A. Carfagna D'Andrea; R. Mazzocco; Giuliano Romani

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SOME CHARACTERIZATIONS OF 2-SYMMETRIC SUBMANIFOLDS IN SPACES OF CONSTANT CURVATURE*

A. CARFAGNA D'ANDREA, R. MAZZOCCO, G. ROMANI, Roma

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INTRODUCTION

The notion of a (locally) symmetric submanifold M of the Euclidean space $\mathbb{R}^n$ was given by Ferus [F] as a submanifold (locally) mapped into itself, for each $x \in M$, by the reflection of $\mathbb{R}^n$ with respect to the (affine) normal space to M at x. Strübing studied locally symmetric submanifolds of any Riemannian manifold [St].

In [K-K] Kowalski and Kulich gave the notion of a k-symmetric submanifold of $\mathbb{R}^n$, $k \geqslant 2$, using suitable isometries of $\mathbb{R}^n$. They observed that a 2-symmetric submanifold M is invariant under the reflections of $\mathbb{R}^n$ with respect to subspaces of the normal spaces of M; so the 2-symmetric submanifolds appear as a generalization of the symmetric submanifolds. It is easy to check that a totally geodesic submanifold of a symmetric submanifold is a 2-symmetric submanifold but generally is not a symmetric submanifold.

In this paper we give a local extension of the notion of a 2-symmetric submanifold M of $\mathbb{R}^n$ considering submanifolds of a space $\overline{M}$ of constant curvature (Definition 2.1).

We remark that Definition 2.1, if $\overline{M} \equiv \mathbb{R}^n$, does not agree with that of Kowalski and Kulich but its local modification. For such 2-symmetric submanifolds there exists a local involutive isometry σ_x of $\overline{M}$, at $x \in M$, which maps M into itself. We prove that if $\overline{M}$ is a standard space form ($\overline{M}$ is a complete, simply connected Riemannian manifold of constant curvature) then the local isometry σ_x is in fact the restriction to a neighbourhood of a geodesic reflection of $\overline{M}$, with respect to a complete totally geodesic submanifold F_x of $\overline{M}$ orthogonally meeting M at x (Proposition 2.2). Since each geodesic reflection of a space form with respect to a totally geodesic submanifold

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is an isometry (Theorem 1.2), we have an alternative definition for 2-symmetric submanifolds in terms of geodesic reflections (Corollary 2.8).

Moreover, for nicely curved submanifolds (Definition 1.3) we prove the following results. Let M be an essential 2-symmetric submanifold (Definition 1.11) of a space form $\overline{M}$. Then we prove that the totally geodesic submanifold F_x coincides with the image of the direct sum of the normal spaces of odd order under the exponential map; otherwise F_x has to contain this image (Corollary 3.5). For 2-symmetric submanifolds of a space of constant curvature we prove (Theorem 3.7) that the derivative of the k-fundamental form $\overset{k}{s}$ of M, $\nabla \overset{k}{s}$, is equal to zero.

In our main result (Theorem 4.23) we characterize a 2-symmetric submanifold of a space of zero curvature by means of the derivatives of $\overset{k}{s}$. Theorem 4.23 is a corollary of Theorem 4.21, which characterizes a 2-symmetric submanifold M of a Euclidean space both in terms of $\nabla \overset{k}{s} = 0$ and in terms of a totally geodesic map v of M in a suitable Grassmannian. Theorem 4.21 is an extension of the analogous theorem for an $\overset{1}{N}$-symmetric submanifold of Euclidean spaces ([R]).

In Section 5 we extend the results of Section 4 to the case of a space of constant positive curvature.

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1. Preliminaries

We now recall some definitions and results which will be useful in the sequel.

1.1. Theorem. *Let $\overline{M}$ be a standard space form. For each point $x \in \overline{M}$ and each subspace $V_x \subset T_x\overline{M}$, $L_x = \exp_x(V_x)$ is the only complete totally geodesic submanifold through x with V_x as the tangent space at x.*

For the proof see [S].

Let M be a topologically embedded submanifold of a Riemannian manifold $\overline{M}$. The geodesic reflection ϱ_M with respect to M is defined as follows. For $z = \exp_y(X_y^\perp)$, where $y \in M$, $X_y^\perp \in (T_yM)^\perp$, let

$$\varrho_M(z) = \exp_y(-X_y^\perp).$$

It is possible to prove that ϱ_M is a well defined local deffeomorfism of a tubular neighbourhood of M ([G]).

1.2. Theorem. *If L is a totally geodesic submanifold topologically embedded in a space of constant curvature $\overline{M}$, then the geodesic reflection ϱ_L is an isometry.*

For the proof see [C-V].

Because each local isometry of a standard space form is the restriction of a global one, if L is a complete totally geodesic submanifold of a standard space form $\overline{M}$, then by Theorem 1.2 we deduce that ϱ_L is an isometry of the whole space $\overline{M}$.

Let now $\overline{M}$ be a Riemannian manifold with a metric $\bar{g}$ and the Levi-Civita connection $\overline{\nabla}$. If M is a submanifold of $\overline{M}$, we shall denote by g the induced metric and by ∇ the Levi-Civita connection of g. The k-normal space of M at $x \in M$ will be denoted by $\overset{k}{N}_x M$. By $\overset{k}{s}$, $\overset{k}{A}$ we shall denote, respectively, the k-fundamental form and the k-Weingarten map of M. For these definitions see [S] (Volume 4, Chapter 7) or [CG-R].

1.3. Definition. A submanifold M of a Riemannian manifold $\overline{M}$ is called nicely curved in $\overline{M}$ if $\dim \overset{k}{N}_x M$ does not depend on the point $x \in M$ for each $k = 1, \ldots$.

We assume $\overset{0}{N}_x M = T_x M$. If M is nicely curved in $\overline{M}$ it is known that there exists an integer l, independent of the point $x \in M$, such that $\overset{l}{N}_x M \neq 0$ and $\overset{l+1}{N}_x M = \overset{l+2}{N}_x M = \ldots = 0$. In the present paper l shall always have this meaning.

Let M be a nicely curved submanifold of $\overline{M}$.

We recall that, for each $x \in M$,

$$\overset{k}{s} : T_x M \times \overset{k}{N}_x M \to \overset{k+1}{N}_x M, \qquad 0 \leqslant k \leqslant l-1$$

and

$$\overset{k}{A} : T_x M \times \overset{k}{N}_x M \to \overset{k-1}{N}_x M, \qquad 1 \leqslant k \leqslant l$$

are bilinear maps.

If $\overset{k}{\xi} \in \Gamma(\overset{k}{N} M)$ is an extension of $\overset{k}{\xi}_x \in \overset{k}{N}_x M$, the connection $\overline{\nabla}$ induces a connection $\overset{k}{\nabla}$ on the vector bundle $\overset{k}{N} M$ by the formula

$$\overset{k}{\nabla}_{X_x} \overset{k}{\xi} = \overset{k}{\Pi}(\overline{\nabla}_{X_x} \overset{k}{\xi}),$$

where $X_x \in T_x M$ and $\overset{k}{\Pi}$ is the orthogonal projection of $T_x \overline{M}$ onto $\overset{k}{N}_x M$. Then we have the Frenet equations of M:

$$\overline{\nabla}_{X_x} \overset{k}{\xi} = -\overset{k}{A}(X_x, \overset{k}{\xi}_x) + \overset{k}{\nabla}_{X_x} \overset{k}{\xi} + \overset{k}{s}(X_x, \overset{k}{\xi}_x). \tag{1.4}$$

The derivatives $\nabla \overset{k}{s}$, $\nabla \overset{k}{A}$ are defined by the formulas

$$(1.5)\qquad (\nabla \overset{k}{s})(X_x, Y_x, \overset{k}{\xi_x}) = \overset{k+1}{\nabla}_{X_x} \overset{k}{s}(Y, \overset{k}{\xi}) - \overset{k}{s}(\nabla_{X_x} Y, \overset{k}{\xi_x}) - \overset{k}{s}(Y_x, \overset{k}{\nabla}_{X_x} \overset{k}{\xi}),$$

$$(1.6)\qquad (\nabla \overset{k}{A})(X_x, Y_x, \overset{k}{\xi_x}) = \overset{k-1}{\nabla}_{X_x} \overset{k}{A}(Y, \overset{k}{\xi}) - \overset{k}{A}(\nabla_{X_x} Y, \overset{k}{\xi_x}) - \overset{k}{A}(Y_x, \overset{k}{\nabla}_{X_x} \overset{k}{\xi}),$$

where $Y \in \Gamma(TM)$ and $\overset{k}{\xi} \in \Gamma(\overset{k}{N} M)$ extend respectively X_x and $\overset{k}{\xi_x}$.

