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SOME CLASS OF ANALYTIC FUNCTIONS RELATED TO CONIC DOMAINS

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ABSTRACT. For $q \in (0, 1)$ let the q-difference operator be defined as follows

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)} \qquad (z \in \mathbb{U}),$$

where $\mathbb{U}$ denotes the open unit disk in a complex plane. Making use of the above operator the extended Ruscheweyh differential operator $R_q^\lambda f$ is defined. Applying $R_q^\lambda f$ a subfamily of analytic functions is defined. Several interesting properties of a defined family of functions are investigated.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f on the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

and let $\mathcal{S}$ be the class of univalent functions in $\mathcal{A}$.

Goodman [5] introduced the class $\mathcal{UCV}$ of uniformly convex functions. Here, a function $f \in \mathcal{A}$ is called *uniformly convex* if every (positively oriented) circular arc of the form $\{z \in \mathbb{U} : |z - \zeta| = r\}$ with $\zeta \in \mathbb{U}$ and $0 < r < |\zeta| + 1$, is mapped by f univalently onto a convex arc. In particular, $\mathcal{UCV}$ is a subset of univalent,

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convex functions. An analytic characterization for the members of $\mathcal{UCV}$ was obtained by Ma and Minda [13] and Rønning [19].

Theorem 1.1. ([13], [19]) *Let $f \in \mathcal{A}$. Then $f \in \mathcal{UCV}$ if and only if*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in \mathbb{U}). \tag{1.2}$$

Applying the well known Alexander relation Rønning in [18], [19] considered the class, denoted $\mathcal{S}_p$, consisting of functions $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (z \in \mathbb{U}). \tag{1.3}$$

Discarding an assertion that ζ have to lie in the unit disk, and that a consideration concerns whole circular arcs with center at ζ, the first author and Wiśniowska in [7] introduced a family of k-uniformly convex functions, denoted k-$\mathcal{UCV}$. Let $0 \le k < \infty$. A function $f \in \mathcal{A}$ is said to be k-uniformly convex in $\mathbb{U}$, if the image of every (positively oriented) circular arc of the form $\{z \in \mathbb{U} : \ |z-\zeta| = r\}$ with $\zeta \in \mathbb{C}$ and $|\zeta| \le k$, is mapped by f univalently onto a convex arc. We note that 1-$\mathcal{UCV} = \mathcal{UCV}$. An analytic characterization for the members of k-$\mathcal{UCV}$ was given in [7] (see also [8], [9], [6], [10] and [11] for related results).

Theorem 1.2. ([7]) *Let $f \in \mathcal{A}$ and let $0 \le k < \infty$. Then $f \in k$-$\mathcal{UCV}$ if and only if*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in \mathbb{U}). \tag{1.4}$$

In [8] a class of k-starlike functions, denoted k-$\mathcal{ST}$, and related with k-$\mathcal{UCV}$ by the Alexander relation was considered. Such a class consists of functions $f \in \mathcal{A}$ which satisfy the inequality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (z \in \mathbb{U}). \tag{1.5}$$

We note that 1-$\mathcal{ST} = \mathcal{ST}$. The classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ have been generalized as follows (see [4], [16]).

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{ST}(k,\alpha)$ of k-starlike functions of order α, $0 \le \alpha < 1$, if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha \qquad (k \ge 0, \ \ 0 \le \alpha < 1). \tag{1.6}$$

Replacing f in (1.6) by $zf'(z)$, we obtain the inequality

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| + \alpha. \tag{1.7}$$

Following the original definition the functions satisfying (1.7) are called k-uniformly convex of order α and denoted $\mathcal{UCV}(k,\alpha)$.

We note that $\mathcal{UCV}(k,0) = k\text{-}\mathcal{UCV}$ and $\mathcal{ST}(k,0) = k\text{-}\mathcal{ST}$. The classes $\mathcal{UCV}(1,\alpha)$ and $\mathcal{ST}(1,\alpha)$ were investigated in [1], [2] and [18].

The aim of this work is to adopt a notion of uniform convexity onto some classes defined by a difference operator. To do this, we first introduce a notion of q-difference operator related to the q-calculus (see, e.g., [3: Ch. 10]). Let $q > 0$. For any non-negative integer n, the *q-integer number n* $[n]$ is defined by:

$$[n] = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}, \qquad [0] = 0. \tag{1.8}$$

The *q-number shifted factorial* is defined by $[0]! = 1$ and $[n]! = [1][2]\cdots[n]$. Clearly, $\lim\limits_{q\to 1^-}[n] = n$ and $\lim\limits_{q\to 1^-}[n]! = n!$. In general we will denote $[t] = \frac{1-q^t}{1-q}$ also for a non-integer number. Throughout this paper we will assume q to be a fixed number between 0 and 1.

