{I}_{\zeta^+}^\beta\ \Psi^*(\mathfrak{w},\ \mho) = \frac{1}{\Gamma(\beta)}\int_\zeta^{\mathfrak{w}} (\mathfrak{w} - m)^{\beta-1}\Psi^*(m,\ \mho)dm, \quad (\mathfrak{w} > \zeta). \tag{24}$$

The RL fractional integral $\widetilde{\Psi}$ of $\mathfrak{w}$ based on left and right end point mappings can be defined in a similar way.

Definition 7. *[75] The $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$ $\widetilde{\Psi}: [\zeta,\ \mathfrak{z}] \to £_o$ is named as a convex $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$ on $[\zeta,\ \mathfrak{z}]$ if*

$$\widetilde{\Psi}(\amalg\mu + (1-\amalg)\mathfrak{z}\,) \leq_{\mathbb{F}} \amalg \odot \widetilde{\Psi}(\mu) \oplus (1-\amalg) \odot \widetilde{\Psi}(\mathfrak{z}), \tag{25}$$

for all $\mu,\ \mathfrak{z} \in [\zeta,\ \mathfrak{z}]$, $\amalg \in [0,\ 1]$, where $\widetilde{\Psi}(\mathfrak{z}) \geq_{\mathbb{F}} \widetilde{0}$ for all $\mathfrak{z} \in [\zeta,\ \mathfrak{z}]$. If (25) is reversed, then $\widetilde{\Psi}$ is named as a concave $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$ on $[\zeta,\ \mathfrak{z}]$. $\widetilde{\Psi}$ is affine if and only if it is both convex and concave $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$.

Remark 2. *If $\Psi_*(\mathfrak{z},\mho) = \Psi^*(\mathfrak{z},\mho)$ and $\mho = 1$, then we obtain the classical convex function.*

Definition 8. *[55] The $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$ $\widetilde{\Psi}: [\zeta,\ \mathfrak{z}] \to £_o$ is named as a pre-invex $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$ on invex interval $[\zeta,\ \mathfrak{z}]$ if*

$$\widetilde{\Psi}(\mu + (1-\amalg)\phi(\mu,\amalg)) \leq_{\mathbb{F}} \amalg \odot \widetilde{\Psi}(\mu) \oplus (1-\amalg) \odot \widetilde{\Psi}(\mathfrak{z}), \tag{26}$$

for all $\mu,\ \mathfrak{z} \in [\zeta,\ \mathfrak{z}]$, $\amalg \in [0,\ 1]$, where all $\widetilde{\Psi}(\mu) \geq_{\mathbb{F}} \widetilde{0}$ for all $\mu \in [\zeta,\ \mathfrak{z}]$. If (26) is reversed, then $\widetilde{\Psi}$ is named as a pre-incave $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$ on $[\zeta,\ \mathfrak{z}]$. $\widetilde{\Psi}$ is affine if and only if it is both pre-invex and pre-incave $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$s.

Definition 9. *[59] Let $\hbar : [0,\ 1] \subseteq [\zeta,\mathfrak{z}] \to \mathbb{R}^+$ such that $\hbar \not\equiv 0$. Then, $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$ $\widetilde{\Psi}: [\zeta,\mathfrak{z}] \to £_o$ is said to be UD-$\hbar$-pre-invex $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$ on $[\zeta,\mathfrak{z}]$ if*

$$\widetilde{\Psi}(\mu + (1-\amalg)\phi(\mu,\mathfrak{z})) \supseteq_{\mathbb{F}} \hbar(\amalg) \odot \widetilde{\Psi}(\mu) \oplus \hbar(1-\amalg) \odot \widetilde{\Psi}(\mathfrak{z}), \tag{27}$$

for all $\mu,\ \mathfrak{z} \in [\zeta,\mathfrak{z}]$, $\amalg \in [0,\ 1]$, where $\widetilde{\Psi}(\mu) \geq_{\mathbb{F}} \widetilde{0}$. If $\widetilde{\Psi}$ is up and $\hbar$-pre-incave on $[\zeta,\mathfrak{z}]$, then inequality (27) is reversed.

Remark 3. *[59] If one attempts to take $\hbar(\amalg) = \amalg$, then from UD-$\hbar$-pre-invex $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$ one achieves UD-pre-invex $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$, that is*

$$\widetilde{\Psi}(\mu + (1-\amalg)\phi(\mu,\mathfrak{z})) \supseteq_{\mathbb{F}} \amalg \odot \widetilde{\Psi}(\mu) \oplus (1-\amalg) \odot \widetilde{\Psi}(\mathfrak{z}),\ \forall\ \mu,\ \mathfrak{z} \in [\zeta,\mathfrak{z}],\ \amalg \in [0,\ 1]. \tag{28}$$

If one attempts to take $\hbar(\amalg) \equiv 1$, then from UD-$\hbar$-pre-invex $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$ one achieves UD-P-pre-invex $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$, that is

$$\widetilde{\Psi}(\mu + (1-\amalg)\phi(\mu,\mathfrak{z})) \supseteq_{\mathbb{F}} \widetilde{\Psi}(\mu) \oplus \widetilde{\Psi}(\mathfrak{z}),\ \forall\ \mu,\ \mathfrak{z} \in [\zeta,\mathfrak{z}],\ \amalg \in [0,\ 1]. \tag{29}$$

Theorem 4. *[59] Let $\hbar : [0,\ 1] \subseteq [\zeta,\mathfrak{z}] \to \mathbb{R}$ be an anon-negative real-valued function such that $\hbar \not\equiv 0$ and let $\widetilde{\Psi}: [\zeta,\mathfrak{z}] \to £_o$ be a $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$, for which the $\mho$-cuts define the family of $\mathfrak{I}\mathcal{V}\mathcal{M}$s $\Psi_\mho : [\zeta,\mathfrak{z}] \to \mathbb{X}_C^+ \subset \mathbb{X}_C$ and are given by*

$$\Psi_\mho(\mathfrak{z}) = [\Psi_*(\mathfrak{z},\mho),\ \Psi^*(\mathfrak{z},\mho)], \tag{30}$$

for all $\mathfrak{z} \in [\zeta,\mathfrak{z}]$ and for all $\mho \in [0,\ 1]$. Then, $\widetilde{\Psi}$ is UD-$\hbar$-pre-invex $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$ on $[\zeta,\mathfrak{z}]$ if and only if, for all $\mho \in [0,\ 1]$, $\Psi_*(\mathfrak{z},\ \mho)$ is a $\hbar$-pre-invex function and $\Psi^*(\mathfrak{z},\ \mho)$ is a $\hbar$-pre-incave function.

Example 1. *If we attempt to take* $\hbar(\text{II}) = \text{II}$, *for* $\text{II} \in [0,\ 1]$ *and the* $\mathcal{FIVM}$ $\widetilde{\Psi} : \left[0,\ 4\right] \to \pounds_o$ *defined by*

$$\widetilde{\Psi}(z)(\theta) = \begin{cases} \frac{\theta}{2e^{z^2}} & \theta \in \left[0,\ 2e^{z^2}\right] \\ \frac{4e^{z^2} - \theta}{2e^{z^2}} & \theta \in \left(2e^{z^2},\ 4e^{z^2}\right] \\ 0 & \textit{otherwise}, \end{cases}$$

then, for each $\mho \in [0,\ 1]$, *we have* $\Psi_{\mho}(z) = \left[2\mho\, e^{z^2}, 2(2-\mho)\, e^{z^2}\right]$. *Since endpoint functions* $\Psi_*(z, \mho)$, $\Psi^*(z, \mho)$ *are* $\hbar$*-pre-invex functions with respect to* $\phi(z, \zeta) = v - \zeta$, *for each* $\mho \in [0,\ 1]$. *Hence,* $\widetilde{\Psi}(z)$ *is UD-*$\hbar$*-pre-invex* $\mathcal{FIVM}$.

3. Fuzzy Riemann–Liouville Fractional Integral Hermite–Hadamard Type Inequality

In the results that follow, we investigate how fuzzy fractional operators can be used to apply up and down functions to integral inequalities; therefore, let us recap the generalized $H.H$ type inequality for $\hbar$-pre-invex $\mathcal{FIVM}$s first.

Theorem 5. *Let* $\widetilde{\Psi} : [\zeta,\ \zeta + \phi(z, \zeta)\big] \to \pounds_o$ *be an UD-*$\hbar$*-pre-invex* $\mathcal{FIVM}$ *on* $[\zeta,\ \zeta + \phi(z, \zeta)]$, *whose* $\mho$*-cuts define the family of* $\mathfrak{IVM}$*s* $\Psi_{\mho} : [\zeta,\ \zeta + \phi(z, \zeta)] \subset \mathbb{R} \to \mathbb{X}_C^+$ *are given by* $\Psi_{\mho}(z) = [\Psi_*(z, \mho),\ \Psi^*(z, \mho)]$ *for all* $z \in [\zeta,\ \zeta + \phi(z, \zeta)]$ *and for all* $\mho \in [0,\ 1]$. *If* ϕ *satisfies Condition C and* $\widetilde{\Psi} \in L\Big([\zeta,\ \zeta + \phi(z, \zeta)],\ \pounds_o\Big)$, *then*

$$\begin{aligned} \frac{1}{\beta\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\Psi}\Big(\tfrac{2\zeta + \phi(z, \zeta)}{2}\Big) &\supseteq_{\mathbb{F}} \frac{\Gamma(\beta)}{(\phi(z, \zeta))^{\beta}} \odot \left[\mathcal{I}^{\beta}_{\zeta^+}\ \widetilde{\Psi}(\zeta + \phi(z, \zeta)) \oplus \mathcal{I}^{\beta}_{\zeta + \phi(z, \zeta)^-}\ \widetilde{\Psi}(\zeta)\right] \\ &\supseteq_{\mathbb{F}} \left[\widetilde{\Psi}(\zeta) \oplus \widetilde{\Psi}(\zeta + \phi(z, \zeta))\right] \odot \int_0^1 \text{II}^{\beta-1}[\hbar(\text{II}) + \hbar(1 - \text{II})]d\text{II} \\ &\supseteq_{\mathbb{F}} \left[\widetilde{\Psi}(\zeta) \oplus \widetilde{\Psi}(z)\right] \odot \int_0^1 \text{II}^{\beta-1}[\hbar(\text{II}) + \hbar(1 - \text{II})]d\text{II} \end{aligned} \tag{31}$$

If $\widetilde{\Psi}(z)$ is pre-incave $\mathcal{FIVM}$, then

$$\begin{aligned} \frac{1}{\beta\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\Psi}\Big(\tfrac{2\zeta + \phi(z, \zeta)}{2}\Big) &\subseteq_{\mathbb{F}} \frac{\Gamma(\beta)}{(\phi(z, \zeta))^{\beta}} \odot \left[\mathcal{I}^{\beta}_{\zeta^+}\ \widetilde{\Psi}(\zeta + \phi(z, \zeta)) \oplus \mathcal{I}^{\beta}_{\zeta + \phi(z, \zeta)^-}\ \widetilde{\Psi}(\zeta)\right] \\ &\subseteq_{\mathbb{F}} \left[\widetilde{\Psi}(\zeta) \oplus \widetilde{\Psi}(\zeta + \phi(z, \zeta))\right] \odot \int_0^1 \text{II}^{\beta-1}[\hbar(\text{II}) - \hbar(1 - \text{II})]d\text{II} \\ &\subseteq_{\mathbb{F}} \left[\widetilde{\Psi}(\zeta) \oplus \widetilde{\Psi}(z)\right] \odot \int_0^1 \text{II}^{\beta-1}[\hbar(\text{II}) - \hbar(1 - \text{II})]d\text{II} \end{aligned} \tag{32}$$

Proof. Let $\widetilde{\Psi} : [\zeta,\ \zeta + \phi(z, \zeta)] \to \pounds_o$ be an *UD-*$\hbar$-pre-invex $\mathcal{FIVM}$. If Condition C holds then, by hypothesis, we have

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\Psi}\left(\frac{2\zeta + \phi(z, \zeta)}{2}\right) \supseteq_{\mathbb{F}} \widetilde{\Psi}(\zeta + (1 - \text{II})\phi(z, \zeta)) \oplus \widetilde{\Psi}(\zeta + \text{II}\phi(z, \zeta)).$$

Therefore, for every $\mho \in [0,\ 1]$, we have

$$\begin{aligned} &\frac{1}{\hbar\left(\frac{1}{2}\right)} \Psi_*\Big(\tfrac{2\zeta + \phi(z, \zeta)}{2},\ \mho\Big) \leq \Psi_*(\zeta + (1 - \text{II})\phi(z, \zeta),\ \mho) + \Psi_*(\zeta + \text{II}\phi(z, \zeta),\ \mho), \\ &\frac{1}{\hbar\left(\frac{1}{2}\right)} \Psi^*\Big(\tfrac{2\zeta + \phi(z, \zeta)}{2},\ \mho\Big) \geq \Psi^*(\zeta + (1 - \text{II})\phi(z, \zeta),\ \mho) + \Psi^*(\zeta + \text{II}\phi(z, \zeta), \mho). \end{aligned}$$

