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SOME CLASSES OF IDEMPOTENT FUNCTIONS AND THEIR COMPOSITIONS

BY

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1. Introduction. In [9] Sierpiński showed that, for any set A, every function $f: A^n \to A$ is the composition of binary functions on A. If $f: A^n \to A$ satisfies $f(a, \ldots, a) = a$ for all $a \in A$, then f is called *idempotent*. In [8] it was shown that if $|A| \geqslant 3$, then every idempotent function on A is the compositon of binary idempotent functions on A, while if $|A| = 2$, then every idempotent function on A is the composition of ternary idempotent functions on A but not of binary idempotent functions (see also [5]). In this paper * we will consider some classes of idempotent functions and prove results similar to those above-mentioned. In the case where A is finite we will use the theory of semi-primal algebras; this theory permits us to determine good bounds on the minimal number of functions of minimal arity needed to generate all functions of a given class. This will partially settle a question raised in [5].

Definition. Let $f: A^n \to A$ and let $K \geqslant 0$; f is *K-idempotent* if, for any $a_1, \ldots, a_n \in A$, $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ whenever $|\{a_1, \ldots, a_n\}| \leqslant K$; f is a *quasi-projection* if, for any $a_1, \ldots, a_n \in A$, $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$.

Thus a 1-idempotent function is idempotent and every function on A is 0-idempotent; a K-idempotent function on a $(K+1)$-element set is a quasi-projection, and an n-ary K-idempotent function is a quasi-projection if $n \leqslant K$. Note that, for fixed K, the set of all K-idempotent functions on A is closed under composition, and, similarly, the set of all quasi-projections on A is closed under composition.

Let F_K be the set of all K-idempotent functions on A, and F the set of all quasi-projections on A. Let $\mathfrak{A}_K = \langle A; F_K \rangle$ and $\mathfrak{A} = \langle A; F \rangle$. Note that every subset of A is a subalgebra of $\mathfrak{A}$ while the proper subalgebras of $\mathfrak{A}_K$ are all subsets of cardinality at most K.

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2. The finite case.

Definition. Given an algebra $\mathfrak{B}$, $\mathscr{S}(\mathfrak{B})$ is the set of subalgebras of $\mathfrak{B}$ and $\mathscr{I}(\mathfrak{B})$ is the set of isomorphisms between not necessarily distinct non-trivial (i.e. with more than one element) subalgebras of $\mathfrak{B}$. We say $f\colon B^n \to B$ *preserves* $\mathscr{S}(\mathfrak{B})$ if, for any $a_1, \ldots, a_n \in B$, $f(a_1, \ldots, a_n)$ lies in the subalgebra generated by $\{a_1, \ldots, a_n\}$.

Definition. Let $\mathfrak{B}$ be a finite non-trivial algebra; $\mathfrak{B}$ is *semi-primal* if every function on B which preserves $\mathscr{S}(\mathfrak{B})$ is a polynomial of $\mathfrak{B}$.

Semi-primal algebras were introduced by Foster and Pixley (cf. [1] and [2]). In sections 2-4 we will assume that $2 \leqslant |A| < \aleph_0$; in this case it is clear that $\mathfrak{A}_K$ and $\mathfrak{A}$ are semi-primal.

Definition. The function $t(x, y, z)$, defined by $t(x, x, z) = z$ and, otherwise, by $t(x, y, z) = x$, is the *ternary discriminator function.*

This function was introduced by Pixley in [7].

The following theorem is an immediate consequence of theorem 3.1 of [2] and theorems 3.1 and 3.2 of [7]:

THEOREM 2.1. *A finite non-trivial algebra $\mathfrak{B}$ is semi-primal iff*

(1) *$t(x, y, z)$ is a polynomial of $\mathfrak{B}$,*

(2) *φ is the identity map on its domain for $\varphi \in \mathscr{I}(\mathfrak{B})$.*

Let $\mathfrak{A}'_K = \langle A; G_K\rangle$ and $\mathfrak{A}' = \langle A; G\rangle$. Suppose $\mathfrak{A}'_K$ is semi-primal, and the proper subalgebras of $\mathfrak{A}'_K$ are the subsets of cardinality at most K, and suppose $\mathfrak{A}'$ is semi-primal and every subset is a subalgebra of $\mathfrak{A}'$. Then, it is clear that, for $k > 0$, every K-idempotent function on A can be obtained by the composition of functions in G_K while every quasi-projection on A is a composition of functions from G. In this section we are interested in the minimal arity functions in G_K and G can have; clearly, G_K must consist of K-idempotent functions and G of quasi-projections.

First, consider the case of K-idempotent functions for $K \geqslant 2$. Recall that the results mentioned in the introduction deal with the cases $K = 0$ and $K = 1$. We may assume $|A| \geqslant K+2$ since, otherwise, we are dealing with the case of quasi-projections. We must have at least one at least $(K+1)$-ary K-idempotent function, else G_K consists of quasi-projections. Since $K \geqslant 2$, we may thus assume that $t(x, y, z)$ belongs to G_K.

To guarantee that the proper subalgebras of $\mathfrak{A}'_K$ have cardinality at most K, let $A = \{1, 2, \ldots, n\}$ and define $f_K\colon A^{K+1} \to A$ by

$$(*)\qquad f_K(a_1, \ldots, a_{K+1}) = \begin{cases} a_1 & \text{if } |\{a_1, \ldots, a_{K+1}\}| \leqslant K, \\ \max\{a_1, \ldots, a_{K+1}\} + 1 \pmod{n} & \\ & \text{if } \max\{a_1, \ldots, a_{K+1}\} - \\ & - