where p is a positive polynomial on $\boldsymbol{R}$ and $k > 0$. Let $\varphi_0, \varphi_1, \ldots$ be the eigenfunctions of L with eigenvalues $\lambda_0, \lambda_1, \ldots$ By Corollary 2.9 there exists an α such that for every $f \in L^1(\boldsymbol{R})$

$$\lim_{n\to\infty} \sum_{\lambda_j < n} (1-j/n)^{\alpha} (f, \varphi_j)\, \varphi_j(x) = f(x) \quad \text{a.e.}$$

In more dimensions our method gives a similar result only for the operators L with the potentials which are sums of squares of polynomials, since in several variables not every positive polynomial is a sum of squares of polynomials.

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Some classes of commuting n-tuples of operators

by

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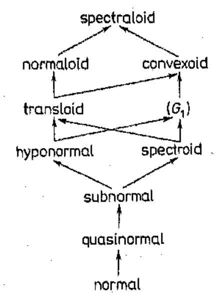
Abstract. In this paper we study the inclusion relations among some classes of operator-families and the topological properties of these classes.

1. Introduction. Throughout this paper, H will be a complex Hilbert space with the scalar product $(\ ,\ )$ and the norm $\|\cdot\|$, and all operators on H will be assumed to be linear and bounded. For an operator T on H, let $\sigma(T)$ denote its spectrum, $r(T)$ its spectral radius, $W(T)$ its numerical range and $w(T)$ its numerical radius.

In case of a single operator, the properties of a normal operator are well known because it has spectral resolution. Thus many authors have discussed some classes of operators which are close to being normal in some sense. It is well known that there exists an inclusion relation among these classes. We



shall indicate it by the diagram above (e. g. see [10]). Here, T is called *normaloid* iff $\|T\| = w(T)$, and *transloid* iff $T-\lambda I$ is normaloid for each $\lambda \in \boldsymbol{C}$. T is called *convexoid* iff $\overline{W(T)} = \operatorname{co}\sigma(T)$. ($\bar{X}$ denotes the closure of the set $X \subset \boldsymbol{C}$ and $\operatorname{co} X$ its convex hull.) T is called *spectraloid* iff $w(T) = r(T)$. *T belongs to* (G_1) iff $\|(T-\lambda I)^{-1}\| = 1/d(\lambda, \sigma(T))$ for each $\lambda \in \boldsymbol{C} - \sigma(T)$, where $d(\lambda, X)$ denotes the distance between λ and the set $X \subset \boldsymbol{C}$. And T is called *spectroid* iff $\sigma(T)$ is a spectral set for T in the sense of von Neumann.

In this paper we study analogous situations for commuting n-tuples of operators. The properties of a commuting n-tuple of normal operators are well known, because there exists a suitable measure space $(\mathfrak{X}, \mu)$ and an n-tuple $\varphi = (\varphi_1, \ldots, \varphi_n)$ of functions in $L^\infty(\mathfrak{X}, \mu)$ such that each A_k, $k = 1, \ldots, n$, is unitarily equivalent to multiplication by φ_k on $L^2(\mathfrak{X}, \mu)$. So we shall introduce some classes of n-tuples of operators similar to the classes of single operator case. In Section 3 we shall give some results on a doubly commuting n-tuple of hyponormal operators. In Section 4 we shall show that there exists an inclusion relation among these classes which is similar to the inclusion relation among the corresponding classes of single operators. And finally, in Section 5, we shall study the topological properties of these classes.

2. Definitions and preliminaries. In the sequel, by an *operator-family* we shall mean a commuting n-tuple of operators and denote the set of all operator-families by $\mathfrak{B}^n(H)$.

Let $A = (A_1, \ldots, A_n) \in \mathfrak{B}^n(H)$. We shall say that a point $z = (z_1, \ldots, z_n)$ of C^n is in the *joint approximate point spectrum* $\sigma_\pi(A)$ of A if there exists a sequence $\{x_i\}$ of unit vectors in H such that

$$\|(z_k - A_k)x_i\| \to 0 \quad (i \to \infty), \; k = 1, \ldots, n.$$

A point $z = (z_1, \ldots, z_n)$ of C^n will be said to be in the *joint approximate compression spectrum* $\sigma_\varrho(A)$ of A if there exists a sequence $\{x_i\}$ of unit vectors in H such that

$$\|(z_k - A_k)^* x_i\| \to 0 \quad (i \to \infty), \; k = 1, \ldots, n.$$

And a point $z = (z_1, \ldots, z_n)$ will be said to be in the *joint point spectrum* $\sigma_p(A)$ of A if there exists a non-zero vector x such that

$$A_k x = z_k x, \qquad k = 1, \ldots, n.$$

Next we shall describe a definition of joint spectrum. There are several definitions of the joint spectrum, but since the concept of Taylor's joint spectrum seems to be the most natural generalization of the usual spectrum of an operator, we shall describe it (cf. [18], [5]).

Let E^n be the complex exterior algebra on n generators $e_1, \ldots, e_n$ with multiplication denoted by $\wedge$. E^n_p will stand for the space of elements of degree p in E^n $(p = 1, \ldots, n)$. Then we denote by $E^n_p(H)$ the tensor product $H \otimes E^n_p$. Define a map $D_p\colon E^n_p(H) \to E^n_{p-1}(H)$ by

$$D_p(x \otimes e_{j_1} \wedge \ldots \wedge e_{j_p}) = \sum_{i=1}^{p} (-1)^{i+1} A_{j_i} x \otimes e_{j_1} \wedge \ldots \wedge \check{e}_{j_i} \wedge \ldots \wedge e_{j_p},$$

where $\check{}$ means deletion. Also define $D_0 = D_{n+1} = 0$. Then we see that every D_p is a continuous linear map and $D_p \circ D_{p+1} = 0$. Thus, we get the sequence $E(H, A)$

$$E(H, A)\colon 0 \xrightarrow{D_{n+1}} E^n_n(H) \xrightarrow{D_n} E^n_{n-1}(H) \xrightarrow{D_{n-1}} \ldots \xrightarrow{D_2} E^n_1(H) \xrightarrow{D_1} E^n_0(H) \xrightarrow{D_0} 0.$$

Then $A = (A_1, \ldots, A_n)$ is said to be *nonsingular* if $E(H, A)$ is exact, i.e., $\ker D_p = \operatorname{ran} D_{p+1}$ for all p. Taylor's *joint spectrum* $\operatorname{Sp}(A)$ of A is the set of points $z = (z_1, \ldots, z_n)$ of C^n such that $z - A = (z_1 - A_1, \ldots, z_n - A_n)$ is singular (but the spectrum of a single operator T will be denoted by $\sigma(T)$).