Definition 1.1. Let $f \in \mathcal{A}$, and let the *q-derivative operator* or *q-difference operator* be defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)} \qquad (z \in \mathbb{U}). \tag{1.9}$$

It is easy to check that for $n \in \mathbb{N} := \{1, 2, \ldots\}$ and $z \in \mathbb{U}$

$$\partial_q z^n = [n] z^{n-1}. \tag{1.10}$$

Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Let the *q-generalized Pochhammer symbol* be defined as

$$[t]_n = [t][t+1][t+2]\cdots[t+n-1],$$

and for $t > 0$ let the *q-gamma function* be defined by

$$\Gamma_q(t+1) = [t]\Gamma_q(t) \qquad \text{and} \qquad \Gamma_q(1) = 1. \tag{1.11}$$

Definition 1.2. For $f \in \mathcal{A}$ let *the Ruscheweyh q-differential operator* be defined as follows

$$R_q^\lambda f(z) = f(z) * F_{q,\lambda+1}(z) \qquad (z \in \mathbb{U}, \ \ \lambda > -1) \tag{1.12}$$

where

$$F_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\lambda)}{[n-1]!\,\Gamma_q(1+\lambda)} z^n = z + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{n-1}}{[n-1]!} z^n \qquad (z \in \mathbb{U}). \tag{1.13}$$

The symbol "$*$" stands for Hadamard product (or convolution).

From (1.12) we obtain that

$$R_q^0 f(z) = f(z), \qquad R_q^1 f(z) = z\partial_q f(z)$$

and

$$R_q^m f(z) = \frac{z\partial_q^m \left(z^{m-1} f(z)\right)}{[m]!} \qquad (m \in \mathbb{N}).$$

Making use of (1.12) and (1.13), the power series of $R_q^\lambda f(z)$ for f of the form (1.1) is given by

$$R_q^\lambda f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)} a_n z^n = z + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{n-1}}{[n-1]!} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.14}$$

Note that

$$\lim_{q\to 1^-} F_{q,\lambda+1}(z) = \frac{z}{(1-z)^{\lambda+1}}$$

and

$$\lim_{q\to 1^-} R_q^\lambda f(z) = f(z) * \frac{z}{(1-z)^{\lambda+1}}.$$

Thus, we can say that Ruscheweyh q-differential operator reduces to the differential operator defined by Ruscheweyh [20] in the case when $q \to 1^-$.

It is easy to check that

$$z\partial_q(F_{q,\lambda+1}(z)) = \left(1 + \frac{[\lambda]}{q^\lambda}\right) F_{q,\lambda+2}(z) - \frac{[\lambda]}{q^\lambda} F_{q,\lambda+1}(z) \qquad (z \in \mathbb{U}). \tag{1.15}$$

Making use of (1.12), (1.15) and the properties of Hadamard product, we obtain the following equality

$$z\partial_q(R_q^\lambda f(z)) = \left(1 + \frac{[\lambda]}{q^\lambda}\right) R_q^{\lambda+1} f(z) - \frac{[\lambda]}{q^\lambda} R_q^\lambda f(z) \qquad (z \in \mathbb{U}). \tag{1.16}$$

If $q \to 1^-$, the equality (1.16) implies

$$z(R^\lambda f(z))' = (1+\lambda) R^{\lambda+1} f(z) - \lambda R^\lambda f(z) \qquad (z \in \mathbb{U})$$

which is the well known recurrent formula for Ruscheweyh differential operator.

Using Ruscheweyh differential operator various new classes of convex and starlike functions have been defined. Therefore it seems natural to use Ruscheweyh q-differential operator to introduce the following class of functions.

Definition 1.3. Let $0 \le \alpha < 1$, $k \ge 0$ and $\lambda > -1$. A function $f \in \mathcal{A}$ is in the class $\mathcal{ST}(k, \alpha, \lambda, q)$ if it satisfies the condition

$$\operatorname{Re}\left\{\frac{z\partial_q\left(R_q^\lambda f(z)\right)}{R_q^\lambda f(z)}\right\} > k \left|\frac{z\partial_q\left(R_q^\lambda f(z)\right)}{R_q^\lambda f(z)} - 1\right| + \alpha \qquad (z \in \mathbb{U}). \tag{1.17}$$

Note that if $\lambda = 0$ and $q \to 1^-$ the class $\mathcal{ST}(k, \alpha, \lambda, q)$ reduces to the class $\mathcal{ST}(k, \alpha)$.

In a present work we study several properties of the family $\mathcal{ST}(k,\alpha,\lambda,q)$, e.g. necessary and sufficient conditions to be a member of $\mathcal{ST}(k,\alpha,\lambda,q)$, coefficients bounds and Fekete-Szegö problem.

