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SOME CHARACTERIZATIONS OF RULED REAL HYPERSURFACES IN A COMPLEX SPACE FORM

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1. Introduction

A complex $n(\geq 2)$-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a complex projective space P_nC, a complex Euclidean space C^n or a complex hyperbolic space H_nC, according as $c > 0$, $c = 0$ or $c < 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ, ξ, η, g).

Now, there exist many studies about real hypersurfaces of $M_n(c)$. One of the first researches is the classification of homogeneous real hypersurfaces of a complex projective space P_nC by Takagi [13], who showed that these hypersurfaces of P_nC could be divided into six types which are said to be of type A_1, A_2, B, C, D, and E, and in [3] Cecil-Ryan and [6] Kimura proved that they were realized as the tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or rank 2. Also Berndt [2] showed recently that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space H_nC are realized as the tubes of constant radius over certain submanifolds when the structure vector field ξ is principal. According to Takagi's classification theorem and Berndt's one, the principal curvatures and their multiplicities of homogeneous real hypersurfaces of $M_n(c)$ are given.

On the other hand, let us denote by $\mathcal{L}_\xi$ the Lie derivative with respect to the structure vector field ξ. Then Okumura [12] and Montiel and Romero [11] proved the followings respectively.

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THEOREM A. *Let M be a real hypersurface of $P_nC, n\geq 3$. If it satisfies*

$$\mathcal{L}_\xi g = 0, \tag{1.1}$$

then M is locally a tube of radius r over one of the following Kaehler submanifolds:

(A_1) a hyperplane $P_{n-1}C$, where $0 < r < \pi/2$,
(A_2) a totally geodesic P_kC $(1\leq k\leq n-2)$, where $0 < r < \pi/2$.

THEOREM B. *Let M be a real hypersurface of $H_nC, n\geq 3$. If it satisfies (1.1), then M is locally one of the following hypersurfaces:*

(A_0) a horosphere in H_nC, i.e., a Montiel tube,
(A_1) a geodesic hypersphere or a tube over a totally geodesic hyper plane $H_{n-1}C$,
(A_2) a tube over a totally geodesic H_kC $(1\leq k\leq n-2)$.

As an example of special real hypersurfaces of $M_n(c)$, $c\neq 0$ different from the above ones, we can give some characterizations of ruled real hypersurfaces in terms of the covariant derivative of the second fundmental form.

On the other hand, Kimura [7] and Ahn,Lee and the second author [1] obtained some properties about ruled real hypersurfaces of P_nC and H_nC, $n\geq 3$ respectively. In particular, an example of minimal ruled hypersurfaces of P_nC and H_nC, $n\geq 3$ is constructed respectively. Now let us define a distribution T_0 by $T_0(x) = \{u\in T_xM : u\perp\xi(x)\}$ of a real hypersurface M of $M_n(c), c\neq 0$, which is orthogonal to the structure vector field ξ and holomorphic with respect to the structure tensor ϕ.

Let us denote by A the second fundamental form of the real hypersurface M of $M_n(c)$. Then we shall calculate the covariant derivative $(\nabla_X A)Y$ of these ruled real hypersurfaces in section 3 and obtain the T_0-component and ξ-component of $(\nabla_X A)Y$, which are given by $g((\nabla_X A)Y, Z) = 0$, so called η-parallel second fundamental tensor, and $g((\nabla_X A)Y,\xi) = f(X,Y)$ for any vector fields X, Y and Z in T_0 and a certain 2-form f respectively.

On the other hand, Kimura and Maeda [8] and the second author [1] gave some characterizations of real hypersurfaces of this type in

$M_n(c)$, $c\neq 0$ with the η-parallel second fundamental tensor and another related conditions respectively. In this paper we consider the ξ-component $g((\nabla_X A)Y,\xi) = f(X,Y)$, X,Y in T_0, which is equivalent to $(\nabla_X A)Y \equiv f(X,Y)\xi$ (*modulo* T_0 – *component*), with which we give another characterization of ruled real hypersurfaces. Namely, we have the following

THEOREM 1. *Let M be a real hypersurface of $M_n(c), c\neq 0$, $n\geq 3$. Assume that the structure vector ξ is not principal. If there is a 1-form θ satisfying*

$$(1.2) \qquad (A\phi - \phi A)X = \theta(X)\xi, \quad X \in T_0,$$

and if it satisfies

$$(1.3) \qquad (\nabla_X A)Y \equiv f(X,Y)\xi \ (mod\ T_0),\ X,Y \in T_0,$$

where $f(X,Y)$ is given by (3.6), then M is locally congruent to a ruled real hypersurface provided that $\eta(A\xi)$ is not constant along the direction of ξ.