For each $x \in M$,

$$\nabla \overset{k}{s} : T_x M \times T_x M \times \overset{k}{N_x} M \to \overset{k+1}{N_x} M,$$
$$\nabla \overset{k}{A} : T_x M \times T_x M \times \overset{k}{N_x} M \to \overset{k-1}{N_x} M$$

are multilinear forms which are symmetric with respect to the first two arguments ([S]).

$\overset{k}{A}$ and $\overset{k-1}{s}$ are "dual" in the following sense:

$$\bar{g}(\overset{k}{A}(X, \overset{k}{\xi}), \overset{k-1}{\eta}) = \bar{g}(\overset{k}{\xi}, \overset{k-1}{s}(X, \overset{k-1}{\eta})),$$

([S]). We deduce

$$\bar{g}((\nabla \overset{k}{A})(X, Y, \overset{k}{\xi}), \overset{k-1}{\eta}) = \bar{g}(\overset{k}{\xi}, (\nabla \overset{k-1}{s})(X, Y, \overset{k-1}{\eta})).$$

We now have the following theorem.

1.7. Theorem. *If M is a nicely curved submanifold of a Riemannian manifold $\overline{M}$, then the condition $\nabla \overset{k}{s} = 0$ for each $k = 0, 1, \ldots, l-1$ is equivalent to the condition $\nabla \overset{k}{A} = 0$ for each $k = 1, 2, \ldots, l$.*

Moreover, under the hypothesis of the previous theorem we have

1.8. Theorem. *If $\sigma \colon \overline{M} \to \overline{M}$ is an isometry such that $\sigma(M) \subset M$, then $\mathrm{d}\sigma\, (\overset{k}{N_x} M) = \overset{k}{N}_{\sigma(x)} M$ for each $x \in M$.*

Finally, if $G(p, \overline{m})$ is the Grassmannian of the p-planes of $\mathbb{R}^{\overline{m}}$, we recall ([V] and [CG-R]) that there exists a canonical isomorphism $\varphi \colon T_\alpha G(p, \overline{m}) \to \mathrm{Hom}(\alpha, \alpha^\perp)$, where $\alpha^\perp$ is the $(\overline{m} - p)$-plane of $\mathbb{R}^{\overline{m}}$ through the origin, orthogonal to the p-plane α.

Let $\psi \colon M \to G(p, \overline{m})$ be a map and $\mathrm{d}\psi$ its differential. We denote $\varphi(\mathrm{d}\psi\,(X_x))(v_x)$ by $\psi_*(X_x, v_x)$ for $X_x \in T_x M$ and $v_x \in \psi(x)$.

Then, if we denote $\psi(x)$ by V_x, we have

$$\psi_*(X_x, v_x) = \Pi_{V_x^\perp} \overline{\nabla}_{X_x} v, \tag{1.9}$$

where $\Pi_{V_x^\perp}$ is the orthogonal projection of $\mathbb{R}^{\overline{m}}$ onto $V_x^\perp$ and v is a vector field of $\mathbb{R}^{\overline{m}}$, defined along a curve $c = c(t)$ of M through $x = c(0)$, with the tangent vector X_x at x, which extends v_x and such that $v_{c(t)} \in V_{c(t)} = \psi(c(t))$.

As $\psi_* \colon T_xM \times V_x \to V_x^\perp$ is a multilinear form for each $x \in M$, we can consider the derivative $\nabla\psi_* \colon T_xM \times T_xM \times V_x \to V_x^\perp$ defined by

$$(\nabla\psi_*)(X_x, Y_x, v_x) = \overset{V^\perp}{\nabla}_{X_x} \psi_*(Y, v) - \psi_*(\nabla_{X_x} Y, v_x) - \psi_*(Y_x, \overset{V}{\nabla}_{X_x} v), \tag{1.10}$$

where $\overset{V}{\nabla}$ and $\overset{V^\perp}{\nabla}$ are the connections on the vector bundles V and $V^\perp$, respectively, induced by the connection $\overline{\nabla}$ of $\mathbb{R}^{\overline{m}}$.

We conclude this section by recalling the following definition.

1.11. Definition. A submanifold M of a space form $\overline{M}$ is called an essential submanifold if it is not contained in any proper totally geodesic submanifold of $\overline{M}$.

1.12. Proposition. *If M is a connected nicely curved submanifold of a space of constant curvature $\overline{M}$ then M is an essential submanifold of a totally geodesic submanifold of $\overline{M}$ of dimension equal to* $\dim(\bigoplus_{k=0}^{l} \overset{k}{N}_x M)$. *In particular, the normal space $\perp_x M$ of M at x is given by*

$$\perp_x M = \bigoplus_{k=1}^{l} \overset{k}{N}_x M$$

([S], Vol. 4, Prop. 67).

2. 2-SYMMETRIC SUBMANIFOLDS OF A SPACE FORM

Let $\overline{M} = \overline{M}(c)$ be a space of constant curvature c and of dimension $\overline{m}$. Let M be a submanifold of $\overline{M}$ of dimension m.

2.1. Definition. M is a 2-symmetric submanifold of $\overline{M}$ if, for each $x \in M$, there exist a neighbourhood $\overline{U}_x$ of x in $\overline{M}$, a neighbourhood U_x of x in M, and an isometry $\sigma_x \colon \overline{U}_x \to \overline{U}_x$ such that

(i) $\sigma_x^2 = \mathrm{Id}_{\overline{U}_x}$,
(ii) $U_x \subset \overline{U}_x$ and $\sigma_x(U_x) \subset U_x$,
(iii) x is an isolated fixed point of σ_x in U_x.

2.2. Proposition. *Let M be a submanifold of a standard space form $\overline{M}$ and $x \in M$. If there exists a neighbourhood $\overline{U}_x$ of x in $\overline{M}$ and an isometry $\sigma_x : \overline{U}_x \to \overline{U}_x$ satisfying the conditions* (i), (ii), (iii) *of Definition* 2.1, *then σ_x can be extended to a geodesic reflection of $\overline{M}$ with respect to a complete totally geodesic submanifold F_x of $\overline{M}$ meeting M orthogonally at x.*

Proof. For the sake of brevity, we shall denote the global extension of σ_x to $\overline{M}$ by the same symbol σ_x. Because of (i) and $\sigma_x(x) = x$, the differential $\mathrm{d}\sigma_x$ of σ_x is an operator of $T_x\overline{M}$ such that $(\mathrm{d}\sigma_x)^2 = \mathrm{Id}_{T_x\overline{M}}$. Then $T_x\overline{M} = V_1(T_x\overline{M}) \oplus V_{-1}(T_x\overline{M})$, where $V_1(T_x\overline{M})$ and $V_{-1}(T_x\overline{M})$ are, respectively, the eigenspaces of 1 and -1.

Condition (ii) implies that T_xM is an invariant subspace. As before, $(\mathrm{d}\sigma_x|_{T_xM})^2 = \mathrm{Id}_{T_xM}$ and $T_xM = V_1(T_xM) \ominus V_{-1}(T_xM)$.

If $X_x \in V_1(T_xM) (\subset T_xM)$, then because σ_x an isometry, we have

$$\sigma_x\big(\exp_x(tX_x)\big) = \exp_{\sigma_x(x)}\big(\mathrm{d}\sigma_x(tX_x)\big) = \exp_x(tX_x).$$

Then $X_x = 0$, otherwise x would not be an isolated fixed point of $U_x = M \cap \overline{U}_x$, as required in condition (iii). We conclude that $V_1(T_xM) = 0$ and hence

$$T_xM \subset V_{-1}(T_x\overline{M}). \tag{2.3}$$

In particular, we have

$$V_1(T_x\overline{M}) \perp T_xM. \tag{2.4}$$

Let now F_x be the totally geodesic submanifold of $\overline{M}$ given by

$$F_x = \exp_x\big(V_1(T_x\overline{M})\big). \tag{2.5}$$

Due to (2.4), F_x meets orthogonally M at x.

We observe that all points of F_x are fixed by σ_x; in fact, if $y \in F_x$, there exists a vector $X_x \in T_xF_x = V_1(T_x\overline{M})$ such that $y = \exp_x(X_x)$ and then

$$\sigma_x(y) = \sigma_x\big(\exp_x(X_x)\big) = \exp_{\sigma_x(x)}(X_x) = \exp_x(X_x) = y.$$

Moreover, in $\overline{M}$ there are no fixed points different from the points of F_x.

In fact, if $y \in \overline{M}$, we can take a vector $X_x \in T_x\overline{M}$ such that $y = \exp_x(X_x)$ and, if we suppose $\sigma_x(y) = y$, then

$$\exp_x(X_x) = y = \sigma_x(y) = \sigma_x\big(\exp_x(X_x)\big) = \exp_{\sigma_x(x)}\big(\mathrm{d}\sigma_x(X_x)\big) = \exp_x\big(\mathrm{d}\sigma_x(X_x)\big)$$

and the local bijectivity of $\exp_x$ implies that $\mathrm{d}\sigma_x(X_x) = X_x$, that is $X_x \in V_1(T_x\overline{M})$, and $y \in F_x$.

