# Some Characterizations of Appell and $q$-Appell Polynomials (*) (**).

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**Summary.** – *Several characterizations are given for the well-known Appell polynomials and for their basic analogues: the $q$-Appell polynomials defined by Equation* (3.3) *below. The main results contained in Theorems* 1, 2 *and* 3 *of the present paper, and the applications considered in Section* 2, *are believed to be new. Some interesting connections with earlier results are also indicated.*

## 1. – Introduction and the main results.

The elementary observation that

$$D_x\left\{\frac{x^n}{n!}\right\}=\frac{x^{n-1}}{(n-1)!}, \quad n=1,2,3,\ldots, \qquad D_x=\frac{d}{dx}, \tag{1.1}$$

has inspired the study of what is now well known as the Appell set of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ defined by

$$D_x\{p_n(x)\}=p_{n-1}(x), \qquad n=1,2,3,\ldots, \tag{1.2}$$

where $p_n(x)$ is a polynomial of *exact* degree $n$ in $x$.

As long ago as 1880, it was Appell himself [2] who showed that a polynomial system $\{p_n(x)\}_{n=0}^{\infty}$ satisfies (1.2) if and only if it is generated by

$$\sum_{n=0}^{\infty} p_n(x)z^n = A(z)e^{xz}, \tag{1.3}$$

where the *determining function* $A(z)$ has a formal power series expansion

$$A(z)=\sum_{n=0}^{\infty} a_n z^n, \qquad a_0\neq 0. \tag{1.4}$$

Several subsequent characterizations of the Appell polynomials in terms of Stieltjes integrals are due, for instance, to THORNE [5], SHEFFER [3], and VARMA [6].

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AL-SALAM [1], on the other hand, has extended the results of Sheffer [3] and Varma [6] to hold for certain $q$-Appell polynomials. In the present note we first prove

THEOREM 1. – *Let $\alpha(t)$ be a function of bounded variation on the interval $0 \leqq t < \infty$, and suppose that $\{\gamma_n(t)\}_{n=0}^{\infty}$ is a given sequence of functions such that the Stieltjes integrals*

$$I_{n,r} = \int_0^{\infty} \gamma_n(t) t^r d\alpha(t), \quad \forall n, r \in \{0, 1, 2, \ldots\}, \tag{1.5}$$

*exist, with*

$$I_{0,0} \neq 0. \tag{1.6}$$

*Also define*

$$K_{n,m}(x, t) = \sum_{k=0}^{n} \gamma_k(t) R_{n-k,m}(x, t), \tag{1.7}$$

*where $m$ is a positive integer, and*

$$R_{n,m}(x, t) = \sum_{k=0}^{[n/m]} \frac{x^{n-mk}}{(n-mk)!} g_k t^k, \quad n = 0, 1, 2, \ldots, \tag{1.8}$$

*the coefficients $g_k \neq 0$ being arbitrary constants.*

*Then*

$$S_n^m(x) = \int_0^{\infty} K_{n,m}(x, t) d\alpha(t), \quad n = 0, 1, 2, \ldots \tag{1.9}$$

*is an Appell set of polynomials.*

PROOF. – From (1.7) and (1.8) we have

$$\begin{aligned}\frac{\partial}{\partial x} K_{n,m}(x, t) &= \frac{\partial}{\partial x} \sum_{j=0}^{n} \gamma_j(t) \sum_{k=0}^{[(n-j)/m]} \frac{x^{n-j-mk}}{(n-j-mk)!} g_k t^k \\ &= \sum_{j=0}^{n-1} \gamma_j(t) \sum_{k=0}^{[(n-j-1)/m]} \frac{x^{n-j-1-mk}}{(n-j-1-mk)!} g_k t^k, \quad \text{by (1.1)}, \\ &= \sum_{j=0}^{n-1} \gamma_j(t) R_{n-j-1,m}(x, t), \quad \text{by (1.8)},\end{aligned}$$

and, on interpreting this last sum by means of (1.7), we obtain

$$\frac{\partial}{\partial x} K_{n,m}(x, t) = K_{n-1,m}(x, t). \tag{1.10}$$

Equations (1.9) and (1.10) evidently yield the relationship

$$D_x\{S_n^m(x)\} = S_{n-1}^m(x), \quad n = 1, 2, 3, \ldots, \tag{1.11}$$

and Theorem 1 follows immediately. ■

REMARK 1. – If in (1.8) we set

$$g_k = \frac{1}{k!}, \qquad \forall k \in \{0, 1, 2, \ldots\}, \tag{1.12}$$

and (for the sake of simplicity) put $m = 1$, then we readily have

$$R_{n,1}(x, t) = \sum_{k=0}^{n} \frac{x^{n-k}}{(n-k)!} \frac{t^k}{k!} = \frac{(x+t)^n}{n!}, \tag{1.13}$$

giving us

$$K_{n,1}(x, t) = \sum_{k=0}^{n} \gamma_k(t) \frac{(x+t)^{n-k}}{(n-k)!}. \tag{1.14}$$

Thus, under these specializations, Theorem 1 reduces to the main result in Varma's paper (cf. [6], p. 594) which is essentially a generalization of Sheffer's characterization [3, pp. 739-740, Theorem 1].

In terms of the generalized hypergeometric functions, (1.14) readily gives us

$$K_{n,1}(x, t) = \frac{x^n}{n!} {}_{r+1}F_s[-n, \varrho_1, \ldots, \varrho_r; \sigma_1, \ldots, \sigma_s; -t/x], \tag{1.15}$$

provided that

$$\gamma_n(t) = \frac{t^n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\prod_{j=1}^{r} (\varrho_j)_{n-k}}{\prod_{j=1}^{s} (\sigma_j)_{n-k}}, \qquad \forall n \in \{0, 1, 2, \ldots\}, \tag{1.16}$$

where $\varrho_1, \ldots, \varrho_r$ and $\sigma_1, \ldots, \sigma_s$ are complex parameters, and $\sigma_j \neq 0, -1, -2, \ldots$; $j = 1, \ldots, s$.

From (1.9) and (2.5) we, therefore, arrive at

THEOREM 2. – *With the function $\alpha(t)$ constrained as in Theorem 1, the polynomials defined explicitly by*

$$S_n^*(x) = \frac{x^n}{n!} \int_0^\infty {}_{r+1}F_s[-n, \varrho_1, \ldots, \varrho_r; \sigma_1, \ldots, \sigma_s; -t/x]\, d\alpha(t) \tag{1.17}$$

*and generated (formally) by*

$$\sum_{n=0}^{\infty} S_n^*(x) z^n = e^{xz} \int_0^\infty {}_rF_s[\varrho_1, \ldots, \varrho_r; \sigma_1, \ldots, \sigma_s; zt]\, d\alpha(t). \tag{1.18}$$

*form an Appell set of polynomials.*

REMARK 2. – For $r = s = 2$, our assertions (1.17) and (1.18) would obviously reduce to the corresponding assertions due to Varma (cf. [6], p. 594, Eq. (1) *et seq.*).

## 2. – An application of Theorem 1.

With a view to applying Theorem 1, we consider a simple situation in which

$$\gamma_j(t) = \delta_{i,i+j}\,, \qquad j = 0, 1, \ldots, n\,, \tag{2.1}$$

where $\delta_{m,n}$ denotes the Kronecker delta. In this case (1.7) reduces to

$$K^*_{n,m}(x, t) = R_{n,m}(x, t)\,, \tag{2.2}$$

whence it follows at once that the polynomials

$$T_n^m(x) = \int_0^\infty R_{n,m}(x, t)\, d\alpha(t) = \sum_{k=0}^{[n/m]} \frac{x^{n-mk}}{(n-mk)!}\,\mu_k g_k\,, \qquad n = 0, 1, 2, \ldots\,, \tag{2.3}$$

form an Appell set of polynomials. Here $\{\mu_n\}_{n=0}^\infty$ is the sequence of *moment constants* defined by [3, p. 739]

$$\mu_n = \int_0^\infty t^n\, d\alpha(t)\,, \qquad n = 0, 1, 2, \ldots, \qquad \mu_0 \neq 0\,, \tag{2.4}$$

and $\alpha(t)$ is a function of bounded variation (on the interval $0 \leqq t < \infty$) for which each $\mu_n$ exists, $n = 0, 1, 2, \ldots$, and $\mu_0 \neq 0$.