Next we shall define the joint inf-spectral set and joint sup-spectral set. Suppose that a closed subset X of C^n includes $\operatorname{Sp}(A)$. Denote the set of all rational functions without singularities on X by $\mathfrak{U}(X)$. For $u(z) = u(z_1, \ldots, z_n) \in \mathfrak{U}(X)$, $u(A) = u(A_1, \ldots, A_n)$ is well defined (see [19]). Then X is said to be a *joint inf-spectral* set for A if

$$\inf\Big\{\sum_{k=1}^{n} \|u_k(A)x\|^2\colon \|x\| = 1\Big\} \geqslant \inf\Big\{\sum_{k=1}^{n} |u_k(z)|^2\colon z \in X\Big\}$$

for all n-tuples $(u_1, \ldots, u_n)$ of elements of $\mathfrak{U}(X)$. And similarly X is said to be a *joint sup-spectral set* for A if

$$\sup\Big\{\sum_{k=1}^{n} \|u_k(A)x\|^2\colon \|x\| = 1\Big\} \leqslant \sup\Big\{\sum_{k=1}^{n} |u_k(z)|^2\colon z \in X\Big\}$$

for all n-tuples $(u_1, \ldots, u_n)$ of elements of $\mathfrak{U}(X)$. In case of $n = 1$, both the joint inf-spectral set and joint sup-spectral set coincide with the usual spectral set (see [12]).

The *joint numerical range* of A is the subset $W(A)$ of C^n such that

$$W(A) = \{((A_1 x, x), \ldots, (A_n x, x))\colon x \in H, \|x\| = 1\}.$$

The *joint operator norm*, *joint spectral radius* and *joint numerical radius* of A, denoted by $\|A\|$, $r(A)$ and $w(A)$ respectively, are defined by

$$\|A\| = \sup\Big\{\Big(\sum_{k=1}^{n} \|A_k x\|^2\Big)^{\frac{1}{2}}\colon \|x\| = 1\Big\},$$

$$r(A) = \sup\Big\{\Big(\sum_{k=1}^{n} |z_k|^2\Big)^{\frac{1}{2}}\colon (z_1, \ldots, z_n) \in \operatorname{Sp}(A)\Big\}$$

and

$$w(A) = \sup\Big\{\Big(\sum_{k=1}^{n} |(A_k x, x)|^2\Big)^{\frac{1}{2}}\colon \|x\| = 1\Big\},$$

respectively.

It is well known that $\sigma_\pi(A)$, $\sigma_\varrho(A)$ and $\operatorname{Sp}(A)$ are non-empty compact sets, and that $\sigma_\pi(A) \cup \sigma_\varrho(A) \subset \operatorname{Sp}(A) \subset \operatorname{co} \overline{W(A)}$ (cf. [21], [20], [18], [3], [9]). Consequently it is evident that $\|A\| \geqslant w(A) \geqslant r(A)$. If, in particular, $A = (A_1, \ldots, A_n)$ is a doubly commuting n-tuple (i.e., $A_k A_j = A_j A_k$ and

$A_k A_j^* = A_j^* A_k$ for $k \neq j$) of hyponormal operators, then $\mathrm{Sp}(A) = \sigma_\varrho(A)$ (cf. [5]). And, moreover, if A is an operator-family of normal operators, then $\mathrm{Sp}(A) = \sigma_\pi(A) = \sigma_\varrho(A)$ and $\mathrm{co}\,\mathrm{Sp}(A) = \overline{W(A)}$ (cf. [7]).

In particular, the spectral mapping theorem also holds for the joint spectrum as follows.

THEOREM A ([19], Theorem 4.8). *Let $A = (A_1, \ldots, A_n) \in \mathfrak{B}^n(H)$ and $u_1, \ldots, u_m \in \mathfrak{U}(\mathrm{Sp}(A))$. Let u: $\mathrm{Sp}(A) \to C^m$ be defined by $u(z) = (u_1(z), \ldots, u_m(z))$ and let $u(A) = (u_1(A), \ldots, u_m(A))$. Then $\mathrm{Sp}(u(A)) = u(\mathrm{Sp}(A))$.*

Next we shall define some classes of operator-families. An operator-family $A = (A_1, \ldots, A_n)$ is called *jointly normaloid* if $\|A\| = w(A)$. An operator-family $A = (A_1, \ldots, A_n)$ is called *jointly transloid* if $A - z = (A_1 - z_1, \ldots, A_n - z_n)$ is jointly normaloid for any point $z = (z_1, \ldots, z_n)$ of C^n. Similarly A are called *jointly spectraloid* and *jointly convexoid* if $w(A) = r(A)$ and $\mathrm{co}\,\mathrm{Sp}(A) = \mathrm{co}\,\overline{W(A)}$, respectively. An operator-family A is said to belong to *joint* (G_1) if

$$\inf\{(\sum_{k=1}^{n} \|(A_k - z_k)x\|^2)^{\frac{1}{2}}: \|x\| = 1\} \geqslant d(z, \mathrm{Sp}(A))$$

for all $z = (z_1, \ldots, z_n) \in C^n$. Especially if for all n-tuples $B = (B_1, \ldots, B_n)$ of linear combinations of $\{A_1, \ldots, A_n\}$

$$\inf\{(\sum_{k=1}^{n} \|(B_k - z_k)x\|^2)^{\frac{1}{2}}: \|x\| = 1\} \geqslant d(z, \mathrm{Sp}(B)),$$

we call A to belong to *complete joint* (G_1). An operator-family A is called *jointly* inf-*spectroid* if the joint spectrum $\mathrm{Sp}(A)$ is a joint inf-spectral set and *jointly* sup-*spectroid* if $\mathrm{Sp}(A)$ is a joint sup-spectral set for A.