2. Properties of the class $\mathcal{ST}(k,\alpha,\lambda,q)$

We begin this section with a sufficient condition for a function f to be in the class $\mathcal{ST}(k,\alpha,\lambda,q)$.

Theorem 2.1. *Let $f \in \mathcal{A}$ be given by (1.1). If the inequality*

$$\sum_{n=2}^{\infty}\left([n](k+1)-k-\alpha\right)\frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)}|a_n| \le 1-\alpha \tag{2.1}$$

holds true for some k $(0 \le k < \infty)$, $\lambda > -1$ and α $(0 \le \alpha < 1)$, then $f \in \mathcal{ST}(k,\alpha,\lambda,q)$. The result is sharp for the function

$$f_n(z) = z - \frac{(1-\alpha)[n-1]!\Gamma_q(1+\lambda)}{\left([n](k+1)-k-\alpha\right)\Gamma_q(n+\lambda)}z^n.$$

P r o o f. Making use of the Definition 1.3 it suffices to prove that

$$k\left|\frac{z\partial_q\left(R_q^\lambda f(z)\right)}{R_q^\lambda f(z)}-1\right| - \operatorname{Re}\left\{\frac{z\partial_q\left(R_q^\lambda f(z)\right)}{R_q^\lambda f(z)}-1\right\} < 1-\alpha.$$

Observe, that

$$\begin{aligned} &k\left|\frac{z\partial_q\left(R_q^\lambda f(z)\right)}{R_q^\lambda f(z)}-1\right| - \operatorname{Re}\left\{\frac{z\partial_q\left(R_q^\lambda f(z)\right)}{R_q^\lambda f(z)}-1\right\} \\ &\le (k+1)\left|\frac{z\partial_q\left(R_q^\lambda f(z)\right)}{R_q^\lambda f(z)}-1\right| \\ &= (k+1)\left|\frac{\sum_{n=2}^{\infty}([n]-1)\frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)}a_n z^{n-1}}{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)}a_n z^{n-1}}\right| \\ &< (k+1)\frac{\sum_{n=2}^{\infty}([n]-1)\frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)}|a_n|}{1-\sum_{n=2}^{\infty}\frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)}|a_n|}. \end{aligned}$$

The last expression is bounded by $1-\alpha$ if inequality (2.1) holds.

It is obvious that the function f_n satisfies inequality (2.1) thus the number $1-\alpha$ can not be replaced by a larger number. Therefore, we only need to show that $f_n \in \mathcal{ST}(k,\alpha,\lambda,q)$. Since

$$k\left|\frac{z\partial_q\left(R_q^\lambda f_n(z)\right)}{R_q^\lambda f_n(z)}-1\right| = k\left|\frac{(1-\alpha)(1-[n])z^{n-1}}{([n](k+1)-k-\alpha)-(1-\alpha)z^{n-1}}\right| < \frac{k(1-\alpha)}{k+1},$$

and

$$\operatorname{Re}\left\{\frac{z\partial_q\left(R_q^\lambda f_n(z)\right)}{R_q^\lambda f_n(z)}\right\} = \operatorname{Re}\left\{\frac{[n](k+1)-k-\alpha-[n](1-\alpha)z^{n-1}}{[n](k+1)-k-\alpha-(1-\alpha)z^{n-1}}\right\} > \frac{k+\alpha}{k+1},$$

the condition (1.17) holds true for $f_n(z)$. Thus, $f_n \in \mathcal{ST}(k,\alpha,\lambda,q)$. □

The next Corollary can be easily obtained from Theorem 2.1.

Corollary 2.1. *Let $f(z) = z + a_n z^n$. If*

$$|a_n| \le \frac{(1-\alpha)[n-1]!\Gamma_q(1+\lambda)}{([n](k+1)-k-\alpha)\,\Gamma_q(n+\lambda)} \qquad (n \ge 2),$$

then $f \in \mathcal{ST}(k,\alpha,\lambda,q)$.

Consider $p(z) = z\partial_q(R_q^\lambda f(z))/R_q^\lambda f(z)$. We can rewrite the condition (1.17) into the form

$$\operatorname{Re} p(z) > k|p(z)-1| + \alpha \qquad (z \in \mathbb{U}). \tag{2.2}$$

It follows that the range of the expression $p(z)$ $(z \in \mathbb{U})$ is a conic domain

$$\Omega_{k,\alpha} = \left\{w \in \mathbb{C}:\ \operatorname{Re} w > k|w-1| + \alpha\right\}, \tag{2.3}$$

or

$$\Omega_{k,\alpha} = \left\{w = u + \mathrm{i}v:\ u > k\sqrt{(u-1)^2+v^2} + \alpha\right\}, \tag{2.4}$$

where $0 \le k < \infty$ and $0 \le \alpha < 1$.