Now we turn to the sum

$$\sum_{n=0}^\infty (\lambda)_n T_n^m(x) z^n = \sum_{n=0}^\infty (\lambda)_n \left\{\int_0^\infty R_{n,m}(x, t)\, d\alpha(t)\right\} z^n = \int_0^\infty \left\{\sum_{n=0}^\infty (\lambda)_n R_{n,m}(x, t) z^n\right\} d\alpha(t)\,, \tag{2.5}$$

where, as usual, $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda)$.

If we define

$$\Phi(\lambda; z) = \sum_{n=0}^\infty (\lambda)_{mn} g_n z^n\,, \qquad g_n \neq 0\,, \tag{2.6}$$

we shall readily observe from (1.8) and (2.5) that

$$\sum_{n=0}^\infty (\lambda)_n T_n^m(x) z^n = (1 - xz)^{-\lambda} \int_0^\infty \Phi(\lambda; \zeta^m t)\, d\alpha(t)\,, \qquad |xz| < 1\,, \tag{2.7}$$

where, for convenience,

$$\zeta = z/(1 - xz)\,. \tag{2.8}$$

In order to derive the determining function $A^*(z)$, corresponding to the general Appell polynomials $T_n^m(x)$ given by (2.3), we replace $z$ on both sides of (2.7) by $z/\lambda$ and formally take their limits as $|\lambda| \to \infty$. We thus obtain

$$A^*(z)e^{xz} = \sum_{n=0}^{\infty} T_n^m(x)\, z^n = e^{xz}\int_0^{\infty} \Psi(z^m t)\, d\alpha(t)\,,$$

which obviously yields the determining function

$$A^*(z) = \int_0^{\infty} \Psi(z^m t)\, d\alpha(t)\,, \tag{2.9}$$

where

$$\Psi(z) = \sum_{n=0}^{\infty} g_n z^n\,, \qquad g_n \neq 0\,. \tag{2.10}$$

## 3. – The $q$-Appell polynomials.

For an arbitrary (real or complex) number $q$, define the $q$-derivative of a function $f(x)$ by

$$D_{x,q}\{f(x)\} = \frac{f(qx) - f(x)}{(q-1)x}\,, \qquad q \neq 1\,, \tag{3.1}$$

so that

$$\lim_{q\to 1} D_{x,q}\{f(x)\} = f'(x) = D_x\{f(x)\}\,. \tag{3.2}$$

Thus, in analogy with (1.2), the class of polynomials $\{p_n(x;q)\}_{n=0}^{\infty}$ defined by (cf. [1], p. 31, Eq. (1.2))

$$D_{x,q}\{p_n(x;q)\} = p_{n-1}(x;q)\,, \qquad n = 1, 2, 3, \ldots\,, \tag{3.3}$$

are called the $q$-Appell polynomials; here, as before, we require that $p_n(x;q)$ be a polynomial of *exact* degree $n$ in $x$.

It follows from (3.3) that *a polynomial system* $\{p_n(x;q)\}_{n=0}^{\infty}$ *is* $q$*-Appell if and only if it is generated by*

$$\sum_{n=0}^{\infty} p_n(x;q)\, z^n = A(z;q)\, e_q(xz)\,, \tag{3.4}$$

*where the determining function* $A(z;q)$ *has a formal power series expansion*

$$A(z;q) = \sum_{n=0}^{\infty} a_{n,q} z^n\,, \qquad a_{0,q} \neq 0\,, \tag{3.5}$$

*and $e_q(z)$ denotes the $q$-exponential function defined by* (*cf.* [4], p. 71)

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]!} = \prod_{n=0}^{\infty} (1-(1-q)q^n z)^{-1}, \tag{3.6}$$