Multiplying both sides by $\amalg^{\beta-1}$ and integrating the obtained result with respect to $\amalg$ over $(0,1)$, we have

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\int_0^1 \amalg^{\beta-1}\Psi_*\left(\frac{2\zeta+\phi(\mathrm{z},\zeta)}{2},\ \mho\right)d\amalg$$
$$\leq \int_0^1 \amalg^{\beta-1}\Psi_*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta),\mho)d\amalg+\int_0^1 \amalg^{\beta-1}\Psi_*(\zeta+\amalg\phi(\mathrm{z},\zeta),\ \mho)d\amalg,$$
$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\int_0^1 \amalg^{\beta-1}\Psi^*\left(\frac{2\zeta+\phi(\mathrm{z},\zeta)}{2},\mho\right)d\amalg$$
$$\geq \int_0^1 \amalg^{\beta-1}\Psi^*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta),\ \mho)d\amalg+\int_0^1 \amalg^{\beta-1}\Psi^*(\zeta+\amalg\phi(\mathrm{z},\zeta),\mho)d\amalg.$$

Let $\mu=\zeta+(1-\amalg)\phi(\mathrm{z},\zeta)$ and $\mathrm{z}=\zeta+\amalg\phi(\mathrm{z},\zeta)$. Then, we have

$$\frac{1}{\beta\hbar\left(\frac{1}{2}\right)}\Psi_*\left(\frac{2\zeta+\phi(\mathrm{z},\zeta)}{2},\ \mho\right)\ \leq \frac{1}{(\phi(\mathrm{z},\zeta))^{\beta}}\ \int_\zeta^{\zeta+\phi(\mathrm{z},\zeta)}(\zeta+\phi(\mathrm{z},\zeta)-\mu)^{\beta-1}\Psi_*(\mu,\mho)d\mu$$
$$+\frac{1}{(\phi(\mathrm{z},\zeta))^{\beta}}\int_\zeta^{\zeta+\phi(\mathrm{z},\zeta)}(\mathrm{z}-\zeta)^{\beta-1}\Psi_*(\mathrm{z},\mho)d\mathrm{z}$$
$$\frac{1}{\beta\hbar\left(\frac{1}{2}\right)}\Psi^*\left(\frac{2\zeta+\phi(\mathrm{z},\zeta)}{2},\ \mho\right)\geq \frac{1}{(\phi(\mathrm{z},\zeta))^{\beta}}\ \int_\zeta^{\zeta+\phi(\mathrm{z},\zeta)}(\zeta+\phi(\mathrm{z},\zeta)-\mu)^{\beta-1}\Psi^*(\mu,\mho)d\mu.$$
$$+\frac{1}{(\phi(\mathrm{z},\zeta))^{\beta}}\int_\zeta^{\zeta+\phi(\mathrm{z},\zeta)}(\mathrm{z}-\zeta)^{\beta-1}\Psi^*(\mathrm{z},\mho)d\mathrm{z},$$
$$\leq \frac{\Gamma(\beta)}{(\phi(\mathrm{z},\zeta))^{\beta}}\left[\mathcal{I}_{\zeta^+}^{\beta}\ \Psi_*(\zeta+\phi(\mathrm{z},\zeta),\ \mho)+\mathcal{I}_{\zeta+\phi(\mathrm{z},\zeta)^-}^{\beta}\ \Psi_*(\zeta,\ \mho)\right]$$
$$\geq \frac{\Gamma(\beta)}{(\phi(\mathrm{z},\zeta))^{\beta}}\left[\mathcal{I}_{\zeta^+}^{\beta}\ \Psi^*(\zeta+\phi(\mathrm{z},\zeta),\ \mho)+\mathcal{I}_{\zeta+\phi(\mathrm{z},\zeta)^-}^{\beta}\ \Psi^*(\zeta,\ \mho)\right],$$

that is

$$\frac{1}{\beta\hbar\left(\frac{1}{2}\right)}\left[\Psi_*\left(\frac{2\zeta+\phi(\mathrm{z},\zeta)}{2},\ \mho\right),\ \Psi^*\left(\frac{2\zeta+\phi(\mathrm{z},\zeta)}{2},\ \mho\right)\right]$$
$$\supseteq_I \frac{\Gamma(\beta)}{(\phi(\mathrm{z},\zeta))^{\beta}}\left[\mathcal{I}_{\zeta^+}^{\beta}\Psi_*(\zeta+\phi(\mathrm{z},\zeta),\ \mho)+\mathcal{I}_{\zeta+\phi(\mathrm{z},\zeta)^-}^{\beta}\Psi_*(\zeta,\ \mho),\ \mathcal{I}_{\zeta^+}^{\beta}\Psi^*(\zeta+\phi(\mathrm{z},\zeta),\ \mho)+\mathcal{I}_{\zeta+\phi(\mathrm{z},\zeta)^-}^{\beta}\Psi^*(\zeta,\ \mho)\right]$$

thus,

$$\frac{1}{\beta\hbar\left(\frac{1}{2}\right)}\ \Psi_\mho\left(\frac{2\zeta+\phi(\mathrm{z},\zeta)}{2}\right)\supseteq_I \frac{\Gamma(\beta)}{(\phi(\mathrm{z},\zeta))^{\beta}}\left[\mathcal{I}_{\zeta^+}^{\beta}\ \Psi_\mho(\zeta+\phi(\mathrm{z},\zeta))+\mathcal{I}_{\zeta+\phi(\mathrm{z},\zeta)^-}^{\beta}\ \Psi_\mho(\zeta)\right]. \tag{33}$$

In a similar way as above, we have

$$\frac{\Gamma(\beta)}{(\phi(\mathrm{z},\zeta))^{\beta}}\left[\mathcal{I}_{\zeta^+}^{\beta}\ \Psi_\mho(\zeta+\phi(\mathrm{z},\zeta))+\mathcal{I}_{\zeta+\phi(\mathrm{z},\zeta)^-}^{\beta}\ \Psi_\mho(\zeta)\right]$$
$$\supseteq_I [\Psi_\mho(\zeta)+\Psi_\mho(\zeta+\phi(\mathrm{z},\zeta))]\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]d\amalg. \tag{34}$$

Combining (33) and (34), we have

$$\frac{1}{\beta\hbar\left(\frac{1}{2}\right)}\Psi_\mho\left(\frac{2\zeta+\phi(\mathrm{z},\zeta)}{2}\right)\supseteq_I \frac{\Gamma(\beta)}{(\phi(\mathrm{z},\zeta))^{\beta}}\left[\mathcal{I}_{\zeta^+}^{\beta}\ \Psi_\mho(\zeta+\phi(\mathrm{z},\zeta))+\mathcal{I}_{\zeta+\phi(\mathrm{z},\zeta)^-}^{\beta}\ \Psi_\mho(\zeta)\right]$$
$$\supseteq_I [\Psi_\mho(\zeta)+\Psi_\mho(\zeta+\phi(\mathrm{z},\zeta))]\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]d\amalg$$

that is

$$\frac{1}{\beta\hbar\left(\frac{1}{2}\right)}\odot\widetilde{\Psi}\left(\frac{2\zeta+\phi(\mathrm{z},\zeta)}{2}\right)\supseteq_{\mathbb{F}} \frac{\Gamma(\beta)}{(\phi(\mathrm{z},\zeta))}\odot\left[\mathcal{I}_{\zeta^+}^{\beta}\ \widetilde{\Psi}(\zeta+\phi(\mathrm{z},\zeta))\oplus\mathcal{I}_{\zeta+\phi(\mathrm{z},\zeta)^-}^{\beta}\ \widetilde{\Psi}(\zeta)\right]$$
$$\supseteq_{\mathbb{F}} \left[\widetilde{\Psi}(\zeta)\oplus\widetilde{\Psi}(\zeta+\phi(\mathrm{z},\zeta))\right]\odot\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]d\amalg$$
$$\supseteq_{\mathbb{F}} \left[\widetilde{\Psi}(\zeta)\oplus\widetilde{\Psi}(\mathrm{z})\right]\odot\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]d\amalg$$

the theorem has been proved. □

Remark 4. *If one attempts to take $\beta = 1$, then from inequality (31) one achieves the result for UD-$\hbar$-pre-invex $\mathcal{F}\mathfrak{T}\mathcal{V}\mathcal{M}$, see* [59]:

$$\begin{gathered}\frac{1}{2\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\Psi}\left(\frac{2\zeta+\phi(z,\zeta)}{2}\right) \\ \supseteq_{\mathbb{F}} \frac{1}{\phi(z,\zeta)} \odot (FR)\int_{\zeta}^{\zeta+\phi(z,\zeta)} \widetilde{\Psi}(z)dz \supseteq_{\mathbb{F}} \left[\widetilde{\Psi}(\zeta) \oplus \widetilde{\Psi}(\zeta+\phi(z,\zeta))\right] \odot \int_0^1 \hbar(\amalg)d\amalg.\end{gathered} \tag{35}$$

If one attempt to take $\hbar(\amalg) = \amalg$, then from inequality (31) one achieves the result for UD-pre-invex $\mathcal{F}\mathfrak{T}\mathcal{V}\mathcal{M}$, see [59]:

$$\widetilde{\Psi}\left(\frac{2\zeta+\phi(z,\zeta)}{2}\right) \supseteq_{\mathbb{F}} \frac{\Gamma(\beta+1)}{2(\phi(z,\zeta))^{\beta}} \odot \left[\mathcal{I}_{\zeta^+}^{\beta}\, \widetilde{\Psi}(\zeta+\phi(z,\zeta)) \oplus \mathcal{I}_{\zeta+\phi(z,\zeta)^-}^{\beta}\, \widetilde{\Psi}(\zeta)\right] \supseteq_{\mathbb{F}} \frac{\widetilde{\Psi}(\zeta) \oplus \widetilde{\Psi}(\zeta+\phi(z,\zeta))}{2} \tag{36}$$

Let one attempt to take $\beta = 1$ and $\hbar(\amalg) = \amalg$. Then, from inequality (31) one acquires the result for UD-pre-invex $\mathcal{F}\mathfrak{T}\mathcal{V}\mathcal{M}$ given in [59]:

$$\widetilde{\Psi}\left(\frac{2\zeta+\phi(z,\zeta)}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\phi(z,\zeta)} \odot (FR)\int_{\zeta}^{\zeta+\phi(z,\zeta)} \widetilde{\Psi}(z)dz \supseteq_{\mathbb{F}} \frac{\widetilde{\Psi}(\zeta) \oplus \widetilde{\Psi}(z)}{2} \tag{37}$$

If one attempt to take $\Psi_*(z,\ \mho) = \Psi^*(z,\ \mho)$ and $\mho = 1$, then from inequality (31) one acquires coming inequality given in [54]:

$$\begin{gathered}\frac{1}{\beta\hbar\left(\frac{1}{2}\right)} \Psi\left(\frac{2\zeta+\phi(z,\zeta)}{2}\right) \leq \frac{\Gamma(\beta)}{(\phi(z,\zeta))^{\beta}}\left[\mathcal{I}_{\zeta^+}^{\beta}\ \Psi(\zeta+\phi(z,\zeta)) + \mathcal{I}_{\zeta+\phi(z,\zeta)^-}^{\beta}\ \Psi(\zeta)\right] \\ \leq [\Psi(\zeta) + \Psi(\zeta+\phi(z,\zeta))]\int_0^1 \amalg^{\beta-1}[\hbar(\amalg) + \hbar(1-\amalg)]d\amalg.\end{gathered} \tag{38}$$

Let one attempt to take $\beta = 1 = \mho$ and $\Psi_*(z,\ \mho) = \Psi^*(z,\ \mho)$. Then, from inequality (31) one acquires coming inequality given in [80]:

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\ \Psi\left(\frac{2\zeta+\phi(z,\zeta)}{2}\right) \leq \frac{1}{\phi(z,\zeta)}\ (R)\int_{\zeta}^{\zeta+\phi(z,\zeta)} \Psi(z)dz \leq [\Psi(\zeta) + \Psi(\zeta+\phi(z,\zeta))]\int_0^1 \hbar(\amalg)d\amalg. \tag{39}$$