Note that $\Omega_{k,\alpha}$ is such that $1 \in \Omega_{k,\alpha}$ and $\partial\Omega_{k,\alpha}$ is a curve defined by

$$\partial\Omega_{k,\alpha} = \left\{w = u + \mathrm{i}v:\ (u-\alpha)^2 = k^2(u-1)^2 + k^2v^2\right\}. \tag{2.5}$$

Elementary computations show that $\partial\Omega_{k,\alpha}$ represents a conic section symmetric about the real axis. It follows that the domain $\Omega_{k,\alpha}$ is bounded by an ellipse for $k > 1$, by a parabola for $k = 1$ and by a hyperbola if $0 < k < 1$. Finally, for $k = 0$, $\Omega_{k,\alpha}$ is the right half plane $\operatorname{Re} w > \alpha$.

From (1.17) we obtain that $f \in \mathcal{ST}(k,\alpha,\lambda,q)$ if and only if

$$\frac{z\partial_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} \in \Omega_{k,\alpha}. \tag{2.6}$$

Making use of the properties of the domain $\Omega_{k,\alpha}$ and (2.6) it follows that if $f \in \mathcal{ST}(k, \alpha, \lambda, q)$, then

$$\operatorname{Re} \frac{z\partial_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} > \frac{k+\alpha}{k+1} \qquad (z \in \mathbb{U})$$

and

$$\left| \operatorname{Arg} \frac{z\partial_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} \right| \leq \begin{cases} \arctan \frac{1-\alpha}{\sqrt{|k^2-\alpha^2|}} & \text{if } 0 \leq \alpha < 1,\ k > 0, \\ \frac{\pi}{2} & \text{if } k = 0. \end{cases}$$

Denote by $\mathcal{P}$ the class of analytic and normalized Carathèodory functions and by $p_{k,\alpha} \in \mathcal{P}$ the function such that $p_{k,\alpha}(\mathbb{U}) = \Omega_{k,\alpha}$. Following Ma and Minda notation [15] let $\mathcal{P}(p_{k,\alpha})$, where $0 \leq k < \infty$ and $0 \leq \alpha < 1$, denotes the following class of functions

$$\mathcal{P}(p_{k,\alpha}) = \big\{p \in \mathcal{P} :\ p(\mathbb{U}) \subset \Omega_{k,\alpha}\big\} = \big\{p \in \mathcal{P} :\ p \prec p_{k,\alpha} \text{ in } \mathbb{U}\big\}.$$

The functions which play the role of extremal functions for the class $\mathcal{P}(p_{k,\alpha})$, may be obtained by a simple modification of related functions described in [7] (see also [11]), and are defined by

$$p_{k,\alpha}(z) = \begin{cases} \frac{1+(1-2\alpha)z}{1-z} & \text{if } k = 0, \\ 1 + \frac{2(1-\alpha)}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 & \text{if } k = 1, \\ \frac{1-\alpha}{1-k^2} \cos\left(A(k) \mathrm{i} \log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right) - \frac{k^2-\alpha}{1-k^2} & \text{if } 0 < k < 1, \\ \frac{1-\alpha}{k^2-1} \sin\left(\frac{\pi}{2K(t)} \int\limits_0^{\frac{u(z)}{\sqrt{t}}} \frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}\right) + \frac{k^2-\alpha}{k^2-1} & \text{if } k > 1. \end{cases} \tag{2.7}$$

with $A(k) = \frac{2}{\pi} \arccos k$,

$$u(z) = \frac{z - \sqrt{t}}{1 - \sqrt{t}z} \qquad (0 < t < 1, \quad z \in \mathbb{U}),$$

where t is chosen such that

$$k = \cosh \frac{\pi K'(t)}{4K(t)},$$

and $K(t)$ is Legendre's complete elliptic integral of the first kind and $K'(t)$ is complementary integral of $K(t)$.

Obviously, if $k = 0$

$$p_{0,\alpha}(z) = 1 + 2(1-\alpha)z + 2(1-\alpha)z^2 + \cdots .$$

For $k = 1$, we have (see [13] and also [19])

$$p_{1,\alpha}(z) = 1 + \frac{8}{\pi^2}(1-\alpha)z + \frac{16}{3\pi^2}(1-\alpha)z^2 + \cdots . \tag{2.8}$$

Using Taylor expansion in [7] and [8], for $0 < k < 1$, we have

$$p_{k,\alpha}(z) = 1 + \frac{1-\alpha}{1-k^2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{2n} 2^l \binom{A(k)}{l} \binom{2n-1}{2n-l} \right] z^n. \tag{2.9}$$