Now let us consider a condition $\mathcal{L}_\xi g(X,Y) = 0$ for any X,Y in T_0, which is equivalent to the condition (1.2) and weaker than the condition (1.1). Obviously, by virtue of Theorems A and B real hypersurfaces of type A satisfy this condition. But until now we do not know "what type of hypersurfaces in $M_n(c)$ except the ones of type A satisfy the condition (1.2)". From this point of view and the motivation of getting a Lie-derivative expression of ruled real hypersurfaces, by using Theorem 1 we have the following

THEOREM 2. *Let M be a real hypersurface of $M_n(c), c\neq 0$ and $n\geq 3$. If it satisfies*

$$(1.4) \qquad \mathcal{L}_\xi g(X,Y) = 0,$$

$$(1.5) \qquad g((\mathcal{L}_\xi A\phi)X,Y) = 0$$

for any vector fields X and Y in the distribution T_0 and if the structure vector field ξ is not principal, then M is locally congruent to a ruled real *hypersurface provided that $\eta(A\xi)$ is not constant along the direction of ξ.*

In section 3 we recall some fundamental properties of ruled real hypersurfaces of $M_n(c)$, $c\neq 0$, and calculate the T_0-component and the ξ-component $f(X,Y)$ of $(\nabla_X A)Y$ for any vector fields X and Y in T_0 respectively.

By paying attention to the ξ-component $f(X,Y)$ a characterization of ruled real hypersurfaces is given in section 4. That is, we shall prove Theorem 1 in this section. Also in section 5 by using Theorem 1 we shall prove Theorem 2 and give another result of real hypersurfaces of type A which is related to this theorem. Also related to this result the linear transformation ϕA is treated in the last section.

2. Preliminaries

We begin with recalling basic properties of real hypersurfaces of a complex space form. Let M be a real hypersurface of $n(\geq 2)$-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$ and let C be a unit normal field on a neighborhood of a point x in M. We denote by J an almost complex structure of $M_n(c)$. For a local vector field X on a neighborhood of x in M, the transformation of X and C under J can be represented as

$$JX = \phi X + \eta(X)C, \qquad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of x in M, respectively. Moreover, it is seen that $g(\xi, X) = \eta(X)$,where g denotes the induced Riemannian metric on M. By properties of the almost complex structure J, the set (ϕ, ξ, η, g) of tensors satisfies

$$\phi^2 = -I + \eta\otimes\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. Accordingly, the set is so called an *almost contact metric structure*. Furthermore the covariant derivative of the structure tensors are given by

$$(2.1) \qquad (\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX,$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to the unit normal C on M.

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given as follows

(2.2)
$$\begin{aligned} R(X,Y)Z =& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ &- 2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY, \end{aligned}$$

(2.3) $$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\},$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

The second fundamental form is said to be η-parallel if the shape operator A satisfies $g((\nabla_X A)Y,Z) = 0$ for any vector fields X,Y and Z in T_0.

Next we suppose that the structure vector field ξ is principal with corresponding principal curvature α. Then it is seen in [4] and [9] that α is constant on M and it satisfies

(2.4) $$A\phi A = \frac{c}{4}\phi + \frac{1}{2}\alpha(A\phi + \phi A).$$

3. Ruled real hypersurfaces

This section is concerned with necessary properties about ruled real hypersurfaces. First of all, we define a ruled real hypersurface M of $M_n(c)$,$c\neq 0$. Let $\gamma : I \to M_n(c)$ be any regular curve. For any $t(\in I)$ let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurface through the point $\gamma(t)$ of $M_n(c)$ which is orthogonal to a holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M = \{x \in M^{(t)}_{n-1}(c) : t \in I\}$. Then the construction of M asserts that M is a real hypersurface of $M_n(c)$. Under this construction the ruled real hypersurface M of $M_n(c)$,$c\neq 0$,has some fundamental properties.