Then for $y \in F_x$ we have

$$T_yF_x = V_1(T_y\overline{M}) \tag{2.6}$$

and

$$(T_yF_x)^{\perp} = V_{-1}(T_y\overline{M}). \tag{2.7}$$

By the definition of the geodesic reflection (see Theorem 1.2) we have, for each $y \in F_x$, $\varrho_{F_x}(y) = y$ and thus $\varrho_{F_x}\big|_{F_x} = \sigma_x\big|_{F_x}$.

If $z \notin F_x$ but z is near F_x, we can write $z = \exp_y(X_y^{\perp})$, where $y \in F_x$ and $X_y^{\perp} \in (T_yF_x)^{\perp}$. Using (2.7), we have

$$\sigma_x(z) = \sigma_x\big(\exp_y(X_y^{\perp})\big) = \exp_y\big(\mathrm{d}\sigma_x(X_y^{\perp})\big) = \exp_y(-X_y^{\perp})$$

and hence, from the definition of ϱ_{F_x}, we obtain $\varrho_{F_x}(z) = \sigma_x(z)$. □

As an immediate consequence of Proposition 2.2 we have the following corollary.

2.8. Corollary. *M is a 2-symmetric submanifold of a standard space form $\overline{M}$ if and only if, for each $x \in M$, there exists a complete totally geodesic submanifold F_x of $\overline{M}$, meeting M at x orthogonally, such that the geodesic reflection ϱ_{F_x} with respect to F_x locally maps M into itself.*

With reference to this corollary, we shall say that M is symmetric with respect to F_x and F_x will be called a submanifold of symmetry for M.

We conclude this section with the following definition, which is equivalent to that of Ferus if $\overline{M} = \mathbb{R}^m$.

2.9. Definition. M is called a (locally) symmetric submanifold of $\overline{M}$ if, for each $x \in M$, M is (locally) mapped into itself by the geodesic reflection of $\overline{M}$ with respect to $\exp_x(\perp_x M)$.

2.10. Remark. By Corollary 2.8, a locally symmetric submanifold M of $\overline{M}$ is just a 2-symmetric submanifold with the submanifold of symmetry $F_x = \exp_x(\perp_x M)$, $x \in M$.

3. Properties of 2-symmetric submanifolds

Henceforth we will consider only nicely curved submanifolds.

3.1. Lemma. *If M is a submanifold of a Riemannian manifold $\overline{M}$ and $\sigma\colon \overline{M} \to \overline{M}$ is an isometry such that $\sigma(M) \subset M$, then, for $x \in M$, $X_x, Y_x \in T_x M$ and $\overset{k}{\xi}_x \in \overset{k}{N}_x M$, we have*

(i) $\mathrm{d}\sigma\left(\overset{k}{s}(X_x, \overset{k}{\xi}_x)\right) = \overset{k}{s}\left(\mathrm{d}\sigma\,(X_x), \mathrm{d}\sigma\,(\overset{k}{\xi}_x)\right)$,

(ii) $\mathrm{d}\sigma\left[(\nabla \overset{k}{s})(X_x, Y_x\, \overset{k}{\xi}_x)\right] = (\nabla \overset{k}{s})\left(\mathrm{d}\sigma\,(X_x), \mathrm{d}\sigma\,(Y_x), \mathrm{d}\sigma\,(\overset{k}{\xi}_x)\right)$.

Proof. These properties are obvious because σ is an isometry and $\sigma(M)$ is an open part of M. □

The following proposition gives a more precise property than property (2.4).

3.2. Proposition. *Let M be a submanifold of a space of constant curvature $\overline{M}$ and $x \in M$. If there exist two neighbourhoods $\overline{U}_x$, U_x of x in $\overline{M}$, M, respectively, and an isometry $\sigma_x\colon \overline{U}_x \to \overline{U}_x$ satisfying the conditions* (i), (ii), (iii) *of Definition* 2.1, *we have*

$$\text{(a)}\qquad V_1(T_x\overline{M}) \supset \bigoplus_{h=0}^{q} \overset{2h+1}{N_x} M, \quad V_{-1}(T_x\overline{M}) \supset \bigoplus_{h=0}^{p} \overset{2h}{N_x} M,$$

where $q = \left[\frac{l-1}{2}\right]$ and $p = \left[\frac{l}{2}\right]$.

In particular, if M is also an essential submanifold, then

$$\text{(b)}\qquad V_1(T_x\overline{M}) = \bigoplus_{h=0}^{q} \overset{2h+1}{N_x} M, \quad V_{-1}(T_x\overline{M}) = \bigoplus_{h=0}^{p} \overset{2h}{N_x} M.$$

Proof. By (2.3) and Lemma 3.1 we have $\mathrm{d}\sigma_x\left(\overset{0}{s}(X_x, Y_x)\right) = \overset{0}{s}\left(\mathrm{d}\sigma_x\,(X_x), \mathrm{d}\sigma_x\,(Y_x)\right) = \overset{0}{s}(X_x, Y_x)$, for $X_x, Y_x \in T_x M$. Then $\overset{0}{s}(X_x, Y_x) \in V_1(T_x\overline{M})$, but $\overset{1}{N_x} M$ is spanned by $\left\{\overset{0}{s}(X_x, Y_x) \mid X_x, Y_x \in T_x M\right\}$ and so

$$\text{(3.3)}\qquad \overset{1}{N_x} M \subset V_1(T_x\overline{M}).$$

Take $X_x \in T_x M$, $\overset{1}{\xi}_x \in \overset{1}{N_x} M$; by (2.3), Lemma 3.1 and (3.3) we have $\mathrm{d}\sigma_x\left(\overset{1}{s}(X_x, \overset{1}{\xi}_x)\right) = \overset{1}{s}\left(\mathrm{d}\sigma_x\,(X_x), \mathrm{d}\sigma_x\,(\overset{1}{\xi}_x)\right) = \overset{1}{s}(-X_x, \overset{1}{\xi}_x) = -\overset{1}{s}(X_x, \overset{1}{\xi}_x)$. This implies $\overset{1}{s}(X_x, \overset{1}{\xi}_x) \in V_{-1}(T_x\overline{M})$, but $\overset{2}{N_x} M$ is spanned by $\left\{\overset{1}{s}(X_x, \overset{1}{\xi}_x) \mid X_x \in T_x M,\ \xi_x \in \overset{1}{N_x} M\right\}$ and so

$$\text{(3.4)}\qquad \overset{2}{N_x} M \subset V_{-1}(T_x\overline{M}).$$

Proceeding by induction, we obtain the statement (a).

If M is also an essential submanifold of $\overline{M}$, then $T_x\overline{M} = \big(\bigoplus_{h=0}^{q} \overset{2h+1}{N_x} M\big) \oplus \big(\bigoplus_{h=0}^{p} \overset{2h}{N_x} M\big)$. However, $T_x\overline{M} = V_1(T_x\overline{M}) \oplus V_{-1}(T_x\overline{M})$, so the inclusions $V_1(T_x\overline{M}) \supset \bigoplus_{h=0}^{q} \overset{2h+1}{N_x} M$ and $V_{-1}(T_x\overline{M}) \supset \bigoplus_{h=0}^{p} \overset{2h}{N_x} M$ give the statement (b). □

Let now $\overline{M}$ be a standard space form. Since the connected component of M through x is contained in $\exp_x \big(\bigoplus_{k=0}^{l} \overset{k}{N_x} M\big)$ ([S]), then putting $\overline{B}_x = \big(\bigoplus_{k=0}^{l} \overset{k}{N_x} M\big)^{\perp}$ we have

3.5. Corollary. *If M is a 2-symmetric submanifold of a standard space form $\overline{M}$, then, for each $x \in M$, a totally geodesic submanifold F_x is a submanifold of symmetry for M through x if and only if it is of the type*

$$F_x = \exp_x \Big(\bigoplus_{h=0}^{q} \overset{2h+1}{N_x} M \oplus B_x\Big),$$

where B_x is an arbitrary subspace of $\overline{B}_x$. In particular, if M is also essential, $F_x = \exp_x \big(\bigoplus_{h=0}^{q} \overset{2h+1}{N_x} M\big)$.

3.6. Remark. If M is a non-essential 2-symmetric submanifold of $\overline{M}$, then we see from Corollary 3.5 that, for each $x \in M$, submanifold of symmetry for M through x is not unique, but each of them induces the same geodesic reflection on the totally geodesic submanifold $\tilde{B} = \exp_x \big(\bigoplus_{k=0}^{l} \overset{k}{N_x} M\big) \supset M$.

The main result we want to prove in this section is the following theorem.