Finally, when $k > 1$

$$\begin{aligned} p_{k,\alpha}(z) = 1 + &\frac{\pi^2(1-\alpha)}{4\sqrt{t}(k^2-1)K^2(t)(1+t)} \\ &\times \left\{ z + \frac{4K^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}K^2(t)(1+t)} z^2 + \cdots \right\}, \end{aligned} \tag{2.10}$$

so that, denoting $p_{k,\alpha}(z) = 1 + P_1 z + P_2 z^2 + \cdots$ $(P_j = P_j(k,\alpha),\ j = 1,2,\ldots)$, we get

$$P_1 = \begin{cases} \frac{8(1-\alpha)(\arccos k)^2}{\pi^2(1-k^2)} & \text{if } 0 \le k < 1, \\ \frac{8(1-\alpha)}{\pi^2} & \text{if } k = 1, \\ \frac{\pi^2(1-\alpha)}{4\sqrt{t}(1+t)K^2(t)(k^2-1)} & \text{if } k > 1. \end{cases} \tag{2.11}$$

Let $f_{k,\alpha}(z) = z + A_2 z^2 + A_3 z^3 + \cdots$ be the extremal function in the class $\mathcal{ST}(k,\alpha,\lambda,q)$. Then, the relation between the extremal functions in the classes $\mathcal{P}(p_{k,\alpha})$ and $\mathcal{ST}(k,\alpha,\lambda,q)$ is given by

$$p_{k,\alpha}(z) = \frac{z\partial_q(R_q^\lambda f_{k,\alpha}(z))}{R_q^\lambda f_{k,\alpha}(z)} \qquad (z \in \mathbb{U}). \tag{2.12}$$

Making use of (1.14), (1.17) and (2.12) we obtain for $p_{k,\alpha}(z)$ the following coefficient relation

$$\frac{q\Gamma_q(n+\lambda)}{[n-2]!\Gamma_q(1+\lambda)} A_n = \sum_{m=1}^{n-1} \frac{\Gamma_q(m+\lambda)}{[m-1]!\Gamma_q(1+\lambda)} A_m P_{n-m}, \qquad A_1 = 1. \tag{2.13}$$

In particular, by a straightforward computation, we get

$$A_2 = \frac{P_1}{q[1+\lambda]}, \tag{2.14}$$

$$A_3 = \frac{qP_2 + P_1^2}{q^2[1+\lambda][2+\lambda]}. \tag{2.15}$$

Since $\lambda > -1$, $q \in (0,1)$ and P_n are nonnegative, it follows that A_n are nonnegative.

Theorem 2.2. *Let $k \in [0,\infty)$, $\lambda > -1$, $q \in (0,1)$ and $\alpha \in [0,1)$. If f of the form* (1.1) *belongs to the class $\mathcal{ST}(k,\alpha,\lambda,q)$, then*

$$|a_2| \le A_2 \qquad \text{and} \qquad |a_3| \le A_3. \tag{2.16}$$

P r o o f. Let $p(z) = z\partial_q(R_q^\lambda f(z))/R_q^\lambda f(z)$. Using the relation (1.14) for $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, we have

$$\frac{\Gamma_q(n+\lambda)}{[n-2]!\Gamma_q(1+\lambda)} q a_n = \sum_{m=1}^{n-1} \frac{\Gamma_q(m+\lambda)}{[m-1]!\Gamma_q(1+\lambda)} a_m p_{n-m}, \qquad a_1 = 1. \tag{2.17}$$

Since $p_{k,\alpha}$ is univalent in $\mathbb{U}$, the function

$$q(z) = \frac{1 + p_{k,\alpha}^{-1}(p(z))}{1 - p_{k,\alpha}^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots,$$

is analytic in $\mathbb{U}$ and $\operatorname{Re} q(z) > 0$. From

$$p(z) = p_{k,\alpha}\left(\frac{q(z)-1}{q(z)+1}\right) = 1 + \frac{1}{2}c_1 P_1 z + \left(\frac{1}{2}c_2 P_1 + \frac{1}{4}c_1^2(P_2 - P_1)\right) z^2 + \cdots,$$

we have

$$|a_2| = \frac{1}{q[1+\lambda]}|p_1| = \frac{1}{2q[1+\lambda]}|c_1 P_1| \le \frac{P_1}{q[1+\lambda]} = A_2, \tag{2.18}$$

where we used the inequality $|c_n| \le 2$ and (2.13). In view of a relation $|p_1|^2 + |p_2| \le P_1^2 + P_2$ (cf. [8]) and (2.14), we obtain

$$\begin{aligned} |a_3| &= \frac{|qp_2 + p_1^2|}{q^2[1+\lambda][2+\lambda]} \le \frac{q\left(|p_2| + |p_1|^2\right) + (1-q)|p_1|^2}{q^2[1+\lambda][2+\lambda]} \\ &\le \frac{q\left(P_2 + P_1^2\right) + (1-q)P_1^2}{q^2[1+\lambda][2+\lambda]} \le \frac{qP_2 + P_1^2}{q^2[1+\lambda][2+\lambda]} = A_3. \end{aligned} \tag{2.19}$$