Let us put $A\xi = \alpha\xi + \beta U$, where U is a unit vector field orthogonal to ξ and α denotes the function $\eta(A\xi)$ and $\beta(\beta\neq 0)$ the length of vector field $A\xi - \alpha\xi$. As is seen in [5], the shape operator A satisfies

$$AU = \beta\xi, \quad AX = 0 \tag{3.1}$$

for any vector field X orthogonal to ξ and U. It turns out to be

$$A\phi X = -\beta g(X, \phi U)\xi, \quad \phi AX = 0, \quad X \in T_0, \tag{3.2}$$

which implies that

$$g((A\phi - \phi A)X, Y) = 0, \quad X, Y \in T_0. \tag{3.3}$$

Because of

$$\begin{aligned}\mathcal{L}_\xi g(X,Y) &= \mathcal{L}_\xi(g(X,Y)) - g(\mathcal{L}_\xi X, Y) - g(X, \mathcal{L}_\xi Y)\\ &= g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi),\end{aligned}$$

the above equation is equivalent to

$$\mathcal{L}_\xi g(X,Y) = 0, \quad X, Y \in T_0. \tag{3.4}$$

Next the covariant derivative $(\nabla_X A)Y$ with respect to X and Y in T_0 is explicitly expressed. The equation (2.3) of Codazzi gives us to

$$(\nabla_X A)\xi - (\nabla_\xi A)X = -\frac{c}{4}\phi X.$$

By the direct calculation of the left hand side of the above relation and using (2.1) and the second formula of (3.2), we get

$$d\alpha(X)\xi + d\beta(X)U + \frac{c}{4}\phi X + \beta\nabla_X U - \nabla_\xi(AX) + A\nabla_\xi X = 0, X \in T_0.$$

Let T_1 be a distribution defined by a subspace $T_1(x) = \{u \in T_0(x) : g(u, U(x)) = g(u, \phi U(x)) = 0\}$. Since AX is expressed as the linear combination of ξ and U, from (3.1),(3.2) and the above equation we can derive the following relations:

$$\beta\nabla_X U = \begin{cases} (\beta^2 - \frac{c}{4})\phi X, & X = U \\ 0, & X = \phi U \\ -\frac{c}{4}\phi X, & X \in T_1 \end{cases}$$

$$d\beta(X) = \begin{cases} 0, & X = U \\ \beta^2 + \frac{c}{4}, & X = \phi U \\ 0, & X \in T_1. \end{cases}$$

Using these relations we can obtain the components of $(\nabla_X A)Y, X, Y \in T_0$, in the direction of ξ. In fact, we have

$$\begin{aligned} g((\nabla_X A)Y, \xi) &= g((\nabla_X A)\xi, Y) = g(\nabla_X(A\xi) - A\nabla_X\xi, Y) \\ &= d\beta(X)g(Y,U) + \beta g(\nabla_X U, Y), \end{aligned}$$

which yields combining with the above equation that

$$(\nabla_X A)Y = f(X,Y)\xi, \quad X, Y \in T_0, \tag{3.5}$$

where we put
(3.6)

$$f(X,Y) = \beta^2\{g(X,U)g(Y,\phi U) + g(X,\phi U)g(Y,U)\} - \frac{c}{4}g(\phi X, Y).$$

From this formula we can consider two different components of $(\nabla_X A)Y$, $X, Y \in T_0$. One is the component of T_0, that is, $g((\nabla_X A)Y, Z) = 0$, with which Kimura and Maeda [8] and the second author [1] have studied some characterizations of ruled real hypersurfaces. The other is the component of ξ, which is given by $g((\nabla_X A)Y, \xi) = f(X,Y)$ for any X, Y in T_0. Thus the purpose of this paper is to give a new characterization of ruled real hypersurfaces with this condition.

4. Proof of Theorem 1

In section 3 we have seen that ruled real hypersurfaces of $M_n(c)$ satisfy the conditions (1.2) and (1.3). Thus in this section as a characterization for ruled real hypersurfaces we consider a converse problem. Let M be the real hypersurface of $M_n(c)$,$c \neq 0$ and assume that the structure vector is not principal. Then we can put $A\xi = \alpha\xi + \beta U$, where the function α is given by $\eta(A\xi)$ and U is a unit vector field in the holomorphic distribution T_0. By the assumption the function β does not vanish identically on M.

Now let us define a vector field V by $\nabla_\xi \xi$. Then, from this definition together with (2.1) it follows $V = \beta\phi U$. Now let us prove Theorem 1 stated in the introduction.