3.7. Theorem. *If M is a 2-symmetric submanifold of a space of constant curvature $\overline{M}$, then $\nabla \overset{k}{s} = 0$ for each $k = 0, 1, \ldots, l-1$ (and hence, by Theorem 1.7, $\nabla \overset{k}{A} = 0$ for each $k = 1, 2, \ldots, l$).*

Proof. We first recall that $(\nabla \overset{k}{s})(X_x, Y_x, \overset{k}{\xi_x}) \in \overset{k+1}{N_x} M$. If k is even, then $k+1$ is odd and, by Proposition 3.2, we have

$$(\nabla \overset{k}{s})(X_x, Y_x, \overset{k}{\xi_x}) \in \overset{k+1}{N_x} M \subset V_1(T_x\overline{M}).$$

Then for $\sigma = \sigma_x$

$$\mathrm{d}\sigma \big((\nabla \overset{k}{s})(X_x, Y_x, \overset{k}{\xi_x})\big) = (\nabla \overset{k}{s})(X_x, Y_x, \overset{k}{\xi_x}), \tag{3.8}$$

but by Lemma 3.1 and Proposition 3.2 this yields

$$\begin{aligned}\mathrm{d}\sigma\left[(\nabla \overset{k}{s})(X_x, Y_x, \overset{k}{\xi_x})\right] &= (\nabla \overset{k}{s})\big(\mathrm{d}\sigma(X_x), \mathrm{d}\sigma(Y_x), \mathrm{d}\sigma(\overset{k}{\xi_x})\big)\\ &= (\nabla \overset{k}{s})(-X_x, -Y_x, -\overset{k}{\xi_x}) = -(\nabla \overset{k}{s})(X_x, Y_x, \overset{k}{\xi_x}).\end{aligned}$$

Hence, we conclude that $\nabla \overset{k}{s} = 0$ for k even.

If k is odd, then $k+1$ is even and, exactly as in the case of k even, we deduce $\nabla \overset{k}{s} = 0$. □

4. Characterization of 2-symmetric submanifolds of a space form of zero curvature

Let M be a submanifold of the Euclidean space $\mathbb{R}^{\overline{m}}$. We denote by V the vector bundle on M whose fiber at $x \in M$ is

$$V_x = \bigoplus_{h=0}^{q} \overset{2h+1}{N_x} M, \qquad q = \left[\frac{l-1}{2}\right]. \tag{4.1}$$

Let $p = \dim V$ and let $G(p, \overline{m})$ be the Grassmannian of the p-planes of $\mathbb{R}^{\overline{m}}$ through the origin. We define a map $\upsilon\colon M \to G(p, \overline{m})$ by

$$\upsilon(x) = V_x, \qquad x \in M. \tag{4.2}$$

The map υ is said to be totally geodesic if $\nabla \upsilon_* = 0$ (see (1.10)).

The following proposition holds.

4.3. Proposition. *If M is a submanifold of $\mathbb{R}^{\overline{m}}$ such that $\nabla \overset{k}{s} = 0$ for each $k = 0, 1, \ldots, l-1$, then the map υ is totally geodesic.*

Proof. Let $\gamma = \gamma(s)$ be the geodesic through $x = \gamma(0) \in M$, with X_x as the tangent vector at x. The tangent vector of γ at the point $\gamma(s)$ will be denoted by $X(s)$.

Let $Y \in \Gamma(TM)$ and $\xi \in \Gamma(V)$ be parallel along γ in TM and V, respectively, i.e.

$$\nabla_{X(s)} Y = 0, \quad \overset{V}{\nabla}_{X(s)} \xi = 0. \tag{4.4}$$

Since $\xi_{\gamma(s)} \in V_{\gamma(s)} = \bigoplus_{h=0}^{q} \overset{2h+1}{N_{\gamma(s)}}$, we have $\xi_{\gamma(s)} = \sum_{h=0}^{q} \overset{2h+1}{\xi_{\gamma(s)}}$ with $\overset{2h+1}{\xi_{\gamma(s)}} \in \overset{2h+1}{N_{\gamma(s)}} M$.

It is easy to check that $\overset{V}{\nabla}_{X(s)}\, \xi = 0$ implies

$$\overset{2h+1}{\nabla}_{X(s)} \overset{2h+1}{\xi} = 0, \qquad h = 0, 1, \ldots, q. \tag{4.5}$$

Then, by (1.10) for $\psi_* = v_*$ and $v_x = \xi_x$ and by (4.4), (4.5) we obtain

$$(\nabla v_*)(X_x, Y_x, \xi_x) = \overset{V^\perp}{\nabla}_{X_x} v_*(Y, \xi) = \Pi_{V^\perp}\big(\overline{\nabla}_{X_x} v_*(Y, \xi)\big).$$

From (1.9) we have

$$v_*(Y, \xi) = \Pi_{V^\perp}(\overline{\nabla}_Y \xi) = \sum_{h=0}^{q} \Pi_{V^\perp}(\overline{\nabla}_Y \overset{2h+1}{\xi})$$

and, using the Frenet equations of M, we obtain

$$v_*(Y, \xi) = \sum_{h=0}^{q} \Pi_{V^\perp}\Big[- \overset{2h+1}{A}(Y, \overset{2h+1}{\xi}) + \overset{2h+1}{\nabla}_Y \overset{2h+1}{\xi} + \overset{2h+1}{s}(Y, \overset{2h+1}{\xi})\Big].$$

But $\overset{2h+1}{A}(Y, \overset{2h+1}{\xi}) \in \overset{2h}{N} M \subset V^\perp$, $\overset{2h+1}{\nabla}_Y \overset{2h+1}{\xi} \in \overset{2h+1}{N} M \subset V$, $\overset{2h+1}{s}(Y, \overset{2h+1}{\xi}) \in \overset{2h+1}{N} M \subset V^\perp$, so we conclude

$$v_*(Y, \xi) = \sum_{h=0}^{q} \Big[- \overset{2h+1}{A}(Y, \overset{2h+1}{\xi}) + \overset{2h+1}{s}(Y, \overset{2h+1}{\xi})\Big]$$

and hence

$$(\nabla v_*)(X_x, Y_x, \xi_x) = \sum_{h=0}^{q} \Pi_{V^\perp}\Big[\overline{\nabla}_{X_x}\big(- \overset{2h+1}{A}(Y, \overset{2h+1}{\xi}) + \overset{2h+1}{s}(Y, \overset{2h+1}{\xi})\big)\Big].$$

Applying again the Frenet equations, we have

$$\begin{aligned}
(\nabla v_*)(X_x, Y_x, \xi_x) &= \Pi_{V^\perp}\Big(\overline{\nabla}_{X_x}\big(- \overset{1}{A}(Y, \overset{1}{\xi})\big) + \overset{0}{s}\big(X_x, - \overset{1}{A}(Y, \overset{1}{\xi})\big)\Big) \\
&+ \sum_{h=1}^{q} \Pi_{V^\perp}\Big(\overset{2h}{A}\big(X_x, \overset{2h+1}{A}(Y_x, \overset{2h+1}{\xi_x})\big) - \overset{2h}{\nabla}_{X_x}\big(\overset{2h+1}{A}(Y, \overset{2h+1}{\xi})\big) \\
&- \overset{2h}{s}\big(X_x, \overset{2h+1}{A}(Y_x, \overset{2h+1}{\xi_x})\big)\Big) + \sum_{h=0}^{q} \Pi_{V^\perp}\Big(- \overset{2h+2}{A}\big(X_x, \overset{2h+1}{s}(Y_x, \overset{2h+1}{\xi_x})\big) \\
&+ \overset{2h+2}{\nabla}_{X_x}\big(\overset{2h+1}{s}(Y, \overset{2h+2}{\xi})\big) + \overset{2h+2}{s}\big(X_x, \overset{2h+1}{s}(Y_x, \overset{2h+1}{\xi_x})\big)\Big) \\
&= \nabla_{X_x}\big(- \overset{1}{A}(Y, \overset{1}{\xi})\big) + \sum_{h=1}^{q} \Big[- \overset{2h}{\nabla}\big(\overset{2h+1}{A}(Y, \overset{2h+1}{\xi})\big) + \overset{2h+2}{\nabla}_{X_x}\big(\overset{2h+1}{s}(Y, \overset{2h+1}{\xi})\big)\Big] = 0.
\end{aligned}$$

□

Our next goal is to prove the following proposition, which is a generalization of a result obtained for $\overset{1}{N}$-symmetric submanifold of Euclidean spaces ([R]).