Example 2. *If we attempt to take $\beta = \frac{1}{2}$, $\hbar(\amalg) = \amalg$, for all $\amalg \in [0,\ 1]$ and the $\mathcal{F}\mathfrak{T}\mathcal{V}\mathcal{M}$ $\widetilde{\Psi}: [\zeta,\ \zeta+\phi(z,\zeta)] = [2,\ 2+\phi(3,2)] \to \pounds_o$, defined by*

$$\widetilde{\Psi}(z)(\theta) = \begin{cases} \frac{\theta-2+z^{\frac{1}{2}}}{1+z^{\frac{1}{2}}}, & \theta \in \left[2-z^{\frac{1}{2}}, 3\right] \\ \frac{2+z^{\frac{1}{2}}-\theta}{z^{\frac{1}{2}}-1}, & \theta \in \left(3,\ 2+z^{\frac{1}{2}}\right] \\ 0, & \textit{otherwise}. \end{cases}$$

Then, for each $\mho \in [0,\ 1]$, we have $\Psi_{\mho}(z) = \left[(1-\mho)\left(2-z^{\frac{1}{2}}\right) + 3\mho, (1-\mho)\left(2+z^{\frac{1}{2}}\right) + 3\mho\right]$. Since left and right end point functions $\Psi_*(z,\mho) = (1-\mho)\left(2-z^{\frac{1}{2}}\right) + 3\mho$, $\Psi^*(z,\ \mho) = (1-\mho)\left(2+z^{\frac{1}{2}}\right) + 3\mho$, are $\hbar$-pre-invex functions for each $\mho \in [0,\ 1]$, then $\widetilde{\Psi}(z)$ is UD-$\hbar$-pre-invex $\mathcal{FIVM}$ with respect to $\phi(z,\zeta) = v - \zeta$. We clearly see that $\widetilde{\Psi} \in L\Big([\zeta,\ \zeta+\phi(z,\zeta)],$ $\pounds_o\Big)$ and

$$\frac{1}{\beta\hbar\left(\frac{1}{2}\right)}\Psi_*\left(\frac{2\zeta+\phi(z,\zeta)}{2},\ \mho\right) = \Psi_*\left(\frac{5}{2},\ \mho\right) = 2(1-\mho)\left(4-\sqrt{10}\right)+12\mho$$

$$\frac{1}{\beta\hbar\left(\frac{1}{2}\right)}\Psi^*\left(\frac{2\zeta+\phi(z,\zeta)}{2},\ \mho\right) = \Psi^*\left(\frac{5}{2},\ \mho\right) = 2(1-\mho)\left(4+\sqrt{10}\right)+12\mho,$$

$$[\Psi_*(\zeta,\mho)+\Psi_*(\zeta+\phi(z,\zeta),\ \mho)]\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]d\amalg = 2(1-\mho)\left(4-\sqrt{2}-\sqrt{3}\right)+12\mho$$

$$[\Psi^*(\zeta,\mho)+\Psi^*(\zeta+\phi((z,\zeta),\ \mho)]\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]d\amalg = 2(1-\mho)\left(4+\sqrt{2}+\sqrt{3}\right)+12\mho\,.$$

Note that

$$\begin{aligned}
&\tfrac{\Gamma(\beta)}{(\phi(z,\zeta))^{\beta}}\left[\mathcal{I}^{\beta}_{\zeta^+}\ \Psi_*(\zeta+\phi(z,\zeta),\ \mho)+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\ \Psi_*(\zeta,\mho)\right]\\
&=\tfrac{\Gamma\left(\frac{1}{2}\right)}{2}\tfrac{1}{\sqrt{\pi}}\int_2^3(3-z)^{\frac{-1}{2}}\ .\ \left((1-\mho)\left(2-z^{\frac{1}{2}}\right)+3\mho\right)dz\\
&+\tfrac{\Gamma\left(\frac{1}{2}\right)}{2}\tfrac{1}{\sqrt{\pi}}\int_2^3(z-2)^{\frac{-1}{2}}\ .\ \left((1-\mho)\left(2-z^{\frac{1}{2}}\right)+3\mho\right)dz\\
&=(1-\mho)\left[\tfrac{7393}{2500}+\tfrac{9501}{2500}\right]+12\mho\\
&=(1-\mho)\tfrac{8447}{2500}+12\mho.
\end{aligned}$$

$$\begin{aligned}
&\tfrac{\Gamma(\beta)}{(\phi(z,\zeta))^{\beta}}\left[\mathcal{I}^{\beta}_{\zeta^+}\ \Psi^*(\zeta+\phi(z,\zeta),\ \mho)+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\ \Psi^*(\zeta,\mho)\right]\\
&=\tfrac{\Gamma\left(\frac{1}{2}\right)}{2}\tfrac{1}{\sqrt{\pi}}\int_2^3(3-z)^{\frac{-1}{2}}\ .\ \left((1-\mho)\left(2+z^{\frac{1}{2}}\right)+3\mho\right)dz\\
&+\tfrac{\Gamma\left(\frac{1}{2}\right)}{2}\tfrac{1}{\sqrt{\pi}}\int_2^3(z-2)^{\frac{-1}{2}}\ .\ \left((1-\mho)\left(2+z^{\frac{1}{2}}\right)+3\mho\right)dz\\
&=(1-\mho)\left[\tfrac{7260}{1000}+\tfrac{7049}{1000}\right]+12\mho\\
&=(1-\mho)\tfrac{14309}{1000}+12\mho.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left[2(1-\mho)\left(4-\sqrt{10}\right)+12\mho,2(1-\mho)\left(4+\sqrt{10}\right)+12\mho\right]\\
&\supseteq_I\ \left[(1-\mho)\tfrac{8447}{2500}+12\mho,(1-\mho)\tfrac{14309}{1000}+12\mho\right]\\
&\supseteq_I\ \left[2(1-\mho)\left(4-\sqrt{2}-\sqrt{3}\right)+12\mho,2(1-\mho)\left(4+\sqrt{2}+\sqrt{3}\right)+12\mho\right]
\end{aligned}$$

and Theorem 5 is verified.

We get various fuzzy fractional integral inequalities connected to fuzzy-interval fractional $H{\cdot}H$-inequalities from Theorems 6 and 7 via products of two UD-$\hbar$-pre-invex $\mathcal{FIVM}s$.

Theorem 6. *Let* $\widetilde{\Psi},\widetilde{\Im}\ :[\zeta,\ \zeta+\phi(z,\zeta)\Big]\to\pounds_o$ *be* UD-$\hbar_1$*-pre-invex and* UD-$\hbar_2$*-pre-invex* $\mathcal{FIVM}s$ *on* $[\zeta,\ \zeta+\phi(z,\zeta)]$*, respectively, whose* $\mho$*-cuts* $\Psi_\mho,\ \Im_\mho:[\zeta,\ \zeta+\phi(z,\zeta)]\subset\mathbb{R}\to\mathbb{X}_C^+$ *are defined by* $\Psi_\mho(z)=[\Psi_*(z,\mho),\ \Psi^*(z,\mho)]$ *and* $\Im_\mho(z)=[\Im_*(z,\mho),\ \Im^*(z,\mho)]$ *for all* $z\in[\zeta,\ \zeta+\phi(z,\zeta)]$ *and* *for all* $\mho \in [0, 1]$. *If* ϕ *satisfies Condition C and* $\widetilde{\Psi} \otimes \widetilde{\Im} \in L\left(\left[\zeta, \zeta + \phi(z, \zeta)\right], \pounds_o\right)$, *then*

$$\begin{aligned}&\frac{\Gamma(\beta)}{(\phi(z,\zeta))^{\beta}} \odot \left[\mathcal{I}^{\beta}_{\zeta^+}\ \widetilde{\Psi}(\zeta + \phi(z, \zeta)) \otimes \widetilde{\Im}(\zeta + \phi(z, \zeta)) \oplus \mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\ \widetilde{\Psi}(\zeta) \otimes \widetilde{\Im}(\zeta)\right] \\ &\supseteq_{\mathbb{F}} \widetilde{\aleph}(\zeta, \zeta + \phi(z, \zeta)) \odot \int_0^1 \text{II}^{\beta-1}[\hbar_1(\text{II})\hbar_2(\text{II}) + \hbar_1(1 - \text{II})\hbar_2(1 - \text{II})]d\text{II} \\ &\oplus \widetilde{\mathcal{M}}(\zeta, \zeta + \phi(z, \zeta)) \odot \int_0^1 \text{II}^{\beta-1}[\hbar_1(\text{II})\hbar_2(1 - \text{II}) + \hbar_1(1 - \text{II})\hbar_2(\text{II})]d\text{II}.\end{aligned} \tag{40}$$

where $\widetilde{\aleph}(\zeta, \zeta + \phi(z, \zeta)) = \widetilde{\Psi}(\zeta) \otimes \widetilde{\Im}(\zeta) \oplus \widetilde{\Psi}(\zeta + \phi(z, \zeta)) \otimes \widetilde{\Im}(\zeta + \phi(z, \zeta))$, $\widetilde{\mathcal{M}}(\zeta, \zeta + \phi(z, \zeta)) = \widetilde{\Psi}(\zeta) \otimes \widetilde{\Im}(\zeta + \phi(z, \zeta)) \oplus \widetilde{\Psi}(\zeta + \phi(z, \zeta)) \otimes \widetilde{\Im}(\zeta)$, *and* $\aleph_{\mho}(\zeta, \zeta + \phi(z, \zeta)) = [\aleph_*((\zeta, \zeta + \phi(z, \zeta)), \mho), \aleph^*((\zeta, \zeta + \phi(z, \zeta)), \mho)]$ *and* $\mathcal{M}_{\mho}(\zeta, \zeta + \phi(z, \zeta)) = [\mathcal{M}_*((\zeta, \zeta + \phi(z, \zeta)), \mho), \mathcal{M}^*((\zeta, \zeta + \phi(z, \zeta)), \mho)]$.

Proof. Since $\widetilde{\Psi}$, $\widetilde{\Im}$ both are UD-$\hbar_1$-pre-invex and UD-$\hbar_2$-pre-invex $\mathcal{F}\mathfrak{T}\mathcal{VM}$ then, for each $\mho \in [0, 1]$ we have

$$\begin{aligned}&\Psi_*(\zeta + (1 - \text{II})\phi(z, \zeta), \mho) \\ &\leq \hbar_1(\text{II})\Psi_*(\zeta, \mho) + \hbar_1(1 - \text{II})\Psi_*(\zeta + \phi(z, \zeta), \mho) \\ &\Psi^*(\zeta + (1 - \text{II})\phi(z, \zeta), \mho) \\ &\geq \hbar_1(\text{II})\Psi^*(\zeta, \mho) + \hbar_1(1 - \text{II})\Psi^*(\zeta + \phi(z, \zeta), \mho).\end{aligned}$$

and

$$\begin{aligned}&\Im_*(\zeta + (1 - \text{II})\phi(z, \zeta), \mho) \\ &\leq \hbar_2(\text{II})\Im_*(\zeta, \mho) + \hbar_2(1 - \text{II})\Im_*(\zeta + \phi(z, \zeta), \mho) \\ &\Im^*(\zeta + (1 - \text{II})\phi(z, \zeta), \mho) \\ &\geq \hbar_2(\text{II})\Im^*(\zeta, \mho) + \hbar_2(1 - \text{II})\Im^*(\zeta + \phi(z, \zeta), \mho).\end{aligned}$$

From the definition of UD-$\hbar$-pre-invex $\mathcal{F}\mathfrak{T}\mathcal{VM}s$ it follows that $\widetilde{0} \leq_{\mathbb{F}} \widetilde{\Psi}(z)$ and $\widetilde{0} \leq_{\mathbb{F}} \widetilde{\Im}(z)$, so

$$\begin{aligned}&\Psi_*(\zeta + (1 - \text{II})\phi(z, \zeta), \mho) \times \Im_*(\zeta + (1 - \text{II})\phi(z, \zeta), \mho) \\ &\leq \hbar_1(\text{II})\hbar_2(\text{II})\Psi_*(\zeta, \mho) \times \Im_*(\zeta, \mho) \\ &+\hbar_1(1 - \text{II})\hbar_2(1 - \text{II})\Psi_*(\zeta + \phi(z, \zeta), \mho) \times \Im_*(\zeta + \phi(z, \zeta), \mho) \\ &+\hbar_1(\text{II})\hbar_2(1 - \text{II})\Psi_*(\zeta, \mho) \times \Im_*(\zeta + \phi(z, \zeta), \mho) \\ &+\hbar_1(1 - \text{II})\hbar_2(\text{II})\Psi_*(\zeta + \phi(z, \zeta), \mho) \times \Im_*(\zeta, \mho) \\ &\Psi^*(\zeta + (1 - \text{II})\phi(z, \zeta), \mho) \times \Im^*(\zeta + (1 - \text{II})\phi(z, \zeta), \mho) \\ &\geq \hbar_1(\text{II})\hbar_2(\text{II})\Psi^*(\zeta, \mho) \times \Im^*(\zeta, \mho) \\ &+\hbar_1(1 - \text{II})\hbar_2(1 - \text{II})\Psi^*(\zeta + \phi(z, \zeta), \mho) \times \Im^*(\zeta + \phi(z, \zeta), \mho) \\ &+\hbar_1(\text{II})\hbar_2(1 - \text{II})\Psi^*(\zeta, \mho) \times \Im^*(\zeta + \phi(z, \zeta), \mho) \\ &+\hbar_1(1 - \text{II})\hbar_2(\text{II})\Psi^*(\zeta + \phi(z, \zeta), \mho) \times \Im^*(\zeta, \mho).\end{aligned} \tag{41}$$