By the assumption (1.2) it turns out to be

$$(A\phi - \phi A)X = -g(X,V)\xi, \quad X \in T_0. \tag{4.1}$$

Then by using Lemma 2.1 in the paper [1] we have

$$g((\nabla_X A)Y, Z) = \mathfrak{S} g(AX,Y)g(Z,V),$$

where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z in T_0, It implies that the shape operator satisfies

$$\begin{aligned}(\nabla_X A)Y =& g(AX,Y)V + g(Y,V)AX + g(X,V)AY \\ &+ \{f_1(X,Y) + f_2(X,Y)\}\xi\end{aligned} \tag{4.2}$$

for any X and Y in T_0, where $f_1(X,Y)$ and $f_2(X,Y)$ are symmetric and skew-symmetric with respect to X and Y respectively. Then taking the inner product (4.2) with ξ and using (3.6) and the assumption (1.3), we have $f_1(X,Y) = 0$. Moreover, it is easily seen that

$$f_2(X,Y) = -\frac{c}{4} g(\phi X, Y), \quad X, Y \in T_0. \tag{4.3}$$

Consider next the assumption (1.3). By combining (1.3) together with (4.2) and (4.3) it reduces to

$$(\nabla_X A)Y = g(AX,Y)V + g(Y,V)AX + g(X,V)AY - \frac{c}{4} g(\phi X, Y)\xi \tag{4.4}$$

for any X and Y in T_0. Differentiating (4.1) covariantly and using (2.1) and (4.4), we can obtain

(4.5)
$$(AX, AY) - \alpha g(AX, Y) - \frac{c}{4}g(X, Y) - g(X, V)g(Y, V) + g(\nabla_X V, Y) = 0$$

for any X and Y in T_0, which yields that

$$g(\nabla_X V, Y) = g(\nabla_Y V, X), \tag{4.6}$$

because other terms of (4.5) except for the last one are symmetric with respect to X and Y. Since we know that for any $X \in T_0$

$$g(A^2X - \alpha AX - \frac{c}{4}X - g(X, V)V + \nabla_X V, \xi) = 0,$$

(4.5) can be written as the following

$$A^2X - \alpha AX - \frac{c}{4}X - g(X, V)V + \nabla_X V = 0, \quad X \in T_0. \tag{4.7}$$

Let us define the covariant derivative $\nabla_X \nabla_Y A$ of $\nabla_Y A$ by

$$(\nabla_X \nabla_Y A)Z = \nabla_X((\nabla_Y A)Z) - (\nabla_{\nabla_X Y} A)Z - (\nabla_Y A)\nabla_X Z$$

for any vector fields X,Y and Z. Differentiating (4.4) and using the definition of $\nabla_X \nabla_Y A$, we have for any vector fields X, Y and Z in T_0

(4.8)
$$\begin{aligned}
&(\nabla_X \nabla_Y A)Z \\
&= g(\phi AX, Y)\{Z\alpha\xi + \alpha\phi AZ + Z\beta U + \beta\nabla_Z U - A\phi AZ \\
&\quad + \frac{c}{4}\phi Z - \beta g(Z, U)V - g(Z, V)A\xi\} \\
&\quad + g(\phi AX, Z)\{Y\alpha\xi + \alpha\phi AY + Y\beta U + \beta\nabla_Y U - A\phi AY \\
&\quad - \beta g(Y, U)V - g(Y, V)A\xi\} \\
&\quad + g((\nabla_X A)Y, Z)V + g(AY, Z)\nabla_X V + g(Z, \nabla_X V)AY \\
&\quad + g(Z, V)(\nabla_X A)Y + g(Y, \nabla_X V)AZ + g(Y, V)(\nabla_X A)Z \\
&\quad - \frac{c}{4}g(\phi Y, Z)\phi AX,
\end{aligned}$$

where we have used various equations obtained already in this section.

Now, we define here a T_0-valued 1-form h on the tangent bundle by

$$h(X) = \{X\alpha - \alpha g(X,V)\}\xi + \{X\beta - \beta g(X,V)\}U - \beta g(X,U)V \\ + \frac{c}{4}\phi X + \alpha\phi AX - A\phi AX + \beta\nabla_X U$$

for any vector field X in T_0. We shall verify that the equation

$$g(AY,Z)h(X) = 0, \quad X,Y,Z\in T_0 \tag{4.9}$$

holds. In fact, by definition we get

$$(\nabla_X\nabla_Y A)Z = \nabla_X\nabla_Y(AZ) - (\nabla_X A)\nabla_Y Z - A(\nabla_X\nabla_Y Z) \\ - (\nabla_{\nabla_X Y}A)Z - (\nabla_Y A)\nabla_X Z,$$