4.6. Proposition. *Let M be a submanifold of $\overline{M} = \mathbb{R}^{\overline{m}}$. If there exists a totally geodesic map $\upsilon\colon M \to G(p, \overline{m})$ such that, for each $x \in M$, $\overset{1}{N}_x M \subset \upsilon(x) \subset \perp_x M$, then M is a 2-symmetric submanifold of $\mathbb{R}^{\overline{m}}$.*

Before proving this proposition, we need some lemmas. Let V be a vector bundle on M, whose fiber V_x at $x \in M$ is such that

$$\overset{1}{N}_x M \subset V_x \subset \perp_x M.$$

We denote by W the vector bundle on M whose fiber W_x is given by $W_x = (T_x M \oplus V_x)^{\perp}$.

Let $\tilde{M}$ denote a submanifold of $\mathbb{R}^{\overline{m}}$ which is a tubular neighbourhood of M in the set $\{x + w \mid x \in M,\ w \in W_x\}$. Then the following lemma holds.

4.7. Lemma. *M is totally geodesic submanifold of $\tilde{M}$.*

Proof. Obviously $T_x\tilde{M} = T_x M \oplus W_x$, $x \in M$. Let $\tilde{\nabla}$ denote the Levi-Civita connection induced on $\tilde{M}$ by the standard connection $\overline{\nabla}$ of $\mathbb{R}^{\overline{m}}$. Then for $X_x \in T_x M$, $Y \in \Gamma(TM)$ we have

$$\tilde{\nabla}_{X_x} Y = \Pi_{T_x\tilde{M}}(\overline{\nabla}_{X_x} Y) = \Pi_{T_x M \oplus W_x}\big(\nabla_{X_x} Y + \overset{0}{s}(X_x, Y_x)\big) = \nabla_{X_x} Y.$$

□

Now we consider the Gauss map $g_{\tilde{M}}\colon \tilde{M} \to G(\tilde{m}, \overline{m})$ and the normal map $\gamma_{\tilde{M}}\colon \tilde{M} \to G(\overline{m} - \tilde{m}, \overline{m})$ of $\tilde{M}$, where $\tilde{m} = \dim \tilde{M}$. They are defined by

$$g_{\tilde{M}}(x) = T_x\tilde{M} \text{ and } \gamma_{\tilde{M}}(x) = \perp_x\tilde{M}. \tag{4.8}$$

Let $\overset{0}{s}_{\tilde{M}}$ denote the second fundamental form of $\tilde{M}$ (as a submanifold of $\mathbb{R}^{\tilde{m}}$), ([V]). Then for $\tilde{X}_x, \tilde{Y}_x \in T_x\tilde{M}$ and $\tilde{Y} \in \Gamma(T\tilde{M})$ the extension of $\tilde{Y}_x$, by (1.9) we have [V]

$$(g_{\tilde{M}})_*(\tilde{X}_x, \tilde{Y}_x) = \overset{0}{s}_{\tilde{M}}(\tilde{X}_x, \tilde{Y}_x). \tag{4.9}$$

Moreover, if $\xi \in \Gamma(\perp\tilde{M})$ is an extension of $\xi_x \in \perp_x\tilde{M}$, then

$$(\gamma_{\tilde{M}})_*(\tilde{X}_x, \xi_x) = \Pi_{T_x\tilde{M}}(\overline{\nabla}_{\tilde{X}_x}\xi). \tag{4.10}$$

Let $\upsilon\colon M \to G(p,\overline{m})$ be a map satisfying the condition

$$\overset{1}{N}_x M \subset \upsilon(x) \subset \perp_x M, \qquad x \in M. \tag{4.11}$$

If we define the vector bundle V on M taking $V_x = \upsilon(x)$, we see that M is a totally geodesic submanifold of $\tilde{M}$ (see Lemma 4.7) and the normal map $\gamma_{\tilde{M}}$ coincides with υ for all points of M:

$$\gamma_{\tilde{M}}(x) = \upsilon(x), \qquad x \in M.$$

Moreover, for $x \in M$, $X_x \in T_xM$ and $\xi_x \in V_x$ we have, by (1.9) and (4.10),

$$(\gamma_{\tilde{M}})_*(X_x, \xi_x) = \upsilon_*(X_x, \xi_x). \tag{4.12}$$

4.13. Lemma. *Let M be a submanifold of $\mathbb{R}^{\overline{m}}$. If the map $\upsilon\colon M \to G(p,\overline{m})$ satisfies (4.11) and is totally geodesic, then $\nabla_{X_x}(g_{\tilde{M}})_*(Y_x, -) = 0$ for each $x \in M$ and $X_x, Y_x \in T_xM$.*

Proof. Let $Y \in \Gamma(TM)$ and $\xi \in \Gamma(V)$ be extensions of Y_x, ξ_x, respectively. Then, by (1.10) and (4.12),

$$(\nabla\upsilon_*)(X_x, Y_x, \xi_x) = \overset{V^\perp}{\nabla}_{X_x}(\gamma_{\tilde{M}})_*(Y,\xi) - (\gamma_{\tilde{M}})_*(\nabla_{X_x}Y, \xi_x) - (\gamma_{\tilde{M}})_*(Y_x, \overset{V}{\nabla}_{X_x}\xi).$$

Since the image of $(\gamma_{\tilde{M}})_*$ is $T_x\tilde{M} = V^\perp$ we obtain by Lemma 4.7

$$(\nabla\upsilon_*)(X_x, Y_x, \xi_x) = \nabla(\gamma_{\tilde{M}})_*(X_x, Y_x, \xi_x). \tag{4.14}$$

Let $\mu\colon G(\overline{m}-p, \overline{m}) \to G(p,\overline{m})$ be the isometry which associates to an $(\overline{m}-p)$-plane through the origin of $\mathbb{R}^{\overline{m}}$ its orthogonal complement. The normal map $\gamma_{\tilde{M}}$ and the Gauss map $g_{\tilde{M}}$ are related by $\gamma_{\tilde{M}} = \mu \circ g_{\tilde{M}}$ and hence

$$\mathrm{d}\gamma_{\tilde{M}} = \mathrm{d}\mu \circ \mathrm{d}g_{\tilde{M}}. \tag{4.15}$$

Since the canonical isomorphism φ (see Section 1) commutes with ∇ ([V]) and μ is an isometry, applying (4.14) and (4.15) we can write

$$\begin{aligned}(\nabla\upsilon_*)(X_x, Y_x, \xi_x) &= \big(\nabla(\gamma_{\tilde{M}})_*\big)(X_x, Y_x, \xi_x) = \big[\big(\nabla_{X_x}(\varphi \circ \mathrm{d}\gamma_{\tilde{M}})\big)(Y_x)\big](\xi_x)\\ &= \big[\varphi \circ \mathrm{d}\mu\,\big((\nabla_{X_x} dg_{\tilde{M}})(Y_x)\big)\big](\xi_x)\\ &= \big[\varphi \circ d\mu \circ \varphi^{-1}\big((\nabla_{X_x}(\varphi \circ \mathrm{d}g_{\tilde{M}}))(Y_x)\big)\big](\xi_x).\end{aligned}$$

Then, if $\nabla\upsilon_* = 0$, we obtain that $\varphi \circ \mathrm{d}\mu \circ \varphi^{-1}\big((\nabla_{X_x}(\varphi \circ \mathrm{d}g_{\tilde{M}}))(Y_x)\big)$ is the zero morphism of $\mathrm{Hom}(V_x, V_x^\perp)$. But $\varphi \circ d\mu \circ \varphi^{-1}$ is an isomorphism, so we conclude that $\big(\nabla_{X_x}(\varphi \circ dg_{\tilde{M}})\big)(Y_x)$ is the zero morphism of $\mathrm{Hom}(V_x^\perp, V_x)$ and hence $\big(\nabla_{X_x}(\varphi \circ \mathrm{d}g_{\tilde{M}})\big)(Y_x) = \big(\nabla(g_{\tilde{M}})_*\big)(X_x, Y_x, -)$. □

If we consider the second fundamental form of $\tilde{M}$, $\overset{0}{s}_{\tilde{M}}$, with values in $\perp\tilde{M}$, we shall denote by $\overset{\perp}{\nabla}\overset{0}{s}_{\tilde{M}}$ its derivative. Then if $\tilde{X}_x, \tilde{Y}_x, \tilde{Z}_x \in T_x\tilde{M}$ and $\tilde{Y}, \tilde{Z} \in \Gamma(T\tilde{M})$ extend $\tilde{Y}_x$, $\tilde{Z}_x$, we have

$$(4.16)\quad (\overset{\perp}{\nabla}\overset{0}{s}_{\tilde{M}})(\tilde{X}_x, \tilde{Y}_x, \tilde{Z}_x) = \overset{\perp\tilde{M}}{\nabla}_{\tilde{X}_x}\big(\overset{0}{s}_{\tilde{M}}(\tilde{Y}, \tilde{Z})\big) - \overset{0}{s}_{\tilde{M}}(\tilde{\nabla}_{\tilde{X}_x}\tilde{Y}, \tilde{Z}_x) - \overset{0}{s}_{\tilde{M}}(\tilde{Y}_x, \tilde{\nabla}_{\tilde{X}_x}\tilde{Z}).$$