It is easily seen that every jointly convexoid operator-family and jointly normaloid operator-family are jointly spectraloid (cf. [2]).

3. A doubly commuting n-tuple of hyponormal operators.

THEOREM 3.1. *Let $A = (A_1, \ldots, A_n)$ be a doubly commuting n-tuple of hyponormal operators and suppose that*

$$r = (r_1, \ldots, r_n) \in \mathrm{Sp}(A^* A) \cup \mathrm{Sp}(AA^*),$$

where $A^* A = (A_1^* A_1, \ldots, A_n^* A_n)$ and $AA^* = (A_1 A_1^*, \ldots, A_n A_n^*)$. Then there exists a point $z = (z_1, \ldots, z_n) \in \mathrm{Sp}(A)$ for which $|z_k| = \sqrt{r_k}$, $k = 1, \ldots, n$.

We shall prepare the following lemmas for the proof of the above theorem.

LEMMA 3.2. ([16], Theorem 7). *Let T be a hyponormal operator and suppose that*

$$r \in \sigma(T^* T) \cup \sigma(TT^*).$$

Then there exists a point $z \in \sigma(T)$ for which $|z| = \sqrt{r}$

LEMMA 3.3 ([1], Theorem 1). *Let $\mathfrak{B}(H)$ be the *-algebra of all operators on H. Then there exist an extension space K of H and a faithful *-representation of $\mathfrak{B}(H)$ into $\mathfrak{B}(K)$: $T \to T^0$ such that $\sigma_\pi(A_1, \ldots, A_n) = \sigma_\pi(A_1^0, \ldots, A_n^0) = \sigma_p(A_1^0, \ldots, A_n^0)$.*

Proof of Theorem 3.1. In what follows, for an operator T we shall denote $(T^* T)^{1/2}$ by $|T|$. We shall prove the theorem by induction.

For $n = 1$, the theorem holds by Lemma 3.2.

For $n \geqslant 2$, assume that it holds for any doubly commuting $(n-1)$-tuple of hyponormal operators. Then we shall prove that the theorem also holds for a doubly commuting n-tuple $A = (A_1, \ldots, A_n)$ of hyponormal operators. We assume that $r = (r_1, \ldots, r_n) \in \mathrm{Sp}(A^* A)$. Since $\mathrm{Sp}(A^* A) = \sigma_\pi(A^* A)$, we have $(\sqrt{r_1}, \ldots, \sqrt{r_n}) \in \sigma_\pi(|A_1|, \ldots, |A_n|)$. Consider the extension space K of H and the faithful *-representation $\mathfrak{B}(H) \to \mathfrak{B}(K)$: $T \to T^0$ in Lemma 3.3. Then $A^0 = (A_1^0, \ldots, A_n^0)$ is a doubly commuting n-tuple of hyponormal operators on K. Let $\mathfrak{M} = \ker(|A_n^0| - \sqrt{r_n})$ $(\neq \{0\})$. Then $\mathfrak{M}$ is a reducing subspace of $A_1^0, \ldots, A_{n-1}^0$, and $(A^0_{1|\mathfrak{M}}, \ldots, A^0_{n-1|\mathfrak{M}})$ is a doubly commuting $(n-1)$-tuple of hyponormal operators. Since

$$\sum_{k=1}^{n} (|A_k| - \sqrt{r_k})^2$$

is not invertible, it follows that

$$\ker(\sum_{k=1}^{n} (|A_k^0| - \sqrt{r_k})^2) = \{\bigcap_{k=1}^{n-1} \ker(|A_k^0| - \sqrt{r_k})\} \cap \mathfrak{M} \neq \{0\}.$$

Hence

$$\sum_{k=1}^{n-1} (|A^0_{k|\mathfrak{M}}| - \sqrt{r_k})^2$$

is not invertible, and so, by the assumption of the induction, there exist $z_1, \ldots, z_{n-1} \in C$ such that $|z_k| = \sqrt{r_k}$, $k = 1, \ldots, n-1$, and

$$\sum_{k=1}^{n-1} (A^0_{k|\mathfrak{M}} - z_k)(A^0_{k|\mathfrak{M}} - z_k)^*$$

is not invertible. Therefore

$$\sum_{k=1}^{n-1} (A_k^0 - z_k)(A_k^0 - z_k)^* + (|A_n^0| - \sqrt{r_n})^2$$

is not invertible, and so

$$\ker\{\sum_{k=1}^{n-1} (A_k^0 - z_k)(A_k^0 - z_k)^* + (|A_n^0| - \sqrt{r_n})^2\} \neq \{0\}.$$

Set

$$\mathfrak{N} = \ker\Big(\sum_{k=1}^{n-1}(A_k^0 - z_k)(A_k^0 - z_k)^*\Big) = \bigcap_{k=1}^{n-1}\ker(A_k^0 - z_k)^*.$$

Then $\mathfrak{N}$ reduces A_n^0. And since $\mathfrak{N}\cap\mathfrak{M}\neq\{0\}$, $\sqrt{r_n}\in\sigma(|A_{n|\mathfrak{N}}^0|)$. Hence by Lemma 3.2 there exists a point $z_n\in\sigma(A_{n|\mathfrak{N}}^0)$ such that $|z_n| = \sqrt{r_n}$. Since $A_{n|\mathfrak{N}}^0$ is hyponormal, $(A_{n|\mathfrak{N}}^0 - z_n)(A_{n|\mathfrak{N}}^0 - z_n)^*$ is not invertible. Hence

$$\sum_{k=1}^{n}(A_k^0 - z_k)(A_k^0 - z_k)^*$$

is not invertible, and so

$$\sum_{k=1}^{n}(A_k - z_k)(A_k - z_k)^*$$

is not invertible. Thus this point $z = (z_1, \ldots, z_n)$ belongs to $\sigma_\varrho(A) = \mathrm{Sp}(A)$ and satisfies $|z_k| = \sqrt{r_k}$, $k = 1, \ldots, n$. In case of $r\in\mathrm{Sp}(AA^*)$, the proof is similar. Consequently, the proof is complete.