from which it follows that the Ricci formula for the shape operator A is given by

$$(\nabla_X\nabla_Y A)Z - (\nabla_Y\nabla_X A)Z = R(X,Y)(AZ) - A(R(X,Y)Z).$$

Using (2.2),(4.4)$\sim$(4.8) and the above Ricci formula, we get for any X, Y and Z in T_0

$$2g(\phi AX,Y)h(Z) = g(\phi AY,Z)h(X) + g(\phi AZ,X)h(Y).$$

Replacing X,Y and Z in the above equation cyclically,we get

$$g(\phi AX,Y)h(Z) = g(\phi AY,Z)h(X) = g(\phi AZ,X)h(Y). \tag{4.10}$$

Putting $Z = X$ and replacing Y into ϕY in (4.10), we have

$$g(AX,Y)h(X) = 0, \tag{4.11}$$

because of $g(\phi AX,X) = 0$. Again, replacing Y into ϕY in (4.10), we have

$$g(AX,Y)h(Z) = -g(AY,Z)h(X).$$

Putting $Z = Y$ in the above equation and using (4.11), we get

$$g(AY, Y)h(X) = 0.$$

Accordingly, by polarization,we can prove that (4.9) holds.

Next, let M_0 be an open set consisting of points x in M such that $\beta \neq 0$. Then from the assumption the subset M_0 is not empty.

In order to get our result, firstly let us consider our discussion on the interior of $M - M_0$, on which $\beta \equiv 0$ and therefore ξ is principal. Then we have

$$(A\phi - \phi A)\xi = 0.$$

For any principal vector X in T_0 with principal curvature λ, the condition (1.2) is reduced to $A\phi X = \lambda\phi X + \theta(X)\xi$. From $A\xi = \alpha\xi$ the inner product of $A\phi X$ and ξ gives us to $\theta(X) = 0$. This means that

$$A\phi - \phi A = 0 \tag{4.12}$$

on the interior of $M - M_0$. It is seen in [4] and [9] that the principal curvature α is constant on the interior of $M - M_0$, because this is a local property. So it satisfies (2.4). Thus, if X is a principal vector field with corresponding principal curvature λ, then we have

$$(2\lambda - \alpha)A\phi X = (\frac{c}{2} + \alpha\lambda)\phi X.$$

Using (4.12) and the above equation we get

$$2\lambda^2 - 2\alpha\lambda - \frac{c}{2} = 0,$$

from which it follows that all principal curvatures are nonzero constant on the interior of $M - M_0$.

Secondly,let us continue our discussion on the open set M_0, on which we can consider the following two cases.

As the first case on M_0 we can consider a non-vanishing 1-form h defined on the distribution T_0. Then (4.9) means that

$$g(AX, Y) = 0, X, Y \in T_0 \tag{4.13}$$

on M_0. So, it follows from this and (1.2) that we get $AX = g(AX,\xi)\xi = \beta g(X,U)\xi$ for any $X \in T_0$, which means that

$$AX = 0, \qquad AU = \beta\xi \tag{4.14}$$

for any $X \in T_0$ orthogonal to U. Consequently we obtain

$$g(A\phi X,Y) = 0, \qquad g(\phi AX,Y) = 0, \quad X,Y \in T_0.$$

As the second case on M_0 we can consider $h(X) = 0$ for any X in T_0. In this case let us prove that (4.14) also holds. Taking the inner product of $h(X)$ and the vector ξ and using (2.1), we have

$$d\alpha(X) = \alpha g(X,V) - 2g(AX,V), \quad X \in T_0,$$

which implies that

$$grad\ \alpha = \nabla\alpha = \alpha V - 2AV + w\xi, \quad X \in T_0, \tag{4.15}$$

where $w = d\alpha(\xi)$. Accordingly it turns out to be $d\alpha(Y) = -2g(AY,V)$ for any Y in T_0 orthogonal to V. Differentiating this equation with respect to X in T_0 orthogonal to V and taking account of (4.7), we have

$$\begin{aligned} XY(\alpha) = &- 2\{g((\nabla_X A)Y,V) + g(A\nabla_X Y,V) \\ &- g(A^3X,Y) + \alpha g(A^2X,Y) + \frac{c}{4}g(AX,Y)\} \end{aligned}$$

for any vector fields X and Y in T_0 orthogonal to V. Taking the skew- symmetric part of the above equation and using the equation of Codazzi (2.3), we have