Analogously, we have

$$\begin{aligned}
&\Psi_*(\zeta+\text{II}\phi(z,\zeta),\ \mho)\Im_*(\zeta+\text{II}\phi(z,\zeta),\ \mho)\\
&\quad\le \hbar_1(1-\text{II})\hbar_2(1-\text{II})\,\Psi_*(\zeta,\mho)\times\Im_*(\zeta,\mho)\\
&\quad+\hbar_1(\text{II})\hbar_2(\text{II})\,\Psi_*(\zeta+\phi(z,\zeta),\ \mho)\times\Im_*(\zeta+\phi(z,\zeta),\ \mho)\\
&\quad+\hbar_1(1-\text{II})\hbar_2(\text{II})\,\Psi_*(\zeta,\mho)\times\Im_*(\zeta+\phi(z,\zeta),\ \mho)\\
&\quad+\hbar_1(\text{II})\hbar_2(1-\text{II})\,\Psi_*(\zeta+\phi(z,\zeta),\ \mho)\times\Im_*(\zeta,\mho)\\
&\Psi^*(\zeta+\text{II}\phi(z,\zeta),\ \mho)\times\Im^*(\zeta+\text{II}\phi(z,\zeta),\ \mho)\\
&\quad\ge \hbar_1(1-\text{II})\hbar_2(1-\text{II})\,\Psi^*(\zeta,\mho)\times\Im^*(\zeta,\mho)\\
&\quad+\hbar_1(\text{II})\hbar_2(\text{II})\,\Psi^*(\zeta+\phi(z,\zeta),\ \mho)\times\Im^*(\zeta+\phi(z,\zeta),\ \mho)\\
&\quad+\hbar_1(1-\text{II})\hbar_2(\text{II})\,\Psi^*(\zeta,\mho)\times\Im^*(\zeta+\phi(z,\zeta),\ \mho)\\
&\quad+\hbar_1(\text{II})\hbar_2(1-\text{II})\,\Psi^*(\zeta+\phi(z,\zeta),\ \mho)\times\Im^*(\zeta,\mho).
\end{aligned}\tag{42}$$

Adding (41) and (42), we have

$$\begin{aligned}
&\Psi_*(\zeta+(1-\text{II})\phi(z,\zeta),\ \mho)\times\Im_*(\zeta+(1-\text{II})\phi(z,\zeta),\ \mho)\\
&\quad+\Psi_*(\zeta+\text{II}\phi(z,\zeta),\ \mho)\times\Im_*(\zeta+\text{II}\phi(z,\zeta),\ \mho)\\
&\le[\hbar_1(\text{II})\hbar_2(\text{II})+\hbar_1(1-\text{II})\hbar_2(1-\text{II})][\Psi_*(\zeta,\mho)\times\Im_*(\zeta,\mho)+\Psi_*(\zeta+\phi(z,\zeta),\ \mho)\times\Im_*(\zeta+\phi(z,\zeta),\ \mho)]\\
&+[\hbar_1(\text{II})\hbar_2(1-\text{II})+\hbar_1(1-\text{II})\hbar_2(\text{II})][\Psi_*(\zeta+\phi(z,\zeta),\ \mho)\times\Im_*(\zeta,\mho)+\Psi_*(\zeta,\mho)\times\Im_*(\zeta+\phi(z,\zeta),\ \mho)]\\
&\Psi^*(\zeta+(1-\text{II})\phi(z,\zeta),\ \mho)\times\Im^*(\zeta+(1-\text{II})\phi(z,\zeta),\ \mho)\\
&\quad+\Psi^*(\zeta+\text{II}\phi(z,\zeta),\ \mho)\times\Im^*(\zeta+\text{II}\phi(z,\zeta),\ \mho)\\
&\ge[\hbar_1(\text{II})\hbar_2(\text{II})+\hbar_1(1-\text{II})\hbar_2(1-\text{II})][\Psi^*(\zeta,\mho)\times\Im^*(\zeta,\mho)+\Psi^*(\zeta+\phi(z,\zeta),\ \mho)\times\Im^*(\zeta+\phi(z,\zeta),\ \mho)]\\
&+[\hbar_1(\text{II})\hbar_2(1-\text{II})+\hbar_1(1-\text{II})\hbar_2(\text{II})][\Psi^*(\zeta+\phi(z,\zeta),\ \mho)\times\Im^*(\zeta,\mho)+\Psi^*(\zeta,\mho)\times\Im^*(\zeta+\phi(z,\zeta),\ \mho)].
\end{aligned}\tag{43}$$

Taking multiplication of (43) with $\text{II}^{\beta-1}$ and integrating the obtained result with respect to II over (0,1), we have

$$\begin{aligned}
&\int_0^1 \text{II}^{\beta-1}\,\Psi_*(\zeta+(1-\text{II})\phi(z,\zeta),\ \mho)\times\Im_*(\zeta+(1-\text{II})\phi(z,\zeta),\ \mho)\\
&\quad+\text{II}^{\beta-1}\,\Psi_*(\zeta+\text{II}\phi(z,\zeta),\ \mho)\times\Im_*(\zeta+\text{II}\phi(z,\zeta),\ \mho)d\text{II}\\
&\le\aleph_*((\zeta,\zeta+\phi(z,\zeta)),\ \mho)\int_0^1 \text{II}^{\beta-1}[\hbar_1(\text{II})\hbar_2(\text{II})+\hbar_1(1-\text{II})\hbar_2(1-\text{II})]d\text{II}\\
&+\mathcal{M}_*((\zeta,\zeta+\phi(z,\zeta)),\ \mho)\int_0^1 \text{II}^{\beta-1}[\hbar_1(\text{II})\hbar_2(1-\text{II})+\hbar_1(1-\text{II})\hbar_2(\text{II})]d\text{II}\\
&\int_0^1 \text{II}^{\beta-1}\,\Psi^*(\zeta+(1-\text{II})\phi(z,\zeta),\ \mho)\times\Im^*(\zeta+(1-\text{II})\phi(z,\zeta),\ \mho)\\
&\quad+\text{II}^{\beta-1}\,\Psi^*(\zeta+\text{II}\phi(z,\zeta),\ \mho)\times\Im^*(\zeta+\text{II}\phi(z,\zeta),\ \mho)d\text{II}\\
&\ge\aleph^*((\zeta,\zeta+\phi(z,\zeta)),\ \mho)\int_0^1 \text{II}^{\beta-1}[\hbar_1(\text{II})\hbar_2(\text{II})+\hbar_1(1-\text{II})\hbar_2(1-\text{II})]d\text{II}\\
&+\mathcal{M}^*((\zeta,\zeta+\phi(z,\zeta)),\ \mho)\int_0^1 \text{II}^{\beta-1}[\hbar_1(\text{II})\hbar_2(1-\text{II})+\hbar_1(1-\text{II})\hbar_2(\text{II})]d\text{II}.
\end{aligned}$$

It follows that

$$\frac{\Gamma(\beta)}{(\phi(z,\zeta))^{\beta}}\left[\mathcal{I}_{\zeta^+}^{\beta}\ \Psi_*(\zeta+\phi(z,\zeta),\ \mho)\times\Im_*(\zeta+\phi(z,\zeta),\ \mho)+\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^{\beta}\ \Psi_*(\zeta,\mho)\times\Im_*(\zeta,\mho)\right]$$
$$\leq\aleph_*((\zeta,\zeta+\phi(z,\zeta)),\ \mho)\int_0^1\Pi^{\beta-1}[\hbar_1(\Pi)\hbar_2(\Pi)+\hbar_1(1-\Pi)\hbar_2(1-\Pi)]d\Pi$$
$$+\mathcal{M}_*((\zeta,\zeta+\phi(z,\zeta)),\ \mho)\int_0^1\Pi^{\beta-1}[\hbar_1(\Pi)\hbar_2(1-\Pi)+\hbar_1(1-\Pi)\hbar_2(\Pi)]d\Pi$$
$$\frac{\Gamma(\beta)}{(\phi(z,\zeta))^{\beta}}\left[\mathcal{I}_{\zeta^+}^{\beta}\ \Psi^*(\zeta+\phi(z,\zeta),\ \mho)\times\Im^*(\zeta+\phi(z,\zeta),\ \mho)+\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^{\beta}\ \Psi^*(\zeta,\mho)\times\Im^*(\zeta,\mho)\right]$$
$$\geq\aleph^*((\zeta,\zeta+\phi(z,\zeta)),\ \mho)\int_0^1\Pi^{\beta-1}[\hbar_1(\Pi)\hbar_2(\Pi)+\hbar_1(1-\Pi)\hbar_2(1-\Pi)]d\Pi$$
$$+\mathcal{M}^*((\zeta,\zeta+\phi(z,\zeta)),\ \mho)\int_0^1\Pi^{\beta-1}[\hbar_1(\Pi)\hbar_2(1-\Pi)+\hbar_1(1-\Pi)\hbar_2(\Pi)]d\Pi.$$

It follows that

$$\frac{\Gamma(\beta)}{(\phi(z,\zeta))^{\beta}}[\mathcal{I}_{\zeta^+}^{\beta}\Psi_*(\zeta+\phi(z,\zeta),\ \mho)\times\Im_*(\zeta+\phi(z,\zeta),\ \mho)+\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^{\beta}\Psi_*(\zeta,\mho)\times\Im_*(\zeta,\mho),\ \mathcal{I}_{\zeta^+}^{\beta}\Psi^*(\zeta+\phi(z,\zeta),\ \mho)$$
$$\times\Im^*(\zeta+\phi(z,\zeta),\ \mho)+\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^{\beta}\Psi^*(\zeta,\mho)\times\Im^*(\zeta,\ \mho)]$$
$$\supseteq_I[\aleph_*((\zeta,\zeta+\phi(z,\zeta)),\ \mho),\ \aleph^*((\zeta,\zeta+\phi(z,\zeta)),\ \mho)]\int_0^1\Pi^{\beta-1}[\hbar_1(\Pi)\hbar_2(\Pi)+\hbar_1(1-\Pi)\hbar_2(1-\Pi)]d\Pi$$
$$+[\mathcal{M}_*((\zeta,\zeta+\phi(z,\zeta)),\ \mho),\ \mathcal{M}^*((\zeta,\zeta+\phi(z,\zeta)),\ \mho)]\int_0^1\Pi^{\beta-1}[\hbar_1(\Pi)\hbar_2(1-\Pi)+\hbar_1(1-\Pi)\hbar_2(\Pi)]d\Pi$$

that is

$$\frac{\Gamma(\beta)}{(\phi(z,\zeta))^{\beta}}\left[\mathcal{I}_{\zeta^+}^{\beta}\ \Psi_\mho(\zeta+\phi(z,\zeta))\times\Im_\mho(\zeta+\phi(z,\zeta))+\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^{\beta}\ \Psi_\mho(\zeta)\times\Im_\mho(\zeta)\right]$$
$$\supseteq_I\aleph_\mho(\zeta,\zeta+\phi(z,\zeta))\int_0^1\Pi^{\beta-1}[\hbar_1(\Pi)\hbar_2(\Pi)+\hbar_1(1-\Pi)\hbar_2(1-\Pi)]d\Pi$$
$$+\mathcal{M}_\mho(\zeta,\zeta+\phi(z,\zeta))\int_0^1\Pi^{\beta-1}[\hbar_1(\Pi)\hbar_2(1-\Pi)+\hbar_1(1-\Pi)\hbar_2(\Pi)]d\Pi.$$