In consequence of Theorem 3.1 we have the following results.

Theorem 3.4. *Let $A = (A_1, \ldots, A_n)$ be a doubly commuting n-tuple of hyponormal operators. Then there exists a $z = (z_1, \ldots, z_n)\in\mathrm{Sp}(A)$ such that $|z| = \|A\|$. That is, A is jointly normaloid.*

Proof. Since A^*A is an operator-family of Hermitian operators, we have $\operatorname{co}\mathrm{Sp}(A^*A) = \overline{W(A^*A)}$. And observing that $\|A\|^2 = \sup\{\sum_{k=1}^{n}(A_k^*A_k x, x)\colon \|x\| = 1\}$, we see that there exists a point $(r_1, \ldots, r_n)\in\mathrm{Sp}(A^*A)$ such that $\sum_{k=1}^{n} r_k = \|A\|^2$. Hence by Theorem 3.1 there exists a point $z = (z_1, \ldots, z_n)\in\mathrm{Sp}(A)$ such that $|z| = \|A\|$. Thus the proof is complete.

Theorem 3.5. *If $A = (A_1, \ldots, A_n)$ is a doubly commuting n-tuple of hyponormal operators, then it follows that*

$$\inf\Big\{\Big(\sum_{k=1}^{n}\|(A_k - z_k)^* x\|^2\Big)^{\frac{1}{2}}\colon \|x\| = 1\Big\} = d\big(z, \mathrm{Sp}(A)\big),$$

for every $z = (z_1, \ldots, z_n)\in \boldsymbol{C}^n$.

Proof. Since $A - z = (A_1 - z_1, \ldots, A_n - z_n)$ is a doubly commuting n-tuple of hyponormal operators, in a similar way as in the proof of Theorem 3.4 there exists a point $\lambda = (\lambda_1, \ldots, \lambda_n)\in\mathrm{Sp}(A - z)$ such that

$$\inf\Big\{\sum_{k=1}^{n}\|(A_k - z_k)^* x\|^2\colon \|x\| = 1\Big\} = |\lambda|^2.$$

Since $\mathrm{Sp}(A - z) = \mathrm{Sp}(A) - z$, it follows that

$$\inf\Big\{\sum_{k=1}^{n}\|(A_k - z_k)^* x\|^2\colon \|x\| = 1\Big\}\geqslant d\big(z, \mathrm{Sp}(A)\big)^2.$$

Conversely, let $\mu = (\mu_1, \ldots, \mu_n)\in\mathrm{Sp}(A)$ be such that $d\big(z, \mathrm{Sp}(A)\big) = |z - \mu|$. Then it is clear that

$$\Big(\sum_{k=1}^{n}\|(A_k - z_k)^* x\|^2\Big)^{\frac{1}{2}}\leqslant\Big(\sum_{k=1}^{n}\|(A_k - \mu_k)^* x\|^2\Big)^{\frac{1}{2}} + d\big(z, \mathrm{Sp}(A)\big)$$

for every unit vector x. So we obtain the opposite inequality, and so the proof is complete.

4. The inclusion relations for classes of operator-families. First we shall show the following theorem.

Theorem 4.1. *Let $A = (A_1, \ldots, A_n)$ be a normal operator-family. Then $\mathrm{Sp}(A)$ is a joint* inf*-spectral set and joint* sup*-spectral set for A.*

Proof. Since A is a commuting n-tuple of normal operators, it is well known that there exists a suitable measure space $(\mathfrak{X}, \mu)$ and an n-tuple $\varphi = (\varphi_1, \ldots, \varphi_n)$ of functions in $L^\infty(\mathfrak{X}, \mu)$ such that each A_k is unitarily equivalent to multiplication by φ_k on $L^2(\mathfrak{X}, \mu)$. That is,

$$A_k f = \varphi_k f \quad \text{for all } f\in L^2(\mathfrak{X}, \mu),\ k = 1, \ldots, n.$$

Then the joint spectrum $\mathrm{Sp}(A)$ of A is the *joint essential range* of $\varphi = (\varphi_1, \ldots, \varphi_n)$ (cf. [8], Theorem 5.2), that is, the set of all points $z = (z_1, \ldots, z_n)$ in C^n such that for every $\varepsilon > 0$

$$\mu\Big(\Big\{t\in\mathfrak{X}\colon \sum_{k=1}^{n}|\varphi_k(t) - z_k| < \varepsilon\Big\}\Big) > 0.$$

Here we claim that $\mu\big(\{t\in\mathfrak{X}\colon (\varphi_1(t), \ldots, \varphi_n(t))\notin\mathrm{Sp}(A)\}\big) = 0$. Because, for each $z = (z_1, \ldots, z_n)\notin\mathrm{Sp}(A)$, there exists an open sphere U_z about z such that

$$\mu\big(\{t\in\mathfrak{X}\colon (\varphi_1(t), \ldots, \varphi_n(t))\in U_z\}\big) = 0.$$

Hence by Lindelöf's theorem it follows that

$$\mu\big(\{t\in\mathfrak{X}\colon (\varphi_1(t), \ldots, \varphi_n(t))\notin\mathrm{Sp}(A)\}\big) = 0.$$

Therefore, for each n-tuple $u = (u_1, \ldots, u_n)$ of elements in $\mathfrak{U}(\mathrm{Sp}(A))$ and $f\in L^2(\mathfrak{X}, \mu)$, $\|f\| = 1$, we have

$$\begin{aligned}\sum_{k=1}^{n}\|u_k(A)f\|^2 &= \sum_{k=1}^{n}\int_{\mathfrak{X}}|u_k(\varphi_1(t), \ldots, \varphi_n(t))|^2|f(t)|^2\,d\mu(t)\\ &= \int_{\{t\in\mathfrak{X}:(\varphi_1(t),\ldots,\varphi_n(t))\in\mathrm{Sp}(A)\}}\sum_{k=1}^{n}|u_k(\varphi_1(t), \ldots, \varphi_n(t))|^2|f(t)|^2\,d\mu(t).\end{aligned}$$