4.17. Lemma. *Let M be a submanifold of $\mathbb{R}^{\overline{m}}$. If $v\colon M \to G(p, \overline{m})$ is totally geodesic, then $\overset{\perp}{\nabla}\overset{0}{s}_{\tilde{M}} = 0$.*

Proof. We fix $x \in M$ and choose $X_x, Y_x \in T_xM$, $\tilde{Z}_x \in T_x\tilde{M}$; by (4.9) ([V]) we obtain

$$(\overset{\perp}{\nabla}\overset{0}{s}_{\tilde{M}})(X_x, Y_x, \tilde{Z}_x) = \big(\nabla(g_{\tilde{M}})_*\big)(X_x, Y_x, \tilde{Z}_x),$$

and since $\nabla v_* = 0$, Lemma 4.13, implies

$$(4.18)\qquad (\overset{\perp}{\nabla}\overset{0}{s}_{\tilde{M}})(X_x, Y_x, \tilde{Z}_x) = 0.$$

But $\overset{0}{s}_{\tilde{M}}$ is symmetric and hence also

$$(4.19)\qquad (\overset{\perp}{\nabla}\overset{0}{s}_{\tilde{M}})(X_x, \tilde{Z}_x, Y_x) = 0.$$

Consider now the following decomposition of the vectors $\tilde{X}_x, \tilde{Y}_x, \tilde{Z}_x \in T_x\tilde{M}$: $\tilde{X}_x = X_x + \eta_x$, $\tilde{Y}_x = Y_x + \vartheta_x$, $\tilde{Z}_x = Z_x + \zeta_x$, where $X_x, Y_x, Z_x \in T_xM$, and $\eta_x, \vartheta_x, \zeta_x \in W_x$. From (4.18), (4.19) we obtain $(\overset{\perp}{\nabla}\overset{0}{s}_{\tilde{M}})(X_x, \tilde{Y}_x, \tilde{Z}_x) = (\overset{\perp}{\nabla}\overset{0}{s}_{\tilde{M}})(X_x, Y_x, Z_x) + (\overset{\perp}{\nabla}\overset{0}{s}_{\tilde{M}})(X_x, \vartheta_x, Z_x) + (\overset{\perp}{\nabla}\overset{0}{s}_{\tilde{M}})(X_x, Y_x, \zeta_x) + (\overset{\perp}{\nabla}\overset{0}{s}_{\tilde{M}})(X_x, \vartheta_x, \zeta_x) = (\overset{\perp}{\nabla}\overset{0}{s}_{\tilde{M}})(X_x, \vartheta_x, \zeta_x)$.

Let γ be a curve of M through x with the tangent vector X_x and suppose that ϑ, ζ are obtained by the parallelism on $\tilde{M}$ along γ from ϑ_x, ζ_x. Then

$$\begin{aligned}(\overset{\perp}{\nabla}\overset{0}{s}_{\tilde{M}})(X_x, \vartheta_x, \zeta_x) &= \overset{\perp\tilde{M}}{\nabla}_{X_x}\big(\overset{0}{s}_{\tilde{M}}(\vartheta, \zeta)\big) - \overset{0}{s}_{\tilde{M}}(\tilde{\nabla}_{X_x}\vartheta, \zeta_x) - \overset{0}{s}_{\tilde{M}}(\vartheta_x, \tilde{\nabla}_{X_x}\zeta) \\ &= \overset{\perp\tilde{M}}{\nabla}_{X_x}\big(\overset{0}{s}_{\tilde{M}}(\vartheta, \zeta)\big).\end{aligned}$$

But for $x' \in \gamma$ we have $\vartheta_{x'}, \zeta_{x'} \in W_{x'}$. In fact $\vartheta_{x'}$, $\zeta_{x'}$ are obtained by the parallelism in $\tilde{M}$ along a curve γ contained in M from the vectors $\vartheta_x, \zeta_x \in W_x$ orthogonal to T_xM. Because M is a totally geodesic submanifold of $\tilde{M}$, the parallel transport on $\tilde{M}$ along a curve of M of any vector of T_xM gives also a vector of TM and hence we have $\vartheta_{x'}, \zeta_{x'} \in \perp_{x'}M \cap T_{x'}\tilde{M} (\subset W_{x'})$, for $x' \in \gamma$.

Then we can extend $\zeta_{x'}$ along the straight line $\{x' + t\vartheta_{x'} \mid t \in \mathbb{R}\}$ of $\tilde{M}$ as a constant vector, and this vector is always tangent to $\tilde{M}$. So we have $\overset{0}{s}_{\tilde{M}}(\vartheta_{x'}, \zeta_{x'}) = 0$ and $\overset{\perp \tilde{M}}{\nabla}_{X_x} \left(\overset{0}{s}_{\tilde{M}}(\vartheta, \zeta) \right) = 0$ and, consequently,

$$(\overset{\perp}{\nabla} \overset{0}{s}_{\tilde{M}})(X_x, \tilde{Y}_x, \tilde{Z}_x) = 0. \tag{4.20}$$

Now, if we use (4.18), (4.19), (4.20) and the symmetry of $\overset{0}{s}_{\tilde{M}}$ and $\overset{\perp}{\nabla} \overset{0}{s}_{\tilde{M}}$, we obtain

$$(\overset{\perp}{\nabla} \overset{0}{s}_{\tilde{M}})(\tilde{X}_x, \tilde{Y}_x, \tilde{Z}_x) = (\overset{\perp}{\nabla} \overset{0}{s}_{\tilde{M}})(\eta_x, \vartheta_x, \zeta_x).$$

However,

$$(\overset{\perp}{\nabla} \overset{0}{s}_{\tilde{M}})(\eta_x, \vartheta_x, \zeta_x) = \overset{\perp \tilde{M}}{\nabla}_{\eta_x} \left(\overset{0}{s}_{\tilde{M}}(\vartheta, \zeta) \right) - \overset{0}{s}_{\tilde{M}}(\tilde{\nabla}_{\eta_x} \vartheta, \zeta_x) - \overset{0}{s}_{\tilde{M}}(\vartheta_x, \tilde{\nabla}_{\eta_x} \zeta)$$

and, as above, we can assume ϑ, ζ constant along the straight line $\{x + t\eta_x \mid t \in \mathbb{R}\}$ of $\tilde{M}$ and tangent to $\tilde{M}$. Hence

$$(\overset{\perp}{\nabla} \overset{0}{s}_{\tilde{M}})(\eta_x, \vartheta_x, \zeta_x) = 0.$$

□

We can now prove Proposition 4.6.

Due to Lemma 4.17, $\overset{\perp}{\nabla} \overset{0}{s}_{\tilde{M}} = 0$ at the points $x \in M$. Then $\overset{0}{s}_{\tilde{M}}$ is a parallel form along each geodesic of M and so all curvatures of each geodesic of M are constant ([St], Theorem I). Moreover, the Frenet vectors of even orders of each geodesic of M are in $\perp \tilde{M} = V$, while those of odd orders are in $T\tilde{M}$. We conclude that, for $x \in M$, the reflection of $\mathbb{R}^{\bar{m}}$ with respect to the affine normal V_x maps each geodesic of M through x, into itself (see [St], Theorem I and Lemma I). Then the above reflection maps M into itself and M is a 2-symmetric submanifold of $\mathbb{R}^{\bar{m}}$.

The results obtained in Theorem 3.7, Proposition 4.3, Proposition 4.6 give the following theorem:

4.21. Theorem. *If M is a submanifold of $\mathbb{R}^{\bar{m}}$, the following properties are equivalent:*

(i) *M is a 2-symmetric submanifold,*

(ii) *$\nabla \overset{k}{s} = 0$, for each $k = 0, 1, \ldots, l-1$,*

(iii) *there exists a totally geodesic map $\upsilon\colon M \to G(p, \bar{m})$ such that $\overset{1}{N}_x M \subset \upsilon(x) \subset \perp_x M$.*

Proof. In fact, if M is a 2-symmetric submanifold of $\mathbb{R}^{\overline{m}}$, Theorem 3.7 implies $\nabla \overset{k}{s} = 0$ for each $k = 0, 1, \ldots, l-1$; then it follows from Proposition 4.3 that the map $v\colon M \to G(p, \overline{m})$ defined by (4.1) and (4.2) satisfies the conditions $\overset{1}{N_x} M \subset v(x) \subset \perp_x M$ and $\nabla v_* = 0$.