Thus,

$$\frac{\Gamma(\beta)}{(\phi(z,\zeta))^{\beta}}\left[\mathcal{I}_{\zeta^+}^{\beta}\ \widetilde{\Psi}(\zeta+\phi(z,\zeta))\otimes\widetilde{\Im}(\zeta+\phi(z,\zeta))\oplus\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^{\beta}\ \widetilde{\Psi}(\zeta)\otimes\widetilde{\Im}(\zeta)\right]$$
$$\supseteq_{\mathbb{F}}\widetilde{\aleph}(\zeta,\zeta+\phi(z,\zeta))\odot\int_0^1\Pi^{\beta-1}[\hbar_1(\Pi)\hbar_2(\Pi)+\hbar_1(1-\Pi)\hbar_2(1-\Pi)]d\Pi$$
$$\oplus\widetilde{\mathcal{M}}(\zeta,\zeta+\phi(z,\zeta))\odot\int_0^1\Pi^{\beta-1}[\hbar_1(\Pi)\hbar_2(1-\Pi)+\hbar_1(1-\Pi)\hbar_2(\Pi)]d\Pi.$$

and the theorem has been established. □

Theorem 7. *Let* $\widetilde{\Psi},\widetilde{\Im}:\left[\zeta,\ \zeta+\phi(z,\zeta)\right]\to\pounds_o$ *be two UD-*$\hbar_1$*-pre-invex and UD-*$\hbar_2$*-pre-invex* $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$*s, respectively, for which the* $\mho$*-cuts define the family of* $\mathfrak{I}\mathcal{V}\mathcal{M}$s. $\Psi_\mho,\ \Im_\mho:[\zeta,\ \zeta+\phi(z,\zeta)]\subset\mathbb{R}\to\mathbb{X}_C^+$ *are given by* $\Psi_\mho(z)=[\Psi_*(z,\mho),\ \Psi^*(z,\mho)]$ *and* $\Im_\mho(z)=[\Im_*(z,\mho),\ \Im^*(z,\mho)]$ *for all* $z\in[\zeta,\ \zeta+\phi(z,\zeta)]$ *and for all* $\mho\in[0,\ 1]$*. If* ϕ *satisfies Condition C and* $\widetilde{\Psi}\otimes\widetilde{\Im}\in L\Big([\zeta,\ \zeta+\phi(z,\zeta)],\ \pounds_o\Big)$*, then*

$$\frac{1}{\beta\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\odot\ \widetilde{\Psi}\left(\frac{2\zeta+\phi(z,\zeta)}{2}\right)\otimes\widetilde{\Im}\left(\frac{2\zeta+\phi(z,\zeta)}{2}\right)$$
$$\supseteq_{\mathbb{F}}\frac{\Gamma(\beta)}{(\phi(z,\zeta))^{\beta}}\odot\left[\mathcal{I}_{\zeta^+}^{\beta}\widetilde{\Psi}(\zeta+\phi(z,\zeta))\otimes\widetilde{\Im}(z)\oplus\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^{\beta}\widetilde{\Psi}(\zeta)\widetilde{\times}\widetilde{\Im}(\zeta)\right]$$
$$\oplus\widetilde{\mathcal{M}}(\zeta,\zeta+\phi(z,\zeta))\odot\int_0^1\left[\Pi^{\beta-1}+(1-\Pi)^{\beta-1}\right]\hbar_1(\Pi)\hbar_2(1-\Pi)d\Pi$$
$$\oplus\widetilde{\aleph}(\zeta,\zeta+\phi(z,\zeta))\odot\int_0^1\left[\Pi^{\beta-1}+(1-\Pi)^{\beta-1}\right]\hbar_1(1-\Pi)\hbar_2(1-\Pi)d\Pi. \quad (44)$$

where $\widetilde{\aleph}(\zeta, \zeta+\phi(\mathrm{z}, \zeta)) = \widetilde{\Psi}(\zeta) \otimes \widetilde{\Im}(\zeta) \oplus \widetilde{\Psi}(\mathrm{z}) \otimes \widetilde{\Im}(\mathrm{z})$, $\widetilde{\mathcal{M}}(\zeta, \zeta+\phi(\mathrm{z}, \zeta)) = \widetilde{\Psi}(\zeta) \otimes \widetilde{\Im}(\mathrm{z}) \oplus \widetilde{\Psi}(\mathrm{z}) \otimes \widetilde{\Im}(\zeta)$, *and* $\aleph_{\mho}(\zeta, \zeta+\phi(\mathrm{z}, \zeta)) = [\aleph_*((\zeta, \zeta+\phi(\mathrm{z}, \zeta)), \mho), \aleph^*((\zeta, \zeta+\phi(\mathrm{z}, \zeta)), \mho)]$ *and* $\mathcal{M}_{\mho}(\zeta, \zeta+\phi(\mathrm{z}, \zeta)) = [\mathcal{M}_*((\zeta, \zeta+\phi(\mathrm{z}, \zeta)), \mho), \mathcal{M}^*((\zeta, \zeta+\phi(\mathrm{z}, \zeta)), \mho)]$.

Proof. Consider $\widetilde{\Psi}, \widetilde{\Im} : \left[\zeta, \ \zeta+\phi(\mathrm{z}, \zeta)\right] \to \pounds_o$ are UD-$\hbar_1$-pre-invex and UD-$\hbar_2$-pre-invex $\mathcal{FIVM}s$. Then, by hypothesis, for each $\mho \in [0, 1]$, we have

$$\begin{aligned}
&\Psi_*\left(\tfrac{2\zeta+\phi(\mathrm{z},\zeta)}{2}, \mho\right) \times \Im_*\left(\tfrac{2\zeta+\phi(\mathrm{z},\zeta)}{2}, \mho\right) \\
&\Psi^*\left(\tfrac{2\zeta+\phi(\mathrm{z},\zeta)}{2}, \mho\right) \times \Im^*\left(\tfrac{2\zeta+\phi(\mathrm{z},\zeta)}{2}, \mho\right) \\
&\le \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \Psi_*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \times \Im_*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \\ +\Psi_*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \times \Im_*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \end{bmatrix} \\
&+ \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \Psi_*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \times \Im_*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \\ +\Psi_*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \times \Im_*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \end{bmatrix} \\
&\ge \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \Psi^*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \times \Im^*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \\ +\Psi^*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \times \Im^*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \end{bmatrix} \\
&+ \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \Psi^*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \times \Im^*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \\ +\Psi^*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \times \Im^*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \end{bmatrix}, \\
&\le \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \Psi_*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \times \Im_*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \\ +\Psi_*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \times \Im_*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \end{bmatrix} \\
&+ \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} (\hbar_1(\amalg)\Psi_*(\zeta, \mho) + \hbar_1(1-\amalg)\Psi_*(\zeta+\phi(\mathrm{z},\zeta),, \mho)) \\ \times(\hbar_2(1-\amalg)\Im_*(\zeta, \mho) + \hbar_2(\amalg)\Im_*(\zeta+\phi(\mathrm{z},\zeta),, \mho)) \\ +(\hbar_1(1-\amalg)\Psi_*(\zeta, \mho) + \hbar_1(\amalg)\Psi_*(\zeta+\phi(\mathrm{z},\zeta),, \mho)) \\ \times(\hbar_2(\amalg)\Im_*(\zeta, \mho) + \hbar_2(1-\amalg)\Im_*(\zeta+\phi(\mathrm{z},\zeta),, \mho)) \end{bmatrix} \\
&\ge \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \Psi^*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \times \Im^*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \\ +\Psi^*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \times \Im^*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \end{bmatrix} \\
&+ \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} (\hbar_1(\amalg)\Psi^*(\zeta, \mho) + \hbar_1(1-\amalg)\Psi^*(\zeta+\phi(\mathrm{z},\zeta),, \mho)) \\ \times(\hbar_2(1-\amalg)\Im^*(\zeta, \mho) + \hbar_2(\amalg)\Im^*(\zeta+\phi(\mathrm{z},\zeta),, \mho)) \\ +(\hbar_1(1-\amalg)\Psi^*(\zeta, \mho) + \hbar_1(\amalg)\Psi^*(\zeta+\phi(\mathrm{z},\zeta),, \mho)) \\ \times(\hbar_2(\amalg)\Im^*(\zeta, \mho) + \hbar_2(1-\amalg)\Im^*(\zeta+\phi(\mathrm{z},\zeta),, \mho)) \end{bmatrix}, \\
&= \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \Psi_*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \times \Im_*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \\ +\Psi_*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \times \Im_*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \end{bmatrix} \\
&+ \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \{\hbar_1(\amalg)\hbar_2(1-\amalg) + \hbar_1(1-\amalg)\hbar_2(\amalg)\}\mathcal{M}_*((\zeta, \zeta+\phi(\mathrm{z},\zeta)), \mho) \\ +\{\hbar_1(\amalg)\hbar_2(\amalg) + \hbar_1(1-\amalg)\hbar_2(1-\amalg)\}\aleph_*((\zeta, \zeta+\phi(\mathrm{z},\zeta)), \mho) \end{bmatrix} \\
&= \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \Psi^*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \times \Im^*(\zeta+(1-\amalg)\phi(\mathrm{z},\zeta), \mho) \\ +\Psi^*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \times \Im^*(\zeta+\amalg\phi(\mathrm{z},\zeta), \mho) \end{bmatrix} \\
&+ \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \{\hbar_1(\amalg)\hbar_2(1-\amalg) + \hbar_1(1-\amalg)\hbar_2(\amalg)\}\mathcal{M}^*((\zeta, \zeta+\phi(\mathrm{z},\zeta)), \mho) \\ +\{\hbar_1(\amalg)\hbar_2(\amalg) + \hbar_1(1-\amalg)\hbar_2(1-\amalg)\}\aleph^*((\zeta, \zeta+\phi(\mathrm{z},\zeta)), \mho) \end{bmatrix}.
\end{aligned} \tag{45}$$

Taking multiplication of (45) with $\text{II}^{\beta-1}$ and integrating over $(0,\ 1)$, we obtain

$$\begin{aligned}
&\frac{1}{\beta\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\Psi_*\left(\frac{2\zeta+\phi(\text{z},\zeta)}{2},\mho\right)\times\Im_*\left(\frac{2\zeta+\phi(\text{z},\zeta)}{2},\mho\right)\\
&\le\frac{\Gamma(\beta)}{(\phi(\text{z},\zeta))^\beta}\left[\mathcal{I}^\beta_{\zeta^+}\ \Psi_*(\zeta+\phi(\text{z},\zeta))\times\Im_*(\zeta+\phi(\text{z},\zeta))+\mathcal{I}^\beta_{\zeta+\phi(\text{z},\zeta)^-}\ \Psi_*(\zeta)\times\Im_*(\zeta)\right]\\
&\quad+\mathcal{M}_*((\zeta,\zeta+\phi(\text{z},\zeta)),\ \mho)\int_0^1\left[\text{II}^{\beta-1}+(1-\text{II})^{\beta-1}\right]\hbar_1(\text{II})\hbar_2(1-\text{II})d\text{II}\\
&\quad+\aleph_*((\zeta,\zeta+\phi(\text{z},\zeta)),\ \mho)\int_0^1\left[\text{II}^{\beta-1}+(1-\text{II})^{\beta-1}\right]\hbar_1(1-\text{II})\hbar_2(1-\text{II})d\text{II}\\
&\frac{1}{\beta\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\Psi^*\left(\frac{2\zeta+\phi(\text{z},\zeta)}{2},\mho\right)\times\Im^*\left(\frac{2\zeta+\phi(\text{z},\zeta)}{2},\mho\right)\\
&\ge\frac{\Gamma(\beta)}{(\phi(\text{z},\zeta))^\beta}\left[\mathcal{I}^\beta_{\zeta^+}\ \Psi^*(\zeta+\phi(\text{z},\zeta))\times\Im^*(\zeta+\phi(\text{z},\zeta))+\mathcal{I}^\beta_{\zeta+\phi(\text{z},\zeta)^-}\ \Psi^*(\zeta)\times\Im^*(\zeta)\right]\\
&\quad+\mathcal{M}^*((\zeta,\zeta+\phi(\text{z},\mho)),\ \text{II})\int_0^1\left[\text{II}^{\beta-1}+(1-\text{II})^{\beta-1}\right]\hbar_1(\text{II})\hbar_2(1-\text{II})d\text{II}\\
&\quad+\aleph^*((\zeta,\zeta+\phi(\text{z},\zeta)),\ \mho)\int_0^1\left[\text{II}^{\beta-1}+(1-\text{II})^{\beta-1}\right]\hbar_1(1-\text{II})\hbar_2(1-\text{II})d\text{II},
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{\beta\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\ \Psi_\mho\left(\frac{2\zeta+\phi(\text{z},\zeta)}{2}\right)\times\Im_\mho\left(\frac{2\zeta+\phi(\text{z},\zeta)}{2}\right)\\
&\supseteq_I\frac{\Gamma(\beta)}{(\phi(\text{z},\zeta))^\beta}\left[\mathcal{I}^\beta_{\zeta^+}\ \Psi_\mho(\zeta+\phi(\text{z},\zeta))\times\Im_\mho(\zeta+\phi(\text{z},\zeta))+\mathcal{I}^\beta_{\zeta+\phi(\text{z},\zeta)^-}\ \Psi_\mho(\zeta)\times\Im_\mho(\zeta)\right]\\
&\quad+\mathcal{M}_\mho(\zeta,\zeta+\phi(\text{z},\zeta))\int_0^1\left[\text{II}^{\beta-1}+(1-\text{II})^{\beta-1}\right]\hbar_1(1-\text{II})\hbar_2(1-\text{II})d\text{II}\\
&\quad+\aleph_\mho(\zeta,\zeta+\phi(\text{z},\zeta))\int_0^1\left[\text{II}^{\beta-1}+(1-\text{II})^{\beta-1}\right]\hbar_1(1-\text{II})\hbar_2(1-\text{II})d\text{II}
\end{aligned}$$

that is

$$\begin{aligned}
&\frac{1}{\beta\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\odot\ \widetilde{\Psi}\left(\frac{2\zeta+\phi(\text{z},\zeta)}{2}\right)\otimes\widetilde{\Im}\left(\frac{2\zeta+\phi(\text{z},\zeta)}{2}\right)\\
&\supseteq_{\mathbb{F}}\frac{\Gamma(\beta)}{(\phi(\text{z},\zeta))^\beta}\odot\left[\mathcal{I}^\beta_{\zeta^+}\ \widetilde{\Psi}(\zeta+\phi(\text{z},\zeta))\otimes\widetilde{\Im}(\zeta+\phi(\text{z},\zeta))\oplus\mathcal{I}^\beta_{\zeta+\phi(\text{z},\zeta)^-}\ \widetilde{\Psi}(\zeta)\otimes\widetilde{\Im}(\zeta)\right]\\
&\quad\oplus\widetilde{\mathcal{M}}(\zeta,\zeta+\phi(\text{z},\zeta))\odot\int_0^1\left[\text{II}^{\beta-1}+(1-\text{II})^{\beta-1}\right]\hbar_1(\text{II})\hbar_2(1-\text{II})d\text{II}\\
&\quad\oplus\widetilde{\aleph}(\zeta,\zeta+\phi(\text{z},\zeta))\odot\int_0^1\left[\text{II}^{\beta-1}+(1-\text{II})^{\beta-1}\right]\hbar_1(1-\text{II})\hbar_2(1-\text{II})d\text{II}.
\end{aligned}$$