And so it follows that

$$\inf\Big\{\sum_{k=1}^{n}\|u_k(A)f\|^2:\ \|f\|=1\Big\}\geqslant\inf\Big\{\sum_{k=1}^{n}|u_k(z)|^2:\ z\in\mathrm{Sp}(A)\Big\}$$

and

$$\sup\Big\{\sum_{k=1}^{n}\|u_k(A)f\|^2:\ \|f\|=1\Big\}\leqslant\sup\Big\{\sum_{k=1}^{n}|u_k(z)|^2:\ z\in\mathrm{Sp}(A)\Big\}.$$

Thus the proof is complete.

Next we cite the following theorem due to R. E. Curto.

THEOREM B ([6], Theorem 1). *Let $A=(A_1,\ldots,A_n)$ be a doubly commuting n-tuple of subnormal operators on H with minimal normal extension $N=(N_1,\ldots,N_n)$ on $K\supset H$. Then* $\mathrm{Sp}(A)\supset\mathrm{Sp}(N)$.

Note that any doubly commuting n-tuple of subnormal operators has a commuting normal extension. The following theorem follows from Theorem 4.1 and Theorem B.

THEOREM 4.2. *A doubly commuting n-tuple of subnormal operators is jointly* inf-*spectroid and jointly* sup-*spectroid.*

Proof. Let $A=(A_1,\ldots,A_n)$ be a doubly commuting n-tuple of subnormal operators on H and $N=(N_1,\ldots,N_n)$ be its minimal commuting normal extension acting on $K\supset H$. Then for any n-tuple $(u_1,\ldots,u_n)$ of elements of $\mathfrak{U}(\mathrm{Sp}(A))$, we have

$$\begin{aligned}\inf\Big\{\sum_{k=1}^{n}\|u_k(A)x\|^2:\ x\in H,\ \|x\|=1\Big\}&\geqslant\inf\Big\{\sum_{k=1}^{n}\|u_k(N)x\|^2:\ x\in K,\ \|x\|=1\Big\}\\&\geqslant\inf\Big\{\sum_{k=1}^{n}|u_k(z)|^2:\ z\in\mathrm{Sp}(N)\Big\}\\&\geqslant\inf\Big\{\sum_{k=1}^{n}|u_k(z)|^2:\ z\in\mathrm{Sp}(A)\Big\}.\end{aligned}$$

Hence A is jointly inf-spectroid. Similarly we can show A is jointly sup-spectroid.

Next we shall prepare a lemma to study jointly convexoid operator-families. In what follows we shall abbreviate $((A_1x,x),\ldots,(A_nx,x))$ and $\big(\sum_{k=1}^{n}\|A_kx\|^2\big)^{1/2}$ to (Ax,x) and $\|Ax\|$, respectively.

LEMMA 4.3. *Let X be a closed convex set of the n-dimensional unitary space $\boldsymbol{C}^n$ such that $p_k(X)$ is contained in the left closed half-plane of $\boldsymbol{C}$, where p_k is the projection of $\boldsymbol{C}^n$ onto the kth coordinate space. Then, if*

$$\|(A-\lambda)x\|^2\geqslant d(\lambda,X)\cdot|((A-\lambda)x,x)|$$

for all $\lambda\in\boldsymbol{C}^n$ and $x\in H$, $p_k(\overline{W(A)})$ is also contained in the left closed half-plane of $\boldsymbol{C}$.

Proof. Let $\lambda_k>0$. Then from the assumption

$$\begin{aligned}\|(A-\lambda)x\|^2&\geqslant\inf\{|z_k-\lambda_k|:\ z_k\in p_k(X)\}\cdot|((A-\lambda)x,x)|\\&\geqslant\lambda_k\cdot\Big(\sum_{i=1}^{n}|(A_ix,x)-\lambda_i|^2\Big)^{\frac{1}{2}}\end{aligned}$$

for every vector x. Squaring both sides of the above inequality, dividing by λ_k^3 and letting $\lambda_k\to\infty$, we obtain

$$-4\cdot\mathrm{Re}(A_kx,x)\geqslant-2\cdot\mathrm{Re}(A_kx,x).$$

This implies $\mathrm{Re}(A_kx,x)\leqslant 0$, and the proof is complete.

THEOREM 4.4. *If, for all n-tuples $B=(B_1,\ldots,B_n)$ of linear combinations of an operator-family $A=(A_1,\ldots,A_n)$,*

$$\|(B-\lambda)x\|^2\geqslant d(\lambda,\mathrm{co}\,\mathrm{Sp}(B))\cdot|((B-\lambda)x,x)|$$

for all $\lambda\in\boldsymbol{C}^n$ and $x\in H$, then A is jointly convexoid.

Proof. Since $\mathrm{co}\,\mathrm{Sp}(A)\subset\mathrm{co}\,\overline{W(A)}$, we have only to show the opposite inclusion. That is, we have only to show that a closed half space which contains $\mathrm{Sp}(A)$, also contains $\overline{W(A)}$, since a closed convex set is the intersection of closed half spaces that contain it. Let π be a hyperplane of support to $\mathrm{co}\,\mathrm{Sp}(A)$ and $\gamma=(\gamma_1,\ldots,\gamma_n)$ a point at which π touches $\mathrm{co}\,\mathrm{Sp}(A)$. π decomposes the entire space $\boldsymbol{C}^n$ into two half spaces. Let S be the closed half space which contains $\mathrm{Sp}(A)$, among the two. Here we may assume $\gamma_k=0$, $k=1,\ldots,n$, because the translation: $z=(z_1,\ldots,z_n)\to z-\gamma=(z_1-\gamma_1,\ldots,z_n-\gamma_n)$ takes γ, $\mathrm{Sp}(B)$ and $W(B)$ into $(0,\ldots,0)$, $\mathrm{Sp}(B-\gamma)$ and $W(B-\gamma)$, respectively, and the following inequality holds:

$$\|\{(B-\gamma)-(\lambda-\gamma)\}x\|^2\geqslant d(\lambda-\gamma,\mathrm{co}\,\mathrm{Sp}(B-\gamma))\cdot|(\{(B-\gamma)-(\lambda-\gamma)\}x,x)|.$$