Thus, the proof of the theorem is completed. □

Theorem 2.3. *Let $0 \le k < \infty$, $\lambda > -1$, $q \in (0,1)$ and $0 \le \alpha < 1$. If f of the form* (1.1) *is in the class $\mathcal{ST}(k,\alpha,\lambda,q)$, then*

$$|a_n| \le \frac{P_1(P_1+q)(P_1+[2]q)\cdots(P_1+[n-2]q)}{q^{n-1}[1+\lambda]_{n-1}}, \qquad n \ge 2. \tag{2.20}$$

P r o o f. The result is clearly true for $n = 2$. Let n be an integer number with $n \ge 2$, and assume that the inequality is true for all $m \le n-1$. Making use of (2.13), we have

$$\begin{aligned} |a_n| &= \frac{[n-2]!\Gamma_q(1+\lambda)}{q\Gamma_q(n+\lambda)}\left| p_{n-1} + \sum_{m=2}^{n-1} \frac{\Gamma_q(m+\lambda)}{[m-1]!\Gamma_q(1+\lambda)} a_m p_{n-m} \right| \\ &\le \frac{[n-2]!\Gamma_q(1+\lambda)}{q\Gamma_q(n+\lambda)}\left\{ P_1 + \sum_{m=2}^{n-1} \frac{\Gamma_q(m+\lambda)}{[m-1]!\Gamma_q(1+\lambda)} |a_m| P_1 \right\} \end{aligned}$$

$$\leq \frac{[n-2]!\Gamma_q(1+\lambda)}{q\Gamma_q(n+\lambda)} P_1 \Bigg\{ 1 + \sum_{m=2}^{n-1} \frac{\Gamma_q(m+\lambda)}{[m-1]!\Gamma_q(1+\lambda)} \times \frac{P_1(P_1+q)\cdots(P_1+[m-2]q)}{q^{m-1}[1+\lambda]_{m-1}} \Bigg\},$$

where we applied the induction hypothesis to $|a_m|$ and the Rogosinski result ([17]) $|p_n| \leq P_1$. Since

$$\frac{\Gamma_q(m+\lambda)}{\Gamma_q(1+\lambda)} = [1+\lambda]_{m-1}$$

we have

$$|a_n| \leq \frac{[n-2]!P_1}{q([1+\lambda])_{n-1}} \left\{ 1 + \sum_{m=2}^{n-1} \frac{P_1(P_1+q)\cdots(P_1+[m-2]q)}{q^{m-1}[m-1]!} \right\}.$$

Applying again mathematical induction, we find that

$$1 + \sum_{m=2}^{n-1} \frac{P_1(P_1+q)\cdots(P_1+[m-2]q)}{q^{m-1}[m-1]!} = \frac{(P_1+q)\cdots(P_1+[n-2]q)}{q^{n-2}[n-2]!}.$$

Consequently, the inequality (2.20) follows. □

To obtain a solution of the Fekete-Szegö problem over the class $\mathcal{ST}(k,\alpha,\lambda,q)$, we need the following lemmas.

Lemma 2.1. ([12], [15]) *If $q(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an analytic function with positive real part in $\mathbb{U}$, then*

$$|c_2 - v c_1^2| \leq 2 \max\{1; |2v-1|\}. \tag{2.21}$$

The result is sharp for the functions $q(z) = \frac{1+z^2}{1-z^2}$ or $q(z) = \frac{1+z}{1-z}$.

Lemma 2.2. ([6]) *If $q(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}(p_k)$ is an analytic function in $\mathbb{U}$, then*

$$|c_2 - v c_1^2| \leq \begin{cases} P_1 - v P_1^2 & \text{if } v \leq 0, \\ P_1 & \text{if } 0 < v < 1, \\ P_1 + (v-1)P_1^2 & \text{if } v \geq 1. \end{cases} \tag{2.22}$$

The result is sharp for the functions $q(z) = \frac{1+z^2}{1-z^2}$ for $0 < v < 1$ and $q(z) = \frac{1+z}{1-z}$ otherwise.