Hence, the required result. □

In upcoming outcomes, we will obtain new versions of $H{\cdot}H$-Fejér inequality using a fuzzy Riemann–Liouville fractional integral. A nontrivial example is also given to discuss the validation of the first and second fuzzy fractional $H{\cdot}H$-Fejér inequalities for UD-$\hbar$-pre-invex $\mathcal{FIVM}$.

Theorem 8. *Let* $\widetilde{\Psi}:\left[\zeta,\ \zeta+\phi(\text{z},\zeta)\right]\to \pounds_o$ be an *UD-$\hbar$-pre-invex $\mathcal{FIVM}$ with $\zeta<\text{v}$, for which the $\mho$-cuts define the family of $\mathcal{IVM}$s. $\Psi_\mho:[\zeta,\ \zeta+\phi(\text{z},\zeta)]\subset\mathbb{R}\to\mathbb{X}_C^+$ are given by $\Psi_\mho(\text{z})=[\Psi_*(\text{z},\mho),\ \Psi^*(\text{z},\mho)]$ for all $\text{z}\in[\zeta,\ \zeta+\phi(\text{z},\zeta)]$ and for all $\mho\in[0,\ 1]$. If* $\widetilde{\Psi}\in$

$L\Big([\zeta,\ \zeta+\phi(z,\zeta)],\ \pounds_o\ \Big)$ *and* $\mathfrak{Z}:[\zeta,\ \zeta+\phi(z,\zeta)]\to\mathbb{R},\ \mathfrak{Z}(z)\geq 0,$ *symmetric with respect to* $\frac{2\zeta+\phi(z,\zeta)}{2}$, *then*

$$\begin{aligned} &\frac{\Gamma(\beta)}{(\phi(z,\zeta))^{\beta}}\odot\Big[\mathcal{I}^{\beta}_{\zeta^{+}}\ \widetilde{\Psi}\mathfrak{Z}(\zeta+\phi(z,\zeta))\oplus\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^{-}}\ \widetilde{\Psi}\mathfrak{Z}(\zeta)\Big]\\ &\supseteq_{\mathbb{F}}\Big(\widetilde{\Psi}(\zeta)\oplus\widetilde{\Psi}(\zeta+\phi(z,\zeta))\Big)\odot\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg\\ &\supseteq_{\mathbb{F}}\Big(\widetilde{\Psi}(\zeta)\oplus\widetilde{\Psi}(z)\Big)\odot\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg. \end{aligned} \tag{46}$$

If $\widetilde{\Psi}$ *is pre-incave* $\mathcal{FIVM}$ *, then inequality (46) is reversed.*

Proof. Let $\widetilde{\Psi}$ be an UD-$\hbar$-pre-invex $\mathcal{FIVM}$ and $\amalg^{\beta-1}\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta))\geq 0$. Then, for each $\mho\in[0,\ 1]$, we have

$$\begin{aligned} &\amalg^{\beta-1}\Psi_*(\zeta+(1-\amalg)\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta))\\ &\leq \amalg^{\beta-1}(\hbar(\amalg)\Psi_*(\zeta,\ \mho)+\hbar(1-\amalg)\Psi_*(\zeta+\phi(z,\zeta),\ \mho))\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta))\\ &\amalg^{\beta-1}\Psi^*(\zeta+(1-\amalg)\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta))\\ &\geq \amalg^{\beta-1}(\hbar(\amalg)\Psi^*(\zeta,\ \mho)+\hbar(1-\amalg)\Psi^*(\zeta+\phi(z,\zeta),\ \mho))\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta)), \end{aligned} \tag{47}$$

and

$$\begin{aligned} &\amalg^{\beta-1}\Psi_*(\zeta+\amalg\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))\\ &\leq \amalg^{\beta-1}(\hbar(1-\amalg)\Psi_*(\zeta,\ \mho)+\hbar(\amalg)\Psi_*(\zeta+\phi(z,\zeta),\ \mho))\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))\\ &\amalg^{\beta-1}\Psi^*(\zeta+\amalg\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))\\ &\geq \amalg^{\beta-1}(\hbar(1-\amalg)\Psi^*(\zeta,\ \mho)+\hbar(\amalg)\Psi^*(\zeta+\phi(z,\zeta),\ \mho))\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta)). \end{aligned} \tag{48}$$

After adding (47) and (48), and integrating over $[0,\ 1]$, we obtain

$$\begin{aligned} &\int_0^1 \amalg^{\beta-1}\Psi_*(\zeta+(1-\amalg)\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta))d\amalg\\ &+\int_0^1 \amalg^{\beta-1}\Psi_*(\zeta+\amalg\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg\\ &\leq \int_0^1\left[\begin{array}{c}\amalg^{\beta-1}\Psi_*(\zeta,\ \mho)\{\hbar(\amalg)\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta))+\hbar(1-\amalg)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))\}\\ +\amalg^{\beta-1}\Psi_*(\zeta+\phi(z,\zeta),\mho)\{\hbar(1-\amalg)\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta))+\hbar(\amalg)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))\}\end{array}\right]d\amalg,\\ &=\Psi_*(\zeta,\ \mho)\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta))d\amalg\\ &+\Psi_*(\zeta+\phi(z,\zeta),\ \mho)\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg,\\ &\int_0^1 \amalg^{\beta-1}\Psi^*(\zeta+\amalg\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg\\ &+\int_0^1 \amalg^{\beta-1}\Psi^*(\zeta+(1-\amalg)\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta))d\amalg\\ &\geq \int_0^1\left[\begin{array}{c}\amalg^{\beta-1}\Psi^*(\zeta,\ \mho)\{\hbar(\amalg)\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta))+\hbar(1-\amalg)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))\}\\ +\amalg^{\beta-1}\Psi^*(\zeta+\phi(z,\zeta),\ \mho)\{\hbar(1-\amalg)\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta))+\hbar(\amalg)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))\}\end{array}\right]d\amalg,\\ &=\Psi^*(\zeta,\ \mho)\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta))d\amalg\\ &+\Psi^*(\zeta+\phi(z,\zeta),\ \mho)\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg. \end{aligned} \tag{49}$$

Taking right hand side of inequality (49), we have

$$\begin{aligned}
&\int_0^1 \text{II}^{\beta-1}\Psi_*(\zeta+(1-\text{II})\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\text{II}\phi(z,\zeta))d\text{II} \\
&+\int_0^1 \text{II}^{\beta-1}\Psi_*(\zeta+\text{II}\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\text{II}\phi(z,\zeta))d\text{II} \\
&=\frac{1}{(\phi(z,\zeta))^\beta}\int_\zeta^{\zeta+\phi(z,\zeta)}(z-\zeta)^{\beta-1}\Psi_*(2\zeta+\phi(z,\zeta)-z,\mho)\mathfrak{Z}(z)dz \\
&+\frac{1}{(\phi(z,\zeta))^\beta}\int_\zeta^{\zeta+\phi(z,\zeta)}(z-\zeta)^{\beta-1}\Psi_*(z,\mho)\mathfrak{Z}(z)dz \\
&=\frac{1}{(\phi(z,\zeta))^\beta}\int_\zeta^{\zeta+\phi(z,\zeta)}(\zeta+\phi(z,\zeta)-z)^{\beta-1}\Psi_*(z,\mho)\mathfrak{Z}(2\zeta+\phi(z,\zeta)-z)dz \\
&+\frac{1}{(\phi(z,\zeta))^\beta}\int_\zeta^{\zeta+\phi(z,\zeta)}(z-\zeta)^{\beta-1}\Psi_*(z,\mho)\mathfrak{Z}(z)dz \\
&=\frac{\Gamma(\beta)}{(\phi(z,\zeta))^\beta}\left[\mathcal{I}_{\zeta^+}^\beta\ \Psi_*\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^\beta\ \Psi_*\mathfrak{Z}(\zeta)\right], \\
&\int_0^1 \text{II}^{\beta-1}\Psi^*(\zeta+(1-\text{II})\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\text{II}\phi(z,\zeta))d\text{II} \\
&+\int_0^1 \text{II}^{\beta-1}\Psi^*(\zeta+\text{II}\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\text{II}\phi(z,\zeta))d\text{II} \\
&=\frac{\Gamma(\beta)}{(\phi(z,\zeta))^\beta}\left[\mathcal{I}_{\zeta^+}^\beta\ \Psi^*\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^\beta\ \Psi^*\mathfrak{Z}(\zeta)\right].
\end{aligned} \tag{50}$$

From (50), we have

$$\begin{aligned}
&\frac{\Gamma(\beta)}{(\phi(z,\zeta))^\beta}\left[\mathcal{I}_{\zeta^+}^\beta\ \Psi_*\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^\beta\ \Psi_*\mathfrak{Z}(\zeta)\right] \\
&\le \frac{\Psi_*(\zeta,\ \mho)+\Psi_*(\zeta+\phi(z,\zeta),\ \mho)}{2}\int_0^1 \text{II}^{\beta-1}\left[\hbar(\text{II})+\hbar(1-\text{II})\right]\mathfrak{Z}(\zeta+\text{II}\phi(z,\zeta)) \\
&\frac{\Gamma(\beta)}{(\phi(z,\zeta))^\beta}\left[\mathcal{I}_{\zeta^+}^\beta\ \Psi^*\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^\beta\ \Psi^*\mathfrak{Z}(\zeta)\right] \\
&\ge \frac{\Psi^*(\zeta,\ \mho)+\Psi^*(\zeta+\phi(z,\zeta),\ \mho)}{2}\int_0^1 \text{II}^{\beta-1}\left[\hbar(\text{II})+\hbar(1-\text{II})\right]\mathfrak{Z}(\zeta+\text{II}\phi(z,\zeta)),
\end{aligned}$$

that is

$$\begin{aligned}
&\frac{\Gamma(\beta)}{(\phi(z,\zeta))^\beta}\left[\mathcal{I}_{\zeta^+}^\beta\Psi_*\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^\beta\Psi_*\mathfrak{Z}(\zeta),\ \mathcal{I}_{\zeta^+}^\beta\Psi^*\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^\beta\Psi^*\mathfrak{Z}(\zeta)\right] \\
&\supseteq_I \left[\frac{\Psi_*(\zeta,\ \mho)+\Psi_*(\zeta+\phi(z,\zeta),\ \mho)}{2},\ \frac{\Psi^*(\zeta,\ \mho)+\Psi^*(\zeta+\phi(z,\zeta),\ \mho)}{2}\right]\int_0^1 \text{II}^{\beta-1}[\hbar(\text{II})+\hbar(1-\text{II})]\mathfrak{Z}(\zeta+\text{II}\phi(z,\zeta))d\text{II}
\end{aligned}$$

hence

$$\begin{aligned}
&\frac{\Gamma(\beta)}{(\phi(z,\zeta))^\beta}\odot\left[\mathcal{I}_{\zeta^+}^\beta\ \widetilde{\Psi}\mathfrak{Z}(\zeta+\phi(z,\zeta))\oplus\mathcal{I}_{\zeta+\phi(z,\zeta)^-}^\beta\ \widetilde{\Psi}\mathfrak{Z}(\zeta)\right] \\
&\supseteq_{\mathbb{F}} \frac{\widetilde{\Psi}(\zeta)\oplus\widetilde{\Psi}(\zeta+\phi(z,\zeta))}{2}\odot\int_0^1 \text{II}^{\beta-1}[\hbar(\text{II})+\hbar(1-\text{II})]\mathfrak{Z}(\zeta+\text{II}\phi(z,\zeta))d\text{II}. \\
&\supseteq_{\mathbb{F}} \frac{\widetilde{\Psi}(\zeta)\oplus\widetilde{\Psi}(z)}{2}\odot\int_0^1 \text{II}^{\beta-1}[\hbar(\text{II})+\hbar(1-\text{II})]\mathfrak{Z}(\zeta+\text{II}\phi(z,\zeta))d\text{II}.
\end{aligned}$$