Then there exists a regular matrix M:

$$M=\begin{bmatrix}\alpha_{11}&\ldots&\alpha_{1n}\\ \cdot&\ldots&\cdot\\ \cdot&\ldots&\cdot\\ \cdot&\ldots&\cdot\\ \alpha_{n1}&\ldots&\alpha_{nn}\end{bmatrix}$$

such that $p_1(M\pi)$ and $p_1(MS)$ are the y-axis and left closed half-plane of the first coordinate space, respectively. Then $p_1(M\,\mathrm{Sp}(A))=p_1(\mathrm{Sp}(D_1,\ldots,D_n))$ is contained in the left closed half-plane of the first coordinate space, where D_k is $\alpha_{k1}A_1+\alpha_{k2}A_2+\ldots+\alpha_{kn}A_n$, $k=1,\ldots,n$. And then, from Lemma 4.3

and the assumption, $p_1(\overline{W(D_1, \ldots, D_n)})$ is contained in the left closed half-plane of it. Therefore, since $M^{-1}W(D_1, \ldots, D_n)$ is $W(A)$, $\mathrm{co}\,\overline{W(A)}$ is contained in S. Hence $\mathrm{co}\,\mathrm{Sp}(A) \supset \mathrm{co}\,\overline{W(A)}$ and the proof is complete.

COROLLARY 4.5. *For an operator-family $A = (A_1, \ldots, A_n)$, the following five conditions are mutually equivalent:*

(i) *A is jointly convexoid,*

(ii) *$d(\lambda, \mathrm{co}\,\mathrm{Sp}(B)) \leqslant \inf\{|((B-\lambda)x, x)|: \|x\| = 1\}$ for all $\lambda \in C^n$ and n-tuples $B = (B_1, \ldots, B_n)$ of linear combinations of $\{A_1, \ldots, A_n\}$,*

(iii) *$d(\lambda, \mathrm{co}\,\mathrm{Sp}(B)) \leqslant \inf\{\|(B-\lambda)x\|: \|x\| = 1\}$ for all $\lambda \in C^n$ and n-tuples $B = (B_1, \ldots, B_n)$ of linear combinations of $\{A_1, \ldots, A_n\}$,*

(iv) *$d(\lambda, \mathrm{co}\,\mathrm{Sp}(B)) \cdot |((B-\lambda)x, x)| \leqslant \|(B-\lambda)x\|^2$ for all $\lambda \in C^n$, $x \in X$ and n-tuples $B = (B_1, \ldots, B_n)$ of linear combinations of $\{A_1, \ldots, A_n\}$,*

(v) *$A-\lambda$ are jointly spectraloid for all points $\lambda = (\lambda_1, \ldots, \lambda_n)$ in C^n.*

Proof. Obviously, (i) implies (ii), (ii) implies (iii) and (iii) implies (iv). And (iv) implies (i) from Theorem 4.4. Thus we have only to show the equivalence between (i) and (v).

Suppose that $w(A-\lambda) = r(A-\lambda)$ for every point λ in C^n. Since a closed convex set of C^n is the intersection of all spheres that contain it, we have

$$\begin{aligned}\mathrm{co}\,\mathrm{Sp}(A) &= \bigcap_{\lambda} \{(\mu_1, \ldots, \mu_n): \Big(\sum_{k=1}^{n} |\mu_k - \lambda_k|^2\Big)^{\frac{1}{2}} \leqslant r(A-\lambda)\} \\ &= \bigcap_{\lambda} \{(\mu_1, \ldots, \mu_n): \Big(\sum_{k=1}^{n} |\mu_k - \lambda_k|^2\Big)^{\frac{1}{2}} \leqslant w(A-\lambda)\} \\ &= \mathrm{co}\,\overline{W(A)}.\end{aligned}$$

Hence (v) implies (i).

Conversely, if A is jointly convexoid, then $w(A-\lambda) = r(A-\lambda)$, for every point $\lambda \in C^n$ since

$$\begin{aligned}\mathrm{co}\,\overline{W(A-\lambda)} &= \mathrm{co}\,\overline{W(A)} - \lambda = \mathrm{co}\,\mathrm{Sp}(A) - \lambda \\ &= \mathrm{co}\,\mathrm{Sp}(A-\lambda).\end{aligned}$$

Hence (i) implies (v).

5. Topological properties for classes of operator-families. First we shall show that the set of all normal operator-families is a very thin subset of complete joint (G_1). For this purpose we prepare the following lemmas.

LEMMA 5.1. *If X is a non-empty compact subset of C^n, then there exists a commuting n-tuple $N = (N_1, \ldots, N_n)$ of normal operators such that $\mathrm{Sp}(N) = X$.*

Proof. Let $\{\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})\}_{i=1}^{\infty}$ be a sequence of points in C^n such that $\{\alpha_i\}$ is dense in X. Set

$$N_k = \begin{bmatrix} \alpha_{1k} & & & 0 \\ & \alpha_{2k} & & \\ & & \alpha_{3k} & \\ 0 & & & \ddots \end{bmatrix}, \quad k = 1, \ldots, n.$$

Then $N = (N_1, \ldots, N_n)$ is a commuting n-tuple of normal operators, and $\mathrm{Sp}(N)$ is the closure of $\{\alpha_i\}$ and so $\mathrm{Sp}(N) = X$.