If M is a submanifold of $\mathbb{R}^{\overline{m}}$ and there exists a totally geodesic map $v\colon M \to G(p, \overline{m})$ such that $\overset{1}{N_x} M \subset v(x) \subset \perp_x M$, then, by Proposition 4.6, M is a 2-symmetric submanifold of $\mathbb{R}^{\overline{m}}$. □

4.22. Remark. If M is a submanifold of $\mathbb{R}^{\overline{m}}$ and we suppose the existence of the totally geodesic map $v\colon M \to G(p, \overline{m})$ satisfying the conditions $\overset{1}{N_x} M \subset v(x) \subset \perp_x M$, then in the proof of Proposition 4.6 we have seen that M is mapped into itself by the reflection of $\mathbb{R}^{\overline{m}}$ with respect to the affine space F_x through each point $x \in M$, parallel to $v(x)$. Then F_x is a submanifold of symmetry of M and, from Corollary 3.5, we conclude $v(x) \supset \bigoplus_{h=0}^{q} \overset{2h+1}{N_x} M$.

We have studied 2-symmetric submanifolds of $\mathbb{R}^{\overline{m}}$. It is now easy to obtain some results also in the case of a general space form $\overline{M}$ of zero curvature. In fact we have the following theorem.

4.23. Theorem. *If M is a submanifold of a space form $\overline{M}$ of zero curvature, then the following conditions are equivalent:*

(i) *M is a 2-symmetric submanifold of $\overline{M}$,*

(ii) *$\nabla \overset{k}{s} = 0$ for each $k = 0, 1, \ldots, l-1$.*

Proof. The implication (i) ⇒(ii) is just Theorem 3.7. In order to have the implication (ii) ⇒(i) we observe that the forms $\overset{k}{s}$ and their derivatives $\nabla \overset{k}{s}$ are invariant under isometries of the ambient space $\overline{M}$. If $\overline{M}$ is a space form of zero curvature, then for each $x \in M$ we can find a neighbourhood $\overline{U}'_x$ of x in $\overline{M}$ and a local isometry $\varphi\colon \overline{U}'_x \to \mathbb{R}^{\overline{m}}$. If we denote by $\overset{k}{s}_\varphi$ the k-fundamental form of $\varphi(M \cap \overline{U}'_x)$ in $\mathbb{R}^{\overline{m}}$, then

$$\nabla \overset{k}{s}_\varphi = 0.$$

It follows from Theorem 4.21 that $\varphi(M \cap \overline{U}'_x)$ is a 2-symmetric submanifold of $\mathbb{R}^{\overline{m}}$. This implies that, for each $y \in \varphi(M \cap \overline{U}'_x)$, there exists a local isometry σ_y such that the conditions (i), (ii), (iii) of Definition 2.1 are satisfied for σ_y. However, then the same conditions follow for the local isometry $\varphi^{-1} \circ \sigma_y \circ \varphi$. □

5. Characterization of 2-symmetric submanifolds of a space form of positive curvature

In this section we want to extend the previous results to the case of a submanifold M of a space of positive constant curvature: $\overline{M} = \overline{M}(c)$, $c > 0$. For the sake of simplicity we consider the case $c = 1$.

We start from $\overline{M} = S^{\overline{m}}(1) \subset \mathbb{R}^{\overline{m}+1}$.

To distinguish the derivative on $\mathbb{R}^{\overline{m}+1}$ from the induced derivative on $S^{\overline{m}}(1)$ we denote the former by $\hat{\nabla}$ and the latter by $\overline{\nabla}$.

The second fundamental form of $S^{\overline{m}}(1)$ in $\mathbb{R}^{\overline{m}+1}$ will be denoted by $\overset{0}{s}_S$. The Gauss equations of $S^{\overline{m}}(1)$ are

$$\hat{\nabla}_{X_x} Y = \overline{\nabla}_{X_x} Y + \overset{0}{s}_S(X_x Y_x), \qquad x \in S^{\overline{m}}(1),\ X_x, Y_x \in T_x S^{\overline{m}}(1),$$

and, recalling that

$$\overset{0}{s}_S(X_x, Y_x) = -\bar{g}(X_x, Y_x)x, \tag{5.1}$$

where $\bar{g}$ is the metric of $\overline{M} = S^{\overline{m}}(1)$ induced by the usual metric of $\mathbb{R}^{\overline{m}+1}$, we can write

$$\hat{\nabla}_{X_x} Y = \overline{\nabla}_{X_x} Y - \bar{g}(X_x, Y_x)x. \tag{5.2}$$

Suppose now that M is a submanifold of $\overline{M} = S^{\overline{m}}(1)$ satisfying the condition

$$\nabla \overset{k}{s} = 0, \qquad k = 0, 1, \dots, l-1, \tag{5.3}$$

where we are using the symbol $\overset{k}{s}$ to denote the k-fundamental form of M as a submanifold of $\overline{M} = S^{\overline{m}}(1)$.

Let $\overset{k}{N_x} M$ and $\perp_x M$, respectively, denote the k-normal space and the normal space at x of $M \subset \overline{M} = S^{\overline{m}}(1)$.

We define

$$V_x = \bigoplus_{h=0}^{q} \overset{2h+1}{N_x} M, \qquad x \in M, \tag{5.4}$$

and

$$\hat{V}_x = V_x \oplus \langle x \rangle, \qquad x \in M. \tag{5.5}$$

If we view M as a submanifold of $\mathbb{R}^{\overline{m}+1}$, we shall denote the k-normal space, the normal space and the k-fundamental form by $\overset{\hat{k}}{N}_x M$, $\hat{\perp}_x M$, $\overset{\hat{k}}{s}$, respectively. For $x \in M$, $X_x, Y_x \in T_x M$ and an extension $Y \in \Gamma(TM)$ of Y_x we have, using (5.2),

$$\overset{\hat{0}}{s}(X_x, Y_x) = \Pi_{\hat{\perp}_x M}(\hat{\nabla}_{X_x} Y) = \Pi_{\hat{\perp}_x M}(\overline{\nabla}_{X_x} Y - \bar{g}(X_x, Y_x)x).$$

Further,

$$\hat{\perp}_x M = \perp_x M \oplus \langle x \rangle, \tag{5.6}$$

and then

$$\overset{\hat{0}}{s}(X_x, Y_x) = \Pi_{\perp_x M}(\overline{\nabla}_{X_x} Y) - \Pi_{\langle x \rangle}(\bar{g}(X_x, Y_x)x) = \overset{0}{s}(X_x, Y_x) - \bar{g}(X_x, Y_x).$$

In particular, it follows that $\overset{\hat{0}}{s}(X_x, Y_x) \in \overset{1}{N}_x M \oplus \langle x \rangle$ and hence $\overset{\hat{1}}{N}_x M \subset \overset{1}{N}_x M \oplus \langle x \rangle$; (5.4) and (5.5) imply that

$$\hat{V}_x \supset \overset{\hat{1}}{N}_x M.$$

Moreover, by (5.6) we can see that

$$\hat{V}_x \subset \hat{\perp}_x M,$$

and then we have the inclusions

$$\overset{\hat{1}}{N}_x M \subset \hat{V}_x \subset \hat{\perp}_x M. \tag{5.7}$$

Let $\hat{v}\colon M \to G(p+1, \overline{m}+1)$ ($p = \dim V$) be the map defined by

$$\hat{v}(x) = \hat{V}_x, \qquad x \in M. \tag{5.8}$$

By (1.9) we have

$$\hat{v}_*(X_x, \hat{\xi}_x) = \Pi_{\hat{V}_x^{\perp}}(\hat{\nabla}_{X_x} \hat{\xi}), \tag{5.9}$$

where $x \in M$, $X_x \in T_x M$, $\hat{\xi}_x \in \hat{V}_x$ and $\hat{\xi} \in \Gamma(\hat{V})$ is an extension of $\hat{\xi}_x$ in the vector bundle $\hat{V}$ on M with the fiber $\hat{V}_x$ at $x \in M$.

We fix $x \in M$, $X_x, Y_x \in T_x M$, $\hat{\xi}_x \in \hat{V}_x$ and consider the geodetic $\gamma = \gamma(s)$ through $x = \gamma(0)$, with the tangent vector X_x at x. We denote by Y the parallel transport of Y_x along γ, and by $\hat{\xi} = \sum_{h=0}^{q} \overset{2h+1}{\xi} + a\gamma$ the parallel transport of $\hat{\xi}_x$ in the vector bundle $\hat{V}$ along γ; the vectors $\overset{2h+1}{\xi}_{\gamma(s)} \in \overset{2h+1}{N}_{\gamma(s)} M$ are the component vectors of $\hat{\xi}_{\gamma(s)}$ in $\overset{2h+1}{N}_{\gamma(s)} M$, and a is a function defined on γ.

It is easy to check that, if $\hat{V}_x$ is defined by (5.5) and (5.4), the following lemma holds.