□

Theorem 9. *Let* $\widetilde{\Psi}:\left[\zeta,\ \zeta+\phi(z,\zeta)\right]\ \to \pounds_o$ *be an UD-*$\hbar$*-pre-invex* $\mathcal{F}\mathfrak{I}\mathcal{V}\mathcal{M}$ *with* $\zeta<\zeta+\phi(z,\zeta)$*, whose* $\mho$*-cuts define the family of* $\mathfrak{I}\mathcal{V}\mathcal{M}$*s.* $\Psi_\mho:[\zeta,\ \zeta+\phi(z,\zeta)]\subset\mathbb{R}\to\mathbb{X}_C^+$ *are given by* $\Psi_\mho(z)=[\Psi_*(z,\mho),\ \Psi^*(z,\mho)]$ *for all* $z\in[\zeta,\ \zeta+\phi(z,\zeta)]$ *and for all* $\mho\in[0,\ 1]$*. Let*

$\widetilde{\Psi} \in L\Big([\zeta,\ \zeta+\phi(z,\zeta)],\ \pounds_o\Big)$ *and* $\mathfrak{Z} : [\zeta,\ \zeta+\phi(z,\zeta)] \to \mathbb{R},\ \mathfrak{Z}(z) \geq 0,$ *symmetric with respect to* $\frac{2\zeta+\phi(z,\zeta)}{2}$. *If* ϕ *satisfies Condition C, then*

$$\begin{aligned}&\frac{1}{2\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\Psi}\Big(\frac{2\zeta+\phi(z,\zeta)}{2}\Big)\Big[\mathcal{I}^{\beta}_{\zeta^+}\mathfrak{Z}(\zeta+\phi(z,\zeta)) + \mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\mathfrak{Z}(\zeta)\Big]\\ &\supseteq_{\mathbb{F}} \Big[\mathcal{I}^{\beta}_{\zeta^+}\widetilde{\Psi}\mathfrak{Z}(\zeta+\phi(z,\zeta)) \oplus \mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\widetilde{\Psi}\mathfrak{Z}(\zeta)\Big]\end{aligned} \tag{51}$$

If $\widetilde{\Psi}$ *is pre-incave* $\mathcal{FIVM}$ *, then inequality (51) is reversed.*

Proof. Since $\widetilde{\Psi}$ is an UD-$\hbar$-pre-invex $\mathcal{FIVM}$, then for $\mho \in [0,\ 1]$, we have

$$\begin{aligned}\Psi_*\Big(\frac{2\zeta+\phi(z,\zeta)}{2},\ \mho\Big) &\leq \hbar\Big(\frac{1}{2}\Big)(\Psi_*(\zeta+(1-\amalg)\phi(z,\zeta),\ \mho) + \Psi_*(\zeta+\amalg\phi(z,\zeta),\ \mho))\\ \Psi^*\Big(\frac{2\zeta+\phi(z,\zeta)}{2},\ \mho\Big) &\geq \hbar\Big(\frac{1}{2}\Big)(\Psi^*(\zeta+(1-\amalg)\phi(z,\zeta),\ \mho) + \Psi^*(\zeta+\amalg\phi(z,\zeta),\ \mho)).\end{aligned} \tag{52}$$

Since $\mathfrak{Z}(\zeta+(1-\amalg)\phi(z,\zeta)) = \mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))$ then by multiplying (52) by $\amalg^{\beta-1}\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))$ and integrating it with respect to $\amalg$ over $[0,\ 1]$, we obtain

$$\begin{aligned}&\Psi_*\Big(\frac{2\zeta+\phi(z,\zeta)}{2},\ \mho\Big)\int_0^1 \amalg^{\beta-1}\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg\\ &\leq \hbar\Big(\frac{1}{2}\Big)\left(\begin{array}{c}\int_0^1 \amalg^{\beta-1}\Psi_*(\zeta+(1-\amalg)\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg\\ +\int_0^1 \amalg^{\beta-1}\Psi_*(\zeta+\amalg\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg\end{array}\right),\\ &\Psi^*\Big(\frac{2\zeta+\phi(z,\zeta)}{2},\ \mho\Big)\int_0^1 \amalg^{\beta-1}\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg\\ &\geq \hbar\Big(\frac{1}{2}\Big)\left(\begin{array}{c}\int_0^1 \amalg^{\beta-1}\Psi^*(\zeta+(1-\amalg)\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg\\ +\int_0^1 \amalg^{\beta-1}\Psi^*(\zeta+\amalg\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg\end{array}\right).\end{aligned} \tag{53}$$

Let $z = \zeta+\amalg\phi(z,\zeta)$. Then, for the right hand side of inequality (54), we have

$$\begin{aligned}&\int_0^1 \amalg^{\beta-1}\Psi_*(\zeta+(1-\amalg)\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg\\ &+\int_0^1 \amalg^{\beta-1}\Psi_*(\zeta+\amalg\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg\\ &= \frac{1}{(\phi(z,\zeta))^{\beta}}\int_{\zeta}^{\zeta+\phi(z,\zeta)}(z-\zeta)^{\beta-1}\Psi_*(2\zeta+\phi(z,\zeta)-z,\mho)\mathfrak{Z}(z)dz\\ &+\frac{1}{(\phi(z,\zeta))^{\beta}}\int_{\zeta}^{\zeta+\phi(z,\zeta)}(z-\zeta)^{\beta-1}\Psi_*(z,\mho)\mathfrak{Z}(z)dz\\ &= \frac{1}{(\phi(z,\zeta))^{\beta}}\int_{\zeta}^{\zeta+\phi(z,\zeta)}(z-\zeta)^{\beta-1}\Psi_*(z,\mho)\mathfrak{Z}(2\zeta+\phi(z,\zeta)-z)dz\\ &+\frac{1}{(\phi(z,\zeta))^{\beta}}\int_{\zeta}^{\zeta+\phi(z,\zeta)}(z-\zeta)^{\beta-1}\Psi_*(z,\mho)\mathfrak{Z}(z)dz\\ &= \frac{\Gamma(\beta)}{(\phi(z,\zeta))^{\beta}}\Big[\mathcal{I}^{\beta}_{\zeta^+}\ \Psi_*\mathfrak{Z}(\zeta+\phi(z,\zeta)) + \mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\ \Psi_*\mathfrak{Z}(\zeta)\Big],\\ &\int_0^1 \amalg^{\beta-1}\Psi^*(\zeta+(1-\amalg)\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg\\ &+\int_0^1 \amalg^{\beta-1}\Psi^*(\zeta+\amalg\phi(z,\zeta),\ \mho)\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta))d\amalg\\ &= \frac{\Gamma(\beta)}{(\phi(z,\zeta))^{\beta}}\Big[\mathcal{I}^{\beta}_{\zeta^+}\ \Psi^*\mathfrak{Z}(\zeta+\phi(z,\zeta)) + \mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\ \Psi^*\mathfrak{Z}(\zeta)\Big].\end{aligned} \tag{54}$$

Then, from (54), (53) we have

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\Psi_*\left(\frac{2\zeta+\phi(z,\zeta)}{2},\mho\right)\left[\mathcal{I}^{\beta}_{\zeta^+}\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\mathfrak{Z}(\zeta)\right]$$
$$\leq\left[\mathcal{I}^{\beta}_{\zeta^+}\Psi_*\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\Psi_*\mathfrak{Z}(\zeta)\right]$$
$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\Psi^*\left(\frac{2\zeta+\phi(z,\zeta)}{2},\mho\right)\left[\mathcal{I}^{\beta}_{\zeta^+}\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\mathfrak{Z}(\zeta)\right]$$
$$\geq\left[\mathcal{I}^{\beta}_{\zeta^+}\Psi^*\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\Psi^*\mathfrak{Z}(\zeta)\right],$$

from which, we have

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\left[\Psi_*\left(\frac{2\zeta+\phi(z,\zeta)}{2},\mho\right),\ \Psi^*\left(\frac{2\zeta+\phi(z,\zeta)}{2},\mho\right)\right]\left[\mathcal{I}^{\beta}_{\zeta^+}\ \mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\ \mathfrak{Z}(\zeta)\right]$$
$$\supseteq_I\left[\mathcal{I}^{\beta}_{\zeta^+}\Psi_*\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\Psi_*\mathfrak{Z}(\zeta),\ \mathcal{I}^{\beta}_{\zeta^+}\Psi^*\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\Psi^*\mathfrak{Z}(\zeta)\right],$$

it follows that

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\Psi_{\mho}\left(\frac{2\zeta+\phi(z,\zeta)}{2}\right)\left[\mathcal{I}^{\beta}_{\zeta^+}\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\mathfrak{Z}(\zeta)\right]\supseteq_I\left[\mathcal{I}^{\beta}_{\zeta^+}\ \Psi_{\mho}\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\Psi_{\mho}\mathfrak{Z}(\zeta)\right]$$

that is

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\odot\widetilde{\Psi}\left(\frac{2\zeta+\phi(z,\zeta)}{2}\right)\left[\mathcal{I}^{\beta}_{\zeta^+}\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\mathfrak{Z}(\zeta)\right]\supseteq_{\mathbb{F}}\left[\mathcal{I}^{\beta}_{\zeta^+}\ \widetilde{\Psi}\mathfrak{Z}(\zeta+\phi(z,\zeta))\oplus\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\widetilde{\Psi}\mathfrak{Z}(\zeta)\right].$$

This completes the proof. □

Remark 5. *If one attempts to take $\mathfrak{Z}(z)=1$, then from (46) and (51) one achieves Theorem 5.*

If one attempts to take $\hbar(\amalg)=\amalg$, then from (46) and (51) one achieves the following inequality.

$$\begin{aligned}&\widetilde{\Psi}\left(\frac{2\zeta+\phi(z,\zeta)}{2}\right)\odot\left[\mathcal{I}^{\beta}_{\zeta^+}\ \mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\mathfrak{Z}(\zeta)\right]\\&\supseteq_{\mathbb{F}}\left[\mathcal{I}^{\beta}_{\zeta^+}\ \widetilde{\Psi}\mathfrak{Z}(\zeta+\phi(z,\zeta))\oplus\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\ \widetilde{\Psi}\mathfrak{Z}(\zeta)\right]\\&\supseteq_{\mathbb{F}}\frac{\widetilde{\Psi}(\zeta)\oplus\widetilde{\Psi}(\zeta+\phi(z,\zeta))}{2}\odot\left[\mathcal{I}^{\beta}_{\zeta^+}\ \mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\ \mathfrak{Z}(\zeta)\right]\\&\supseteq_{\mathbb{F}}\frac{\widetilde{\Psi}(\zeta)\oplus\widetilde{\Psi}(z)}{2}\odot\left[\mathcal{I}^{\beta}_{\zeta^+}\mathfrak{Z}(\zeta+\phi(z,\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(z,\zeta)^-}\ \mathfrak{Z}(\zeta)\right]\end{aligned}\tag{55}$$

Let one attempt to take $\hbar(\amalg)=\amalg$ and $\beta=1$. Then, from (46) and (51) one achieves coming inequality for UD-pre-invex $\mathcal{FIVM}$, see [59].

$$\widetilde{\Psi}\left(\frac{2\zeta+\phi(z,\zeta)}{2}\right)\supseteq_{\mathbb{F}}\frac{1}{\int_{\zeta}^{\zeta+\phi(z,\zeta)}\mathfrak{Z}(z)dz}\odot(FR)\int_{\zeta}^{\zeta+\phi(z,\zeta)}\widetilde{\Psi}(z)\mathfrak{Z}(z)dz\supseteq_{\mathbb{F}}\frac{\widetilde{\Psi}(\zeta)\oplus\widetilde{\Psi}(z)}{2}\tag{56}$$