LEMMA 5.2. *If $A = (A_1, \ldots, A_n)$ is any commuting n-tuple of operators on H, then there exist a Hilbert space K and a commuting n-tuple $N = (N_1, \ldots, N_n)$ of normal operators on K such that $A \oplus N = (A_1 \oplus N_1, \ldots, A_n \oplus N_n)$ belongs to joint (G_1).*

Proof. There exist a Hilbert space K and a commuting n-tuple of normal operators $N = (N_1, \ldots, N_n)$ on K such that $\mathrm{Sp}(N) = \mathrm{co}\,\overline{W(A)}$, by the above lemma. Then, since $\mathrm{Sp}(A \oplus N) = \mathrm{Sp}(A) \cup \mathrm{Sp}(N) = \mathrm{Sp}(N)$, it follows that for every $z \in C^n$

$$\begin{aligned}\inf\{\|(A \oplus N - z)x\|: \|x\| = 1\} &= \inf\{\|(A-z)x_1 \oplus (N-z)x_2\|: \|x_1\|^2 + \|x_2\|^2 = 1\} \\ &\geqslant \min\{\inf_{\|x\|=1} \|(A-z)x\|, \inf_{\|x\|=1} \|(N-z)x\|\} \\ &\geqslant \min\{d(z, \overline{W(A)}), d(z, \mathrm{Sp}(N))\} \\ &\geqslant d(z, \mathrm{Sp}(A \oplus N)).\end{aligned}$$

Thus $A \oplus N$ belongs to joint (G_1) and the proof is complete.

Remark. The operator-family $A \oplus N$ constructed in the proof of Lemma 5.2 belongs not only to joint (G_1) but also to complete joint (G_1).

THEOREM 5.3. *The set $\mathfrak{N}$ of all commuting n-tuples of normal operators is nowhere dense in complete joint (G_1) when $\dim H = \infty$.*

Proof. Since $\mathfrak{N}$ is closed, to show that $\mathfrak{N}$ is a nowhere dense subset of complete joint (G_1), it suffices to show that $\mathfrak{N}$ has empty interior in complete joint (G_1). Let $A = (A_1, \ldots, A_n) \in \mathfrak{N}$.

Here we cite the notations used in the proof of Theorem 4.1. Let ε be an arbitrary positive number. For each point $z = (z_1, \ldots, z_n) \in \mathrm{Sp}(A)$ we denote the open ε-sphere about z by $U(z, \varepsilon)$. Since $\{U(z, \varepsilon)\}_{z \in \mathrm{Sp}(A)}$ is an open cover of $\mathrm{Sp}(A)$ and $\mathrm{Sp}(A)$ is compact, there exists a finite subcover of $\mathrm{Sp}(A)$. Consequently, since $\dim L^2(\mathfrak{X}, \mu) = \infty$, there exists a subset

$\mathfrak{X}_0 = \{t\colon (\varphi_1(t), \ldots, \varphi_n(t)) \in U(z^0, \varepsilon)\}$ of $\mathfrak{X}$ such that $\dim L^2(\mathfrak{X}_0, \mu) = \infty$. Then set

$$\psi_k(t) = \begin{cases} \varphi_k(t) & \text{if } t \in \mathfrak{X} - \mathfrak{X}_0, \\ z_k^0 & \text{if } t \in \mathfrak{X}_0, \end{cases}$$

and let T_k be the multiplication operator induced by ψ_k, $k = 1, \ldots, n$. And then set

$$S_1 = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix}, \quad S_k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad k = 2, \ldots, n.$$

Since $S = (S_1, \ldots, S_n)$ is commuting, in the same way as in Lemma 5.2 we can construct an n-tuple $N = (N_1, \ldots, N_n)$ of normal operators such that $S \oplus N = (S_1 \oplus N_1, \ldots, S_n \oplus N_n)$ acts on $L^2(\mathfrak{X}_0, \mu)$ and belongs to complete joint (G_1). Let $Z = (Z_1, \ldots, Z_n)$ be the n-tuple of the zero operator on $L^2(\mathfrak{X} - \mathfrak{X}_0, \mu)$ and $B = T + (Z \oplus S \oplus N)$. Then B is a non-normal operator-family belonging to complete joint (G_1). And since

$$\mathrm{Sp}(S \oplus N) = \mathrm{co}\, \overline{W(S)} \quad \text{and} \quad \|S \oplus N\| < \varepsilon,$$

we have

$$\|A - B\| \leqslant \|A - T\| + \|T - B\| \leqslant \varepsilon + \varepsilon.$$

Therefore, since $\varepsilon > 0$ is arbitrary, A is not contained in the interior of $\mathfrak{N}$ in complete joint (G_1). Hence the interior of $\mathfrak{N}$ in complete joint (G_1) is empty and the proof is complete.

Next we shall study the continuity of the joint spectrum. In general, the joint spectrum $\mathrm{Sp}(A)$ is not a continuous function of A in $\mathfrak{B}^n(H)$, but $\mathrm{Sp}(A)$ is upper semicontinuous (cf. [17] or it follows from Corollary 3.5 in Curto [5]). Moreover, if we restrict A to normal operator-families, then $\mathrm{Sp}(A)$ is continuous (cf. Theorem 3 in [4]). Furthermore, Janas [14, Theorem 1] showed that the rationally convex hull of $\mathrm{Sp}(A)$ is continuous if we restrict A to doubly commuting hyponormal operator-families.

The following theorem is an improvement on these results.

