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SOME CHARACTERIZATIONS OF REFLEXIVITY

IVAN SINGER[1]

ABSTRACT. The results of R. C. James on characterizations of reflexivity of Banach spaces with an unconditional basis in terms of c_0 and l^1 are extended to arbitrary Banach spaces. Some consequences are obtained.

R. C. James has proved [4, Theorem 2] that a Banach space E with an unconditional basis is reflexive if and only if E contains no subspace isomorphic to c_0 or l^1. This result has been shown to remain valid for any subspace E of a space with an unconditional basis (of any power) by C. Bessaga and A. Pełczyński [1], [2], who have also proved [2] that such a space E is reflexive if and only if E^* contains no subspace isomorphic to l^1.

The above conditions are clearly necessary for the reflexivity of any Banach space E. In the present note we shall show that one can add a certain necessary condition to them, which is also satisfied by any (not necessarily reflexive) subspace of a space with an unconditional basis (of any power) in such a way that these conditions together will be also sufficient for reflexivity. Thus, we shall obtain extensions of the above results to characterizations of reflexivity of an arbitrary Banach space E.

Following A. Pełczyński [6], a Banach space E is said to have *property* (u), if for every weak Cauchy sequence $\{x_n\} \subset E$ there exists a sequence $\{y_n\} \subset E$ such that (a) the series $\Sigma_{i=1}^{\infty} y_i$ is weakly unconditionally Cauchy and (b) the sequence $\{x_n - \Sigma_{i=1}^{n} y_i\}$ converges weakly to 0.

Theorem. *For a Banach space E the following statements are equivalent*:

1°. *E is reflexive.*

2°. *E has property* (u) *and E contains no subspace isomorphic to c_0 or l^1.*

3°. *E has property* (u) *and E^* contains no subspace isomorphic to l^1.*

Proof. Assume 1°. Then, by [6, Proposition 2], E has property (u). Also, E^* is reflexive and hence E^* contains no subspace isomorphic to l^1. Thus, $1° \Rightarrow 3°$.

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Assume now 3°. If E contains a subspace G isomorphic to c_0, then E^* has a quotient space $E^*/G^\perp$ isomorphic to l^1 and hence (see e.g. [3, p. 63, exercise 2]) E^* has a complemented subspace isomorphic to l^1, in contradiction with 3°. On the other hand, if E contains a subspace isomorphic to l^1, then so does E^* [7, Proposition 3.3], in contradiction with 3°. Thus, $3° \Rightarrow 2°$.

Assume now 2°. Then, since E has property (u) and contains no subspace isomorphic to c_0, by [6, Theorem 1] (for a proof see [9, p. 450]), E is weakly complete. Hence, since E contains no subspace isomorphic to l^1, from [8, Corollary 1] it follows that E is reflexive. Thus, $2° \Rightarrow 1°$.

Since every Banach space E with an unconditional basis and every subspace of such a space have property (u) (by [6, Theorem 3 and Corollary 1]; for a proof see [9, pp. 445–449]), from the above Theorem we obtain, in particular, the results of R. C. James [4] and C. Bessaga and A. Pełczyński [1], [2] mentioned in the introduction.

Corollary 1. *A separable Banach space E is reflexive if and only if* (i) *E has property* (u) *and* (ii) *E^{**} is separable.*

Proof. Clearly, (i) and (ii) imply 3° of the above Theorem.

Remark 1. Combining [8] with [6, Corollary 5], it follows that *if a Banach space E has property* (u) *and contains no subspace isomorphic to l^1, then E^* is weakly complete.* This result yields other proofs of Corollary 1 and the implication $3° \Rightarrow 1°$ of the Theorem.

Corollary 2. *The following two conjectures are equivalent*:

1° [5, p. 165]. *Every infinite dimensional Banach space contains an infinite dimensional subspace that is either reflexive or is isomorphic to c_0 or l^1.*

2°. *Every infinite dimensional Banach space contains an infinite dimensional subspace with property* (u),

Proof. c_0, l^1 and every reflexive space have property (u), so $1° \Rightarrow 2°$. Conversely, if $G \subset E$ has property (u), then by the above Theorem either G is reflexive or G contains a subspace isomorphic to c_0 or l^1. Thus, $2° \Rightarrow 1°$.

Remark 2. The conjecture of Corollary 2, if substantiated, would have some interesting consequences, e.g., that *every* (*infinite dimensional*) *second conjugate space E^{**} contains a reflexive subspace* (*of infinite dimension*)—or, equivalently, that *every conjugate Banach space E^* has a reflexive quotient space.* Indeed, if $E \supset G$ reflexive, then $E^{**} \supset E \supset G$; if $E \underset{\sim}{\supset} c_0$, then $E^{**} \underset{\sim}{\supset} l^\infty \supset l^2$; finally, if $E \underset{\sim}{\supset} l^1$, then $E^{**} \underset{\sim}{\supset} (l^\infty)^* \supset l^2$.

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DÉPARTEMENT D'INFORMATIQUE, UNIVERSITÉ DE MONTRÉAL, MONTRÉAL, CANADA

INSTITUTE OF MATHEMATICS, ACADEMY OF SCIENCES, BUCHAREST, ROMANIA