Let one attempt to take $\hbar(\amalg) = \amalg$ and $\beta = 1 = \mathfrak{Z}(z)$. Then, from (46) and (51) one achieves the following inequality for UD-pre-invex $\mathcal{FIVM}$ given in [59]:

$$\widetilde{\Psi}\left(\frac{2\zeta+\phi(z,\zeta)}{2}\right) \supseteq_{\mathbb{F}} (FR)\int_{\zeta}^{\zeta+\phi(z,\zeta)} \widetilde{\Psi}(z)dz \supseteq_{\mathbb{F}} \frac{\widetilde{\Psi}(\zeta) \oplus \widetilde{\Psi}(z)}{2} \tag{57}$$

If one attempts to take $\Psi_*(z, \mho) = \Psi^*(z, \mho)$ and $1 = \mho$ and $\hbar(\amalg) = \amalg$, then from (46) and (51) one achieves the following inequality given in [81]:

$$\begin{aligned} \Psi\left(\tfrac{2\zeta+\phi(z,\zeta)}{2}\right)\left[\mathcal{I}_{\zeta^+}^{\beta}\ \mathfrak{Z}(z) + \mathcal{I}_{z^-}^{\beta}\ \mathfrak{Z}(\zeta)\right] &\le \left[\mathcal{I}_{\zeta^+}^{\beta}\ \Psi\mathfrak{Z}(\zeta+\phi(z,\zeta)) + \mathcal{I}_{\zeta+\phi(z,\zeta)^-}^{\beta}\ \Psi\mathfrak{Z}(\zeta)\right] \\ &\le \tfrac{\Psi(\zeta)+\Psi(\zeta+\phi(z,\zeta))}{2}\left[\mathcal{I}_{\zeta^+}^{\beta}\ \mathfrak{Z}(\zeta+\phi(z,\zeta)) + \mathcal{I}_{\zeta+\phi(z,\zeta)^-}^{\beta}\ \mathfrak{Z}(\zeta)\right] \\ &\le \tfrac{\Psi(\zeta)+\Psi(z)}{2}\left[\mathcal{I}_{\zeta^+}^{\beta}\ \mathfrak{Z}(\zeta+\phi(z,\zeta)) + \mathcal{I}_{\zeta+\phi(z,\zeta)^-}^{\beta}\ \mathfrak{Z}(\zeta)\right] \end{aligned} \tag{58}$$

If one attempts to take $\Psi_*(z, \mho) = \Psi^*(z, \mho)$ and $\beta = 1 = \mho$ and $\hbar(\amalg) = \amalg$, then from (46) and (51) one achieves the classical HH-Fejér inequality.

If one attempts to take $\Psi_*(z, \mho) = \Psi^*(z, \mho)$ and $\mathfrak{Z}(z) = \beta = 1 = \mho$ and $\hbar(\amalg) = \amalg$, then from (46) and (51) one achieves the classical HH-inequality.

Example 3. *If we attempt to take* $\mathcal{FIVM}$ $\widetilde{\Psi}: [0, 2] \to £_o$ *defined by,*

$$\widetilde{\Psi}(z)(\theta) = \begin{cases} \frac{\theta-2+\sqrt{z}}{\sqrt{z}-\frac{1}{2}}, & \theta \in \left[2-\sqrt{z}, \frac{3}{2}\right], \\ \frac{2+\sqrt{z}-\theta}{\frac{1}{2}+\sqrt{z}}, & \theta \in \left(\frac{3}{2}, 2+\sqrt{z}\right], \\ 0, & \text{otherwise}. \end{cases}$$

Then, for each $\mho \in [0, 1]$, we have $\Psi_{\mho}(z) = \left[(1-\mho)\left(2-\sqrt{z}\right) + \frac{3}{2}\mho, (1-\mho)\left(2+\sqrt{z}\right) + \frac{3}{2}\mho\right]$. Since endpoint functions $\Psi_*(z,\mho)$, $\Psi^*(z,\mho)$ are $\hbar$-pre-invex functions for each $\mho \in [0, 1]$, then $\widetilde{\Psi}(z)$ is UD-$\hbar$-pre-invex $\mathcal{FIVM}$. If

$$\mathfrak{Z}(z) = \begin{cases} \sqrt{z}, & \theta \in [0,1], \\ \sqrt{2-z}, & \theta \in (1, 2], \end{cases}$$

then $\mathfrak{Z}(2-z) = \mathfrak{Z}(z) \ge 0$, for all $z \in [0, 2]$. Since $\Psi_*(z,\mho) = (1-\mho)\left(2-\sqrt{z}\right) + \frac{3}{2}\mho$ and $\Psi^*(z,\mho) = (1-\mho)\left(2+\sqrt{z}\right) + \frac{3}{2}\mho$. If $\hbar(\amalg) = \amalg$ and $\beta = \frac{1}{2}$, then we compute the following:

$$\begin{aligned} &[\Psi_*(\zeta, \mho) + \Psi_*(\zeta+\phi(z,\zeta), \mho)]\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta)) \\ &\quad = (1-\mho)\sqrt{\pi}\left(\tfrac{4-\sqrt{2}}{2}\right) + \tfrac{3}{2}\sqrt{\pi}\mho, \\ &[\Psi^*(\zeta, \mho) + \Psi^*(\zeta+\phi(z,\zeta), \mho)]\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta)) \\ &\quad = (1-\mho)\sqrt{\pi}\left(\tfrac{4+\sqrt{2}}{2}\right) + \tfrac{3}{2}\sqrt{\pi}\mho, \end{aligned} \tag{59}$$

$$\begin{aligned} &[\Psi_*(\zeta, \mho) + \Psi_*(\zeta+\phi(\zeta+\phi(z,\zeta),\zeta), \mho)]\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta)) \\ &\quad = (1-\mho)\sqrt{\pi}\left(\tfrac{4-\sqrt{2}}{2}\right) + \tfrac{3}{2}\sqrt{\pi}\mho, \\ &[\Psi^*(\zeta, \mho) + \Psi^*(\zeta+\phi(\zeta+\phi(z,\zeta),\zeta), \mho)]\int_0^1 \amalg^{\beta-1}[\hbar(\amalg)+\hbar(1-\amalg)]\mathfrak{Z}(\zeta+\amalg\phi(z,\zeta)) \\ &\quad = (1-\mho)\sqrt{\pi}\left(\tfrac{4+\sqrt{2}}{2}\right) + \tfrac{3}{2}\sqrt{\pi}\mho, \end{aligned}$$

$$\frac{\Gamma(\beta)}{(\phi(\mathrm{Z},\zeta))^{\beta}}\left[\mathcal{I}^{\beta}_{\zeta^{+}}\Psi_{*}\mathfrak{Z}(\zeta+\phi(\mathrm{z},\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(\mathrm{Z},\zeta)^{-}}\Psi_{*}\mathfrak{Z}(\zeta)\right]=\frac{1}{\sqrt{2}}(1-\mho)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)+\frac{3}{2\cdot\sqrt{2}}\pi\mho,$$
$$\frac{\Gamma(\beta)}{(\phi(\mathrm{Z},\zeta))^{\beta}}\left[\mathcal{I}^{\beta}_{\zeta^{+}}\Psi^{*}\mathfrak{Z}(\zeta+\phi(\mathrm{z},\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(\mathrm{Z},\zeta)^{-}}\Psi^{*}\mathfrak{Z}(\zeta)\right]=\frac{1}{\sqrt{2}}(1-\mho)\left(2\pi+\frac{8\sqrt{2}-4}{3}\right)+\frac{3}{2\cdot\sqrt{2}}\pi\mho. \tag{60}$$

From (59) and (60), we have

$$\frac{1}{\sqrt{2}}\left[(1-\mho)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)+\frac{3}{2}\pi\mho,\ (1-\mho)\left(2\pi+\frac{8\sqrt{2}-4}{3}\right)+\frac{3}{2}\pi\mho\right]$$
$$\supseteq_{I}\sqrt{\pi}\left[(1-\mho)\left(\frac{4-\sqrt{2}}{2}\right)+\frac{3}{2}\mho,\ (1-\mho)\left(\frac{4+\sqrt{2}}{2}\right)+\frac{3}{2}\mho\right],\text{ for each }\mho\in[0,\ 1].$$

Hence, (46) is verified.
For (51), we have

$$\begin{aligned}
&\mathcal{I}^{\beta}_{\zeta^{+}}\ \Psi_{*}\mathfrak{Z}(\zeta+\phi(\mathrm{z},\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(\mathrm{Z},\zeta)^{-}}\Psi_{*}\mathfrak{Z}(\zeta)\\
&=\frac{1}{\sqrt{\pi}}\int_{0}^{2}(2-\mathrm{z})^{\frac{-1}{2}}\ \mathfrak{Z}(\mathrm{z})\left((1-\mho)(2-\sqrt{\mathrm{z}})+\frac{3}{2}\mho\right)d\mathrm{z}\\
&+\frac{1}{\sqrt{\pi}}\int_{0}^{2}(\mathrm{z})^{\frac{-1}{2}}\mathfrak{Z}(\mathrm{z})\left((1-\mho)(2-\sqrt{\mathrm{z}})+\frac{3}{2}\mho\right)d\mathrm{z}\\
&=\frac{1}{\sqrt{\pi}}\left[(1-\mho)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)+\frac{3}{2\cdot}\pi\mho\right]\\
&\mathcal{I}^{\beta}_{\zeta^{+}}\ \Psi^{*}\mathfrak{Z}(\zeta+\phi(\mathrm{z},\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(\mathrm{Z},\zeta)^{-}}\ \Psi^{*}\mathfrak{Z}(\zeta)\\
&=\frac{1}{\sqrt{\pi}}\int_{0}^{2}(2-\mathrm{z})^{\frac{-1}{2}}\ \mathfrak{Z}(\mathrm{z})\left((1-\mho)(2+\sqrt{\mathrm{z}})+\frac{3}{2}\mho\right)d\mathrm{z}\\
&+\frac{1}{\sqrt{\pi}}\int_{0}^{2}(\mathrm{z})^{\frac{-1}{2}}\mathfrak{Z}(\mathrm{z})\left((1-\mho)(2+\sqrt{\mathrm{z}})+\frac{3}{2}\mho\right)d\mathrm{z}\\
&=\frac{1}{\sqrt{\pi}}\left[(1-\mho)\left(2\pi+\frac{8\sqrt{2}-4}{3}\right)+\frac{3}{2\cdot}\pi\mho\right].
\end{aligned} \tag{61}$$

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\Psi_{*}\left(\frac{2\zeta+\phi(\mathrm{Z},\zeta)}{2},\ \mho\right)\left[\mathcal{I}^{\beta}_{\zeta^{+}}\ \mathfrak{Z}(\zeta+\phi(\mathrm{z},\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(\mathrm{Z},\zeta)^{-}}\ \mathfrak{Z}(\zeta)\right]=(1-\mho)\sqrt{\pi}+\frac{3}{2}\sqrt{\pi}\mho,$$
$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\Psi^{*}\left(\frac{2\zeta+\phi(\mathrm{Z},\zeta)}{2},\ \mho\right)\left[\mathcal{I}^{\beta}_{\zeta^{+}}\ \mathfrak{Z}(\zeta+\phi(\mathrm{z},\zeta))+\mathcal{I}^{\beta}_{\zeta+\phi(\mathrm{Z},\zeta)^{-}}\ \mathfrak{Z}(\zeta)\right]=3(1-\mho)\sqrt{\pi}+\frac{3}{2}\sqrt{\pi}\mho. \tag{62}$$

From (61) and (62), we have

$$\left[(1-\mho)\sqrt{\pi}+\frac{3}{2}\sqrt{\pi}\mho,\ 3(1-\mho)\sqrt{\pi}+\frac{3}{2}\sqrt{\pi}\mho\right]$$
$$\supseteq_{I}\frac{1}{\sqrt{\pi}}\left[(1-\mho)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)+\frac{3}{2\cdot}\pi\mho,\ (1-\mho)\left(2\pi+\frac{8\sqrt{2}-4}{3}\right)+\frac{3}{2\cdot}\pi\mho\right],\text{ for each }\mho\in[0,\ 1].$$

4. Conclusions

Fuzzy-number valued mapping is a good method for incorporating uncertainty into prediction systems. We demonstrated fractional versions of the Hermite–Hadamard-, Fejér-, and Pachpatte-type inequalities using a novel concept from [59]. We showed that our results can lead to a few new results for the h-preinvex mapping and the h-convex mapping in fuzzy-number valued settings. The well-known Riemann–Liouville fractional integral was used in a novel method for solving *UD*-fuzzy ordered inequalities. Some numerical examples were also looked at to help explain the findings. Future work could adapt this strategy to include other fractional operators such as tempered, Atangana–Baleanu, Caputo–Fabrizio, and generalized fractional integral operators. Various non-symmetric functions can also be used using these methods.

Future presentations of various inequalities, including those of the Hermite–Hadamard, Ostrowski, Jensen–Mercer, Bullen, and Simpson types, can be obtained using this new this idea. A variety of interval-valued quantum calculus, fuzzy calculus, and fractional calculus can all be used to establish related inequalities.

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