$$(XY - YX)\alpha = -2g((\nabla_X Y - \nabla_Y X), AV).$$

Substituting (4.5) into the above equation and using the fact that $(XY - YX)\alpha = g(\nabla_X Y - \nabla_Y X, \nabla\alpha)$ to the obtained equation, we see

$$wg(\nabla_X Y - \nabla_Y X,\xi) + \alpha g(\nabla_X Y - \nabla_Y X, V) = 0.$$

From this by using (2.1),(4.6),(4.15) and the assumption (1.2) we have

$$wg(A\phi X, Y) = 0.$$

From this and the assumption that the function $\alpha = \eta(A\xi)$ is not constant alnog the direction of ξ it follows $g(A\phi X, Y) = 0$ for any vector fields X and Y in T_0 orthogonal to V. So we have by virtue of (1.2)

$$AX = g(AX, \xi)\xi + g(AX, U)U.$$

Thus we see $AU = \beta\xi + \gamma U$. Namely

$$AX = g(X, U)AU$$

for any vector field X in T_0 orthogonal to V. By the assumption (4.1) and the form of AU, we have $AV = \gamma V$. Differentiating this equation with respect to X orthogonal to ξ,U and V and taking account of (4.4) and (4.7), we have

$$4d\gamma(X)V = c\gamma X = 0,$$

which means that

$$AU = \beta\xi.$$

From these facts we have known that (4.14) also holds for this case.

By means of the continuity of principal curvatures, (4.12) and (4.14) lead a contradiction. It shows that the interior of $M - M_0$ must be empty. Thus the open set M_0 is dense. By the continuity of principal curvatures again we see that the shape operator satisfies the condition (4.14) on the whole M. Accordingly, for any vector fields X and Y in T_0 we get

$$g(\nabla_X Y, \xi) = -g(\nabla_X \xi, Y) = -g(\phi AX, Y) = 0$$

by (2.1), which means that $\nabla_X Y - \nabla_Y X$ is also contained in T_0. Hence the distribution T_0 is integrable on M. Moreover, the integral manifold of T_0 can be regarded as the submanifold of codimension 2 in $M_n(c)$ whose normal vectors are ξ and C. Since we have

$$\bar{g}(\bar{\nabla}_X Y, \xi) = g(\nabla_X Y, \xi) = 0$$

and

$$\bar{g}(\bar{\nabla}_X Y, C) = -\bar{g}(\bar{\nabla}_X C, Y) = g(AX, Y) = 0$$

for any vector fields X and Y in T_0 by (2.1) and (4.14), where $\bar{\nabla}$ denotes the Riemannian connection of $M_n(c)$, it is seen that the submanifold is totally geodesic in $M_n(c)$. Since T_0 is also J-invariant , its integral manifold is a complex submanifold and therefore it is a complex space form $M_{n-1}(c)$. Thus M is a ruled real hypersurface.

REMARK 1. Ruled real hypersurfaces in P_nC and H_nC with its function $\eta(A\xi)$ is not constant along the direction of ξ are explicitly described in [8] and [1],respectively. Moreover, Kimura [7] and the second author [1] also constructed examples of minimal ruled real hypersurfaces in P_nC and H_nC, respectively.

REMARK 2. Kimura and Maeda [8] proved that for a real hypersurface M of P_nC if the distribution T_0 is integrable and if the second fundamental form A is η-parallel, then M is locally congruent to a ruled real hypersurface.

5. The linear transformation $A\phi$

In this section as an application of Theorem 1 we shall prove Theorem 2 stated in the introduction. Namely, real hypersurfaces M of $M_n(c)$, $c \neq 0$ satisfying the conditions of (1.4) and (1.5) will be determined. First of all, let us investigate the conditions equivalent to the assumption (1.5) of Theorem 2.