5.10. Lemma. *The condition* $\overset{\hat{V}}{\nabla}_{X(s)} \hat{\xi} = 0$ *implies* $\overset{2h+1}{\nabla}_{X(s)} \overset{2h+1}{\xi} = 0$, $h = 0, 1, \ldots, q$ *and* $a = \text{const}$, *where* $X(s)$ *is the tangent vector of* γ *at* $\gamma(s)$.

Now we compute $(\nabla \hat{v}_*)(X_x, Y_x, \hat{\xi}_x)$ for $x \in M$, $X_x, Y_x \in T_x M$, $\hat{\xi}_x \in \hat{V}_x$. We obtain

$$
(5.11)\qquad
\begin{aligned}
(\nabla \hat{v}_*)(X_x, Y_x, \hat{\xi}_x) &= \overset{\hat{V}^{\perp}}{\nabla}_{X_x} \big(\hat{v}_*(Y, \hat{\xi})\big) - v_*(\nabla_{X_x} Y, \hat{\xi}_x) - \hat{v}_*(Y_x, \overset{\hat{V}}{\nabla}_{X_x} \hat{\xi}) \\
&= \overset{\hat{V}^{\perp}}{\nabla}_{X_x} \big(\hat{v}_*(Y, \hat{\xi})\big) = \Pi_{\hat{V}^{\perp}} \Big(\hat{\nabla}_{X_x} \big(\hat{v}_*(Y, \hat{\xi})\big)\Big).
\end{aligned}
$$

However, due to (1.9), (5.2), (5.10) we have, along γ,

$$
\begin{aligned}
\hat{v}_*(Y, \hat{\xi}) &= \Pi_{\hat{V}^{\perp}} \left(\hat{\nabla}_Y \left(\sum_{h=0}^{q} \overset{2h+1}{\xi} + a\gamma\right)\right) \\
&= \sum_{h=0}^{q} \Pi_{\hat{V}^{\perp}} \left(\hat{\nabla}_Y \overset{2h+1}{\xi}\right) + a \Pi_{\hat{V}^{\perp}} (\hat{\nabla}_Y \gamma) \\
&= \sum_{h=0}^{q} \Pi_{\hat{V}^{\perp}} \left[\overline{\nabla}_Y \overset{2h+1}{\xi} - \bar{g}\left(Y, \overset{2h+1}{\xi}\right)\gamma\right] + a \Pi_{\hat{V}^{\perp}}(Y).
\end{aligned}
$$

Now, because $Y \in TM$, we conclude that $Y \perp \overset{2h+1}{\xi}$ and $Y \in \hat{V}^{\perp}$. So applying the Frenet equations for M as a submanifold of $\overline{M} = S^{\overline{m}}(1)$, we have

$$
\begin{aligned}
\hat{v}_*(Y, \hat{\xi}) &= \sum_{h=0}^{q} \Pi_{\hat{V}^{\perp}} \left[\overline{\nabla}_Y \overset{2h+1}{\xi}\right] + aY \\
&= \sum_{h=0}^{q} \Pi_{\hat{V}^{\perp}} \left[-\overset{2h+1}{A}\left(Y, \overset{2h+1}{\xi}\right) + \overset{2h+1}{\nabla}_Y \overset{2h+1}{\xi} + \overset{2h+1}{s}\left(Y, \overset{2h+1}{\xi}\right)\right] + aY.
\end{aligned}
$$

If we recall that $\overset{2h+1}{A}(Y, \overset{2h+1}{\xi}) \in \overset{2h}{N} M \perp \hat{V}$, $\overset{2h+1}{s}(Y, \overset{2h+1}{\xi}) \in \overset{2h+2}{N} M \perp \hat{V}$, $\overset{2h+1}{\nabla}_Y \overset{2h+1}{\xi} \in \overset{2h+1}{N} M \subset \hat{V}$, we can write

$$
\hat{v}_*(Y, \hat{\xi}) = \sum_{h=0}^{q} \left(-\overset{2h+1}{A}\left(Y, \overset{2h+1}{\xi}\right) + \overset{2h+1}{s}(Y, \overset{2h+1}{\xi})\right) + aY.
$$

Coming back to Formula 5.11, we obtain, using also (5.2),

$$
\begin{aligned}
(\nabla\hat{v}_*)(X_x,Y_x,\hat{\xi}_x) &= \Pi_{V^\perp}\Big\{\hat{\nabla}_{X_x}\Big[\sum_{h=0}^{q}\Big(-\overset{2h+1}{A}\Big(Y,\overset{2h+1}{\xi}\Big)+\overset{2h+1}{s}\Big(Y,\overset{2h+1}{\xi}\Big)\Big)+aY\Big]\Big\}\\
&= \Pi_{V^\perp}\Big\{\bar{\nabla}_{X_x}\Big[\sum_{h=0}^{q}\Big(-\overset{2h+1}{A}\Big(Y,\overset{2h+1}{\xi}\Big)+\overset{2h+1}{s}\Big(Y,\overset{2h+1}{\xi}\Big)\Big)+aY\Big]\\
&\quad -\bar{g}\Big(X_x,\sum_{h=0}^{q}\Big(-\overset{2h+1}{A}\Big(Y_x,\overset{2h+1}{\xi_x}\Big)+\overset{2h+1}{s}\Big(Y_x,\overset{2h+1}{\xi_x}\Big)\Big)+aY_x\Big)x\Big\}.
\end{aligned}
$$

Now $x \in \hat{V}_x$. Moreover, since $\nabla\overset{k}{s} = 0$ for $k = 0,1,\dots,l-1$ (and hence also $\nabla\overset{k}{A} = 0$ for $k = 1,\dots,l$) and Y, $\overset{2h+1}{\xi}$ are parallel along γ due to Lemma 5.10, we have $\nabla^{2h}_{X_x}\big(\overset{2h+1}{A}(Y,\overset{2h+1}{\xi})\big) = 0$ and $\nabla^{2h+1}_{X_x}\big(\overset{2h+1}{s}(Y,\overset{2h+1}{\xi})\big) = 0$. Then by the same computation as in the Proposition 4.3 we obtain $(\nabla\hat{v}_x)(X_x,Y_x,\hat{\xi}_x) = 0$ and we arrive at the following proposition:

5.12. Proposition. *If M is a submanifold of $S^{\bar{m}}(1)$ such that $\nabla\overset{k}{s} = 0$ for each $k = 0,1,\dots,l-1$, then the map $\hat{v}$ (defined in* (5.4), (5.5), (5.8)*) is totally geodesic.*

For submanifolds of $S^{\bar{m}}(1)$ we have the following theorem analogous to Theorem 4.21.

5.13. Theorem. *If M is a submanifold of $S^{\bar{m}}(1)$, the following properties are equivalent:*

(i) *M is a 2-symmetric submanifold of $S^{\bar{m}}(1)$,*

(ii) *$\nabla\overset{k}{s} = 0$, for each $k = 0,1,\dots,l-1$,*

(iii) *the map $\hat{v}$ (defined in* (5.4), (5.5), (5.8)*) is totally geodesic.*

Proof. The implication (i) ⇒(ii) is always true by Theorem 3.7. The implication (ii) ⇒(iii) is just Proposition 5.12.

In order to verify implication (iii) ⇒(i) we observe the following facts.

Because of the definition of $\hat{v}$ and in particular of (5.7), it is possible to apply Proposition 4.6 to M as to a submanifold of $\mathbb{R}^{\bar{m}+1}$ ($\supset S^{\bar{m}}(1)$). Then we can conclude that M is a 2-symmetric submanifold of $\mathbb{R}^{\bar{m}+1}$. By Remark 4.22, M is mapped into itself by the reflection of $\mathbb{R}^{\bar{m}+1}$ with respect to $\hat{v}(x)$ for each $x \in M$. Each such reflection induces a geodesic reflection on $S^{\bar{m}}(1)$ with respect to the totally geodesic submanifold $\hat{v}(x) \cap S^{\bar{m}}(1)$ of $S^{\bar{m}}(1)$, which maps M into itself, and hence M will be also a 2-symmetric submanifold of $S^{\bar{m}}(1)$. □

A similar argument as in the proof of Theorem 4.23 can be used for proving the following theorem:

5.14. Theorem. *If M is a submanifold of a space form $\overline{M} = \overline{M}(c)$ of positive constant curvature c, then the following conditions are equivalent:*

(i) *M is a 2-symmetric submanifold of $\overline{M}$,*

(ii) $\nabla \overset{k}{s} = 0$ *for each* $k = 0, 1, \dots, l-1$.

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Authors' addresses: *Antonella Carfagna D'Andrea*, Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Universitá degli Studi di Roma "La Sapienza", Via A. Scarpiia 10, 00161 Rome, Italy; *Renzo Mazzocco* and *Giuliano Romani*, Dipartimento di Matematica Instituto "G. Castelnuovo", Universitá degli Studi di Roma "La Sapienza", P. le A. Moro 5, 00185 Roma, Italy.