Theorem 2.4. *Let $0 \leq k < \infty$, $\lambda > -1$, $q \in (0,1)$ and $0 \leq \alpha < 1$. Suppose that the function f given by (1.1) is in the class $\mathcal{ST}(k,\alpha,\lambda,q)$. Then, for a complex number μ*

$$|a_3 - \mu a_2^2| \leq \frac{2}{q[1+\lambda][2+\lambda]} \max\left\{1; \left|\frac{2\mu[2+\lambda]}{q[1+\lambda]} - 3\right|\right\}. \tag{2.23}$$

Moreover for a real parameter μ, we obtain more rigorous bounds, as follows.

$$|a_3 - \mu a_2^2| \leq \frac{1}{[1+\lambda][2+\lambda]} \begin{cases} P_1 - \frac{1}{q}\left(1 - \mu\frac{[2+\lambda]}{[1+\lambda]}\right) P_1^2 & \text{if } \mu \geq \frac{[1+\lambda]}{[2+\lambda]} \\ P_1 & \text{if } \mu \in \frac{[1+\lambda]}{[2+\lambda]}(1-q,1) \\ P_1 + \frac{1}{q}\left(1 - q - \mu\frac{[1+\lambda]}{[2+\lambda]}\right) P_1^2 & \text{if } \mu \leq (1-q)\frac{[1+\lambda]}{[2+\lambda]}, \end{cases}$$

where P_1 given by (2.11). *The results are sharp.*

P r o o f. From (2.18) and (2.19), it follows that

$$a_2 = \frac{p_1}{q[1+\lambda]} \tag{2.24}$$

and

$$a_3 = \frac{qp_2 + p_1^2}{q^2[1+\lambda][2+\lambda]}. \tag{2.25}$$

In view of (2.24) and (2.25), for a complex number μ, we have

$$|a_3 - \mu a_2^2| = \frac{1}{q[1+\lambda][2+\lambda]}\left|p_2 - \frac{p_1^2}{q}\left(\mu\frac{[2+\lambda]}{[1+\lambda]} - 1\right)\right|.$$

Applying Lemma 2.1, we get

$$|a_3 - \mu a_2^2| \leq \frac{2}{q[1+\lambda][2+\lambda]}\max\left\{1; \left|\frac{2\mu[2+\lambda]}{q[1+\lambda]} - 3\right|\right\},$$

which is the thesis. The sharpness of (2.23) follows from the sharpness of (2.22).

Assume now, that a parameter μ is real. Since

$$|a_3 - \mu a_2^2| = \left|\frac{qp_2 + p_1^2}{q^2[1+\lambda][2+\lambda]} - \mu\frac{p_1^2}{q^2[1+\lambda]^2}\right|, = \frac{1}{[1+\lambda][2+\lambda]}\left|p_2 + \left(\frac{1}{q} - \mu\frac{[2+\lambda]}{q[1+\lambda]}\right)p_1^2\right|$$

therefore, making use of (2.22) from Lemma 2.2 the thesis follows. □

A necessary and sufficient condition for a function $f \in \mathcal{A}$ to be in the class $\mathcal{ST}(k,\alpha,\lambda,q)$ in terms of Hadamard product is given in the following theorem.

Theorem 2.5. *Let $0 \leq k < \infty$, $\lambda > -1$, $q \in (0,1)$ and $0 \leq \alpha < 1$. Then the function f belongs to the class $\mathcal{ST}(k,\alpha,\lambda,q)$ if and only if $(f * H_{q,\lambda})(z)/z \neq 0$ in $\mathbb{U}$, where*

$$H_{q,\lambda}(z) = F_{q,\lambda+2}(z)\left\{1 - \left(1 - \frac{F_{q,\lambda+1}(z)}{F_{q,\lambda+2}(z)}\right)\frac{w(t)q^\lambda + [\lambda]}{q^\lambda(w(t)-1)}\right\} \tag{2.26}$$

with

$$w(t) = (kt+\alpha)^{\pm}\mathrm{i}\sqrt{t^2 - (kt+\alpha-1)^2}$$

and $t^2 - (kt+\alpha-1)^2 \geq 0$.

P r o o f. From (2.6) we have that the values of $z\partial_q(R_q^\lambda f(z))/R_q^\lambda f(z)$ lie in $\Omega_{k,\alpha}$. Therefore

$$\frac{z\partial_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} \neq (kt+\alpha)^{\pm}\mathrm{i}\sqrt{t^2-(kt+\alpha-1)^2} = w(t) \tag{2.27}$$

with $z \in \mathbb{U}$, $t^2-(kt+\alpha-1)^2 \geq 0$.

Applying the definition of $R_q^\lambda f$ and the properties of Hadamard product, the condition (2.27) will hold if

$$f(z) * \big[z\partial_q(F_{q,\lambda+1}(z)) - w(t)F_{q,\lambda+1}(z)\big]/z \neq 0. \tag{2.28}$$

Making use of (1.15), it follows from (2.28), that $(f * H_{q,\lambda})(z)/z \neq 0$, where $H_{q,\lambda}(z)$ is given by (2.26).