By definition the Lie derivative of the tensor $A\phi$ with respect to ξ is given by

$$\begin{aligned}(\mathcal{L}_\xi A\phi)X &= \mathcal{L}_\xi(A\phi X) - A\phi(\mathcal{L}_\xi X)\\ &= (\nabla_\xi A)\phi X + A(\nabla_\xi \phi)X + A\phi(\nabla_\xi X)\\ &\quad - \nabla_{A\phi X}\xi - A\phi(\nabla_\xi X - \nabla_X \xi)\end{aligned}$$

for any vector field X. By using (2.1) again it gives us to

$$\begin{aligned}(\mathcal{L}_\xi A\phi)X =&(\nabla_\xi A)\phi X + A\{g(X,\xi)A\xi - g(A\xi, X)\xi\}\\ &- \phi A^2\phi X + A\phi^2 AX\\ =&(\nabla_\xi A)\phi X + g(X,\xi)A^2\xi - \phi A^2\phi X - A^2 X\end{aligned}$$

for any vector field X. By the assumption $g((\mathcal{L}_\xi A\phi)X,\ Y) = 0,\ X,$ $Y \in T_0$, we have

$$g((\nabla_\xi A)\phi X, Y) = g((A^2 + \phi A^2 \phi)X, Y), \quad X, Y \in T_0.$$

Replacing X into ϕX, we get

$$g((\nabla_\xi A)X, Y) = -g((A^2\phi - \phi A^2)X, Y), \quad X, Y \in T_0. \tag{5.1}$$

By (2.3) we obtain

$$g((\nabla_X A)\xi, Y) = g((\nabla_X A)Y, \xi) = -g((A^2\phi - \phi A^2 + \frac{c}{4}\phi)X, Y) \tag{5.2}$$

for any X and Y in T_0. Consequently it is seen that this condition is equivalent to (1.5).

Under the assumption that ξ is principal, i.e., $A\xi = \alpha\xi$, where α is constant it follows

$$\begin{aligned}(\nabla_X A)\xi &= \nabla_X(A\xi) - A\nabla_X\xi = \nabla_X(\alpha\xi) - A\phi AX \\ &= \alpha\phi AX - A\phi AX.\end{aligned}$$

By this result combined with (5.2) we get

$$g((\alpha\phi A - A\phi A)X, Y) = -g((A^2\phi - \phi A^2 + \frac{c}{4}\phi)X, Y)$$

for any X and Y in T_0, from which together with the assumption of $A\xi = \alpha\xi$ we have

$$\alpha\phi A - A\phi A + A^2\phi - \phi A^2 + \frac{c}{4}\phi = 0.$$

By (2.4) we get

$$\frac{1}{2}\alpha(\phi A - A\phi) + A^2\phi - \phi A^2 = 0. \tag{5.3}$$

Thus we can show that (1.5) and (5.3) are equivalent to each other under the situation that ξ is principal. This means that the real hypersurfaces of type A satisfies the condition (1.5). So, conversely, we prove the following

PROPOSITION 5.1. *Let M be a real hypersurfaces of $M_n(c)$, $c\neq 0$, $n\geq 3$. If ξ is principal and if it satisfies (1.5), then M is of type A.*

Proof. Let X in T_o be a principal vector corresponding to principal curvature λ. Then,by (2.4), we get

$$A\phi X = \mu\phi X,$$

where the principal curvature μ satisfies

$$(2\lambda-\alpha)\mu = \alpha\lambda + \frac{1}{2}c.$$

By (5.3) we have

$$(\lambda-\mu)(\lambda+\mu-\frac{1}{2}\alpha) = 0,$$

which means that all principal curvature of M are constant. Consequently, according to Takagi's classification theorem and Berndt's one, the principal curvatures and the multiplicities are given.

Suppose that M is not of type A. Then

$$\lambda\neq\mu, \qquad \lambda+\mu = \frac{1}{2}\alpha\neq 0.$$

We consider the case where the curvature c is positive. Without loss of generality, we may suppose $c=4$. By Takagi's classification theorem we may take

$$\alpha = 2cot2\theta, \quad \lambda = cot(\theta-\frac{\pi}{4}), \quad \mu = -tan(\theta-\frac{\pi}{4}),$$

where $0<\theta<\frac{\pi}{2}$. Then we have

$$\lambda+\mu = -2\alpha,$$

a contradiction.

In the case where c is negative, we may suppose that $c=-4$ without loss of generality. Since M is not of type A, Berndt's classification theorem means that $\alpha^2<4$ and

$$\alpha = 2tanh2\theta, \quad \lambda = coth\theta, \quad \mu = tanh\theta.$$

Thus we get

$$\lambda + \mu = \frac{4}{\alpha},$$

a contradiction. This completes the proof. □

Lastly we shall prove Theorem 2 in the introduction.