*with* $[\lambda] = (1-q^\lambda)/(1-q)$, *and*

$$[n]! = [1]\,[2]\,[3]\ldots[n], \qquad [0]! = 1. \tag{3.7}$$

In order to give characterizations of $q$-Appell polynomials, analogous to those contained in Theorems 1 and 2, we shall need the definition of the $q$-binomial coefficient

$$\begin{bmatrix}\lambda\\0\end{bmatrix} = 1, \qquad \begin{bmatrix}\lambda\\n\end{bmatrix} = \frac{[\lambda][\lambda-1]\ldots[\lambda-n+1]}{[n]!} \tag{3.8}$$

so that, for arbitrary real or complex $\lambda$,

$$\begin{bmatrix}\lambda+1\\n\end{bmatrix} = \begin{bmatrix}\lambda\\n-1\end{bmatrix} + q^n \begin{bmatrix}\lambda\\n\end{bmatrix}, \qquad n = 1, 2, 3, \ldots \tag{3.9}$$

and

$$\begin{bmatrix}\lambda\\n\end{bmatrix} = (-1)^n q^{\frac{1}{2}n(2\lambda-n+1)} \frac{(q^{-\lambda}; q)_n}{(q; q)_n}, \qquad n = 0, 1, 2, \ldots, \tag{3.10}$$

where, in analogy with the Pochhammer symbol $(\lambda)_n$ used in (2.5),

$$(\lambda; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-\lambda)(1-\lambda q)\ldots(1-\lambda q^{n-1}), & \forall n \in \{1, 2, 3, \ldots\}. \end{cases} \tag{3.11}$$

Notice that, if $m$ is a positive integer,

$$\begin{bmatrix}m\\n\end{bmatrix} = \begin{bmatrix}m\\m-n\end{bmatrix} = \frac{(q; q)_m}{(q; q)_{m-n}(q; q)_n}, \qquad 0 \leqq n \leqq m. \tag{3.12}$$

We shall also need the $q$-binomial theorem

$$[x+y; q]^n = \sum_{k=0}^{n} \begin{bmatrix}n\\k\end{bmatrix} x^{n-k} y^k = [y+x; q]^n \tag{3.13}$$

and the following $q$-analogues of the generalized hypergeometric function occurring in (1.15) and (1.18):

$${}_r\Phi_s[\varrho_1, \ldots, \varrho_r; \sigma_1, \ldots, \sigma_s; q, z] = \sum_{n=0}^{\infty} \Omega_{n,q} \frac{z^n}{(q; q)_n} \tag{3.14}$$

and

$$ {}_r\Phi_s^*[\varrho_1, \ldots, \varrho_r; \sigma_1, \ldots, \sigma_s; q, z] = \sum_{n=0}^{\infty} \Omega_{n,q} q^{\frac{1}{2}n(n-1)} \frac{z^n}{(q;q)_n}, \tag{3.15} $$

where, for convenience,

$$ \Omega_{n,q} = \frac{\prod_{j=1}^{r} (\varrho_j; q)_n}{\prod_{j=1}^{s} (\sigma_j; q)_n}, \quad n = 0, 1, 2, \ldots. \tag{3.16} $$

Our proofs of the characterizations (1.9) and (1.17), detailed in Section 1, can be applied *mutatis mutandis* to derive the desired $q$-analogues of Theorems 1 and 2, given by

THEOREM 3. – *Let* $\beta(t) \equiv \beta(t; q)$ *be a function of bounded variation on the interval* $0 \leqq t < \infty$, *and suppose that* $\{\delta_n(t; q)\}_{n=0}^{\infty}$ *is a given sequence of functions such that the Stieltjes integrals*

$$ \mathscr{I}_{n,r} = \int_0^{\infty} \delta_n(t; q) t^r d\beta(t), \quad \forall n, r \in \{0, 1, 2, \ldots\}, \tag{3.17} $$

*exist, with*

$$ \mathscr{I}_{0,0} \neq 0. \tag{3.18} $$

*Also, for every positive integer* $m$, *let* $N$ *denote the largest integer in* $n/m$, *and define*

$$ \mathscr{K}_{n,m}(x, t; q) = \sum_{k=0}^{n} \delta_k(t; q) \mathscr{R}_{n-k,m}(x, t; q), \tag{3.19} $$