Conversely, if $(f*H_{q,\lambda})(z)/z \neq 0$ in $\mathbb{U}$, then the values of $z\partial_q(R_q^\lambda f(z))/R_q^\lambda f(z)$ lie completely inside $\Omega_{k,\alpha}$ or its complement. Since

$$\left.\frac{z\partial_q(R_q^\lambda f(z))}{R_q^\lambda f(z)}\right|_{z=0} = 1 \in \Omega_{k,\alpha}$$

we obtain $z\partial_q(R_q^\lambda f(z))/R_q^\lambda f(z) \in \Omega_{k,\alpha}$ which shows that $f \in \mathcal{ST}(k,\alpha,\lambda,q)$. □

Theorem 2.6. *Let $0 \leq k < \infty$, $\lambda > -1$, $q \in (0,1)$ and $0 \leq \alpha < 1$. The coefficients h_n of the function $H_{q,\lambda}$ given by (2.26) satisfy the inequality*

$$|h_n| \leq \frac{\Gamma_q(n+\lambda)[1-\alpha+[n](k+1)]}{(1-\alpha)[n-1]!\Gamma_q(1+\lambda)}, \qquad n \geq 2. \tag{2.29}$$

P r o o f. From the power series of the function $H_{q,\lambda}$ we have

$$h_n = \frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)}\,\frac{[n]-w(t)}{1-w(t)}$$

and therefore

$$\begin{aligned}|h_n|^2 &= \left(\frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)}\right)^2\left(1-\frac{2k([n]-1)}{t}+([n]-1)\frac{[n]+1-2\alpha}{t^2}\right)\\ &=: \left(\frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)}\right)^2 V(t).\end{aligned}$$

The function $V(t)$ is decreasing in the interval $\left\langle\frac{1-\alpha}{1+k}, t_0\right)$ and increasing in (t_0,∞), where $t_0 = \frac{[n]+1-2\alpha}{k}$, with its minimum at t_0. The limit of $V(t)$ as t tends to infinity is equal to 1 and

$$V\left(\frac{1-\alpha}{k+1}\right) = 1-2k([n]-1)\frac{1+k}{1-\alpha}+([n]-1)([n]+1-2\alpha)\frac{(1+k)^2}{(1-\alpha)^2} \geq 1.$$

Thus, the maximum value of $V(t)$ is attained at the point $\frac{1-\alpha}{k+1}$. Since

$$V\left(\frac{1-\alpha}{k+1}\right) \leq \left[\frac{1-\alpha+[n](k+1)}{1-\alpha}\right]^2$$

the coefficients of $H_{q,\lambda}$ satisfy the inequality (2.29). □

Corollary 2.2. *Let* $g(z) = z + a_n z^n$. *If*

$$|a_n| \leq \frac{(1-\alpha)[n-1]!\Gamma_q(1+\lambda)}{(1-\alpha+[n](k+1))\,\Gamma_q(n+\lambda)} \qquad (n \geq 2),$$

then $g \in \mathcal{ST}(k,\alpha,\lambda,q)$

P r o o f. Since

$$\left|\frac{(g * H_{q,\lambda})(z)}{z}\right| = |1 + h_n a_n z^{n-1}| \geq 1 - |h_n||a_n||z| \geq 1 - |z| > 0 \qquad (z \in \mathbb{U})$$

it follows that $g \in \mathcal{ST}(k,\alpha,\lambda,q)$. □

Remark 2.1. The q-analogue of the Leibniz rule is

$$\partial_q(f(z)g(z)) = g(z)\partial_q f(z) + f(qz)\partial_q g(z).$$

Replacing $R_q^\lambda f(z)$ in (1.17) by $z\partial_q(R_q^\lambda f(z))$ we can obtain a new class of functions which is the analogue of the class $\mathcal{UCV}(k,\alpha)$ of k-uniformly convex functions of order α.

Definition 2.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UCV}(k,\alpha,\lambda,q)$ if it satisfies the inequality

$$\operatorname{Re}\left\{1 + q\frac{z\partial_q^2(R_q^\lambda f(z))}{\partial_q(R_q^\lambda f(z))}\right\} > k\left|q\frac{z\partial_q^2(R_q^\lambda f(z))}{\partial_q(R_q^\lambda f(z))}\right| + \alpha.$$

Note that if $\lambda = 0$ and $q \to 1^-$, the class $\mathcal{UCV}(k,\alpha,\lambda,q)$ reduces to the class $\mathcal{UCV}(k,\alpha)$.

Remark 2.2. Making use of the properties of the functions in $\mathcal{ST}(k,\alpha,\lambda,q)$ we can obtain easily the properties of the functions that belong to the class $\mathcal{UCV}(k,\alpha,\lambda,q)$.

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