Proof of Theorem 2. By the assumption $g((\mathcal{L}_\xi A\phi)X, Y) = 0$ the equation (5.2) holds. So, it is deformed as follows:

$$\begin{aligned} g((\nabla_X A)Y, \xi) &= -g((A^2\phi - \phi A^2 + \frac{c}{4}\phi)X, Y) \\ &= -g(A(A\phi - \phi A)X + (A\phi - \phi A)AX, Y) - \frac{c}{4}g(\phi X, Y) \\ &= -g((A\phi - \phi A)X, AY) - g(AX, (A\phi - \phi A)Y) - \frac{c}{4}g(\phi X, Y). \end{aligned}$$

By the assumption $\mathcal{L}_\xi g(X, Y) = 0$ the equation (3.3) holds. Thus we have

$$(A\phi - \phi A)X = -g(X, V)\xi.$$

Accordingly we get

(5.4)
$$g(\nabla_X A)Y, \xi) = \beta\{g(X, U)g(Y, V) + g(X, V)g(Y, U)\} - \frac{c}{4}g(\phi X, Y).$$

By Theorem 1 it implies that M must be a ruled real hypersurface. □

REMARK 3. The ruled real hypersurface M of $M_n(c)$ satisfies (1.4) and moreover the last equation (5.4) holds. From these together with the equation of Codazzi (2.3) it follows (5.1), which is equivalent to the condition (1.5).

REMARK 4. If the real hypersurface M is of A-type or ruled, then it satisfies (1.4). Moreover, it can be easily verified that the condition (1.4) is equivalent to the condition $g((\mathcal{L}_\xi \phi)X, Y) = 0$ for any vector fields $X, Y \in T_0$.

6. The linear transformation ϕA

In relation to Proposition 5.1 the following proposition is proved in this section. But contrary to Proposition 5.1 this proposition can be acquired without the condition that the structure vector field ξ is principal.

PROPOSITION 6.1. *Let M be a real hypersurface of $M_n(c), c \neq 0$, $n \geq 3$. If $g(A\xi, \xi) \neq 0$ and if it satisfies*

$$g((\mathcal{L}_\xi \phi A)X, Y) = 0, \quad X, Y \in T_0,$$

then M is of type A.

Proof. We can first calculate the Lie derivative $\mathcal{L}_\xi(\phi A)$ of the structure tensor ϕA with respect to ξ. By definition we have

$$\begin{aligned}(\mathcal{L}_\xi \phi A)X =& \mathcal{L}_\xi(\phi AX) - \phi A \mathcal{L}_\xi X \\ =& (\nabla_\xi \phi)AX + \phi(\nabla_\xi A)X + \phi A \nabla_\xi X - \nabla_{\phi AX}\xi \\ & - \phi A(\nabla_\xi X - \nabla_X \xi).\end{aligned}$$

Consequently, by both equations of (2.1) it is reformed to

$$(\mathcal{L}_\xi \phi A)X = \beta g(X, U)A\xi - g(AX, A\xi)\xi + \phi(\nabla_\xi A)X, \quad X \in T_0$$

Hence, by the assumption of the theorem, it turns out to be

$$g(\phi(\nabla_\xi A)X, \phi Y) = -\beta g(X, U)g(A\xi, \phi Y) = \beta g(X, U)g(Y, V),$$

which means that

$$g((\nabla_\xi A)X, Y) = \beta g(X, U)g(Y, V), \quad X, Y \in T_0. \tag{6.1}$$

Since the left hand side is symmetric with respect to X and Y, we have

$$\beta g(X, U)g(Y, V) = \beta g(X, V)g(Y, U), \quad X, Y \in T_0.$$

Putting $X = U$ and $Y = V$ in the above equation, we get $\beta = 0$, which means that ξ is principal, i.e., $A\xi = \alpha\xi$, where the principal curvature α is constant. From the property combined with (6.1) it follows that

$$g((\nabla_\xi A)X, Y) = 0, \quad X, Y \in T_0. \tag{6.2}$$

From (2.3) and (6.2), we have

$$g((\nabla_X A)\xi, Y) = -\frac{c}{4} g(\phi X, Y), \quad X, Y \in T_0.$$

On the other hand, we have

$$\begin{aligned}g((\nabla_X A)\xi, Y) &= g(\nabla_X(A\xi), Y) - g(A\nabla_X \xi, Y) \\ &= g(\alpha \nabla_X \xi - A\nabla_X \xi, Y)\end{aligned}$$

for any X and Y in T_0, because α is constant. By these equations together with (2.4) we have

$$\alpha(A\phi - \phi A) = 0.$$

This completes the proof. □

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