*where*

$$ \mathscr{R}_{n,m}(x, t; q) = \sum_{k=0}^{N} \frac{x^{n-mk}}{[n-mk]!} h_{k,q} t^k, \quad n = 0, 1, 2, \ldots, \tag{3.20} $$

*the coefficients* $h_{k,q} \neq 0$ *being arbitrary constants.*

*Then*

$$ \mathscr{S}_n^m(x; q) = \int_0^{\infty} \mathscr{K}_{n,m}(x, t; q) d\beta(t), \quad n = 0, 1, 2, \ldots \tag{3.21} $$

*is a* $q$-*Appell set of polynomials.*

*Furthermore, in terms of the* $q$-*hypergeometric functions* ${}_r\Phi_s$ *and* ${}_r\Phi_s^*$ *given by* (3.14) *and* (3.15), *respectively, the polynomials defined explicitly by*

$$ \mathscr{S}_n^*(x; q) = \frac{x^n}{[n]!} \int_0^{\infty} {}_{r+1}\Phi_s[q^{-n}, \varrho_1, \ldots, \varrho_r; \sigma_1, \ldots, \sigma_s; q, -q^n t/x] d\beta(t) \tag{3.22} $$

*and generated (formally) by*

$$\sum_{n=0}^{\infty} \mathscr{S}_n^*(x; q) z^n = e_q(xz) \int_0^{\infty} {}_r\Phi_s^*[\varrho_1, \ldots, \varrho_r; \sigma_1, \ldots, \sigma_s; q, zt]\, d\beta(t) \tag{3.23}$$

*form a $q$-Appell set of polynomials.*

REMARK 3. – In its special case when

$$m = 1 \quad \text{and} \quad h_{k,q} = \frac{1}{[k]!}, \qquad k = 0, 1, 2, \ldots, \tag{3.24}$$

our main assertion (3.21) of Theorem 3 corresponds to that of Al-Salam [1, p. 42, Eq. (5.2)], since it is readily verified from (3.13) and (3.20) that

$$\mathscr{R}_{n,1}(x, t; q) = \sum_{k=0}^{n} \frac{x^{n-k}}{[n-k]!} \frac{t^k}{[k]!} = \frac{[x+t; q]^n}{[n]!}, \tag{3.25}$$

so that (3.19) becomes

$$\mathscr{K}_{n,1}(x, t; q) = \sum_{k=0}^{n} \delta_{n-k}(t; q) \frac{[x+t; q]^k}{[k]!}. \tag{3.26}$$

It may be of interest to notice also that, in terms of the $q$-hypergeometric function defined by (3.14), if we let

$$\mathscr{K}_{n,1}(x, t; q) = \frac{x^n}{[n]!} {}_{r+1}\Phi_s[q^{-n}, \varrho_1, \ldots, \varrho_r; \sigma_1, \ldots, \sigma_s; q, -q^n t/x] \tag{3.27}$$

then, by (3.10) and (3.26), we have

$$\sum_{k=0}^{n} \delta_{n-k}(t; q) \frac{t^k}{[k]!} = \Omega_{n,q}\, q^{\frac{1}{2}n(n-1)} \frac{t^n}{[n]!} \tag{3.28}$$

or, equivalently,

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \delta_k(t; q) \frac{[k]!}{t^k} = \Omega_{n,q}\, q^{\frac{1}{2}n(n-1)}, \tag{3.29}$$

which, upon inversion, yields

$$\delta_n(t; q) = \frac{t^n}{[n]!} q^{\frac{1}{2}n(n-1)} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \Omega_{n-k,q}\, q^{k(k-n)}, \tag{3.30}$$

where $\Omega_{n,q}$ is defined by (3.16).

Thus we are led to our assertions (3.22) and (3.23) which, when $r = s = 2$, would provide the *corrected* (and appropriately modified) version of the corresponding assertions due to Al-Salam (cf. [1], p. 42, Eq. (5.3) *et seq.*).

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