THEOREM 5.4. *If $\{A_i = (A_{i1}, \ldots, A_{in})\}$ is a sequence of doubly commuting operator-families of hyponormal operators approaching an operator-family $A = (A_1, \ldots, A_n)$ in joint operator norm, then $\mathrm{Sp}(A_i) \to \mathrm{Sp}(A)$ as $i \to \infty$.*

The proof of this theorem is directly deduced from the following:

LEMMA 5.5. *Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be two doubly commuting n-tuples of hyponormal operators. Then*

$$\mathrm{Sp}(B) \subset \mathrm{Sp}(A) + (\|A^* - B^*\|).$$

Proof. Let $z = (z_1, \ldots, z_n) \in C^n$ be such that $d(z, \mathrm{Sp}(A)) > \|A^* - B^*\|$. Then, by Theorem 3.5, it follows that

$$\begin{aligned} d(z, \mathrm{Sp}(A)) &\leqslant \Big(\sum_{k=1}^n \|(A_k - z_k)^* x\|^2\Big)^{\frac{1}{2}} \\ &\leqslant \Big(\sum_{k=1}^n \|(A_k - B_k)^* x\|^2\Big)^{\frac{1}{2}} + \Big(\sum_{k=1}^n \|(B_k - z_k)^* x\|^2\Big)^{\frac{1}{2}} \\ &\leqslant \|A^* - B^*\| + \Big(\sum_{k=1}^n \|(B_k - z_k)^* x\|^2\Big)^{\frac{1}{2}} \end{aligned}$$

for every unit vector x. Therefore,

$$0 < d(z, \mathrm{Sp}(A)) - \|A^* - B^*\| \leqslant \inf\Big\{\Big(\sum_{k=1}^n \|(B_k - z_k)^* x\|^2\Big)^{\frac{1}{2}} \colon \|x\| = 1\Big\}.$$

Hence $z = (z_1, \ldots, z_n) \notin \mathrm{Sp}(B) = \sigma_\varrho(B)$ and the proof is complete.

Finally we shall investigate whether each class of operator-families dealt with above is closed or not in $\mathfrak{B}^n(H)$. We note that if a sequence $\{A_i = (A_{i1}, \ldots, A_{in})\}_{i=1}^\infty$ of operator-families approaches $A = (A_1, \ldots, A_n)$ in joint operator norm, then A_i^*, $\|A_i\|$ and $W(A_i)$ also approach A^*, $\|A\|$ and $W(A)$, respectively. Moreover, if $B_i = (B_{i1}, \ldots, B_{in}) \to B = (B_1, \ldots, B_n)$, then $A_i B_i \to AB$, where AB means $(A_1 B_1, \ldots, A_n B_n)$. From these facts we have the following:

PROPOSITION 5.6. *The sets of normal, doubly commuting quasinormal, doubly commuting hyponormal, jointly normaloid and jointly spectraloid operator-families are all closed in $\mathfrak{B}^n(H)$.*

PROPOSITION 5.7. *The set of all operator-families belonging to joint (G_1) is closed in $\mathfrak{B}^n(H)$.*

Proof. Let $\{A_i = (A_{i1}, \ldots, A_{in})\}$ be a sequence of operator-families belonging to joint (G_1) and $A_i \to A = (A_1, \ldots, A_n)$, $i \to \infty$. Then by the upper semicontinuity of the joint spectrum

$$\liminf_i d(z, \mathrm{Sp}(A_i)) \geqslant d(z, \mathrm{Sp}(A)).$$

Since $\|(A_i - z)x\| \to \|(A - z)x\|$, $i \to \infty$, and by the assumption $\|(A_i - z)x\| \geqslant d(z, \mathrm{Sp}(A_i))$ for every unit vector x and $i = 1, 2, \ldots$, it follows that

$$\|(A - z)x\| = \lim_i \|(A_i - z)x\| \geqslant \liminf_i d(z, \mathrm{Sp}(A_i)) \geqslant d(z, \mathrm{Sp}(A)).$$

Hence A belongs to joint (G_1) and the proof is complete.

The set of all operator-families belonging to complete joint (G_1) is also closed; this is proved in the same way as the proposition above.

PROPOSITION 5.8. *The set of all jointly convexoid operator-families is closed in* $\mathfrak{B}^n(H)$.

Proof. Let $\{A_i = (A_{i1}, \ldots, A_{in})\}$ be a sequence of jointly convexoid operator-families and $A_i \to A = (A_1, \ldots, A_n)$ as $i \to \infty$. Then $\operatorname{co} \overline{W(A_i)} \to \operatorname{co} \overline{W(A)}$ as $i \to \infty$. Let $\varepsilon > 0$. Then by the upper semicontinuity of the joint spectrum, there exists a positive integer N such that

$$\operatorname{co} \operatorname{Sp}(A_i) \subset \operatorname{co} \operatorname{Sp}(A) + (\varepsilon)$$

for all $i \geqslant N$. Hence

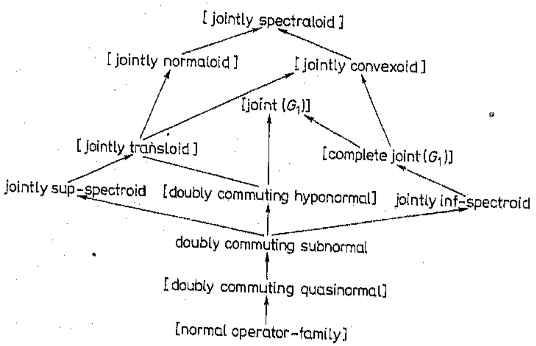
$$\operatorname{co} \overline{W(A)} = \lim_i \operatorname{co} \overline{W(A_i)} = \lim_i \operatorname{co} \operatorname{Sp}(A_i) \subset \operatorname{co} \operatorname{Sp}(A) + (\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, $\operatorname{co} \overline{W(A)} \subset \operatorname{co} \operatorname{Sp}(A)$. On the other hand, in general, $\operatorname{co} \operatorname{Sp}(A) \subset \operatorname{co} \overline{W(A)}$. Hence A is jointly convexoid and the proof is complete.

From this proposition we can see that the convex hull of the joint spectrum $\operatorname{co} \operatorname{Sp}(A)$ is continuous if we restrict A to the convexoid operator-families.

COROLLARY 5.8. *If $\{A_i = (A_{i1}, \ldots, A_{in})\}$ is a sequence of jointly convexoid operator-families approaching an operator-family $A = (A_1, \ldots, A_n)$, then $\operatorname{co} \operatorname{Sp}(A_i)$ approaches $\operatorname{co} \operatorname{Sp}(A)$.*

Setting the foregoing results in order we get the following chart on the classes of operator-families:



where the symbol $\to$ indicates the inclusion relation and the symbol $[\mathfrak{A}]$ denotes that the class $\mathfrak{A}$ of operator-families is closed in $\mathfrak{B}^n(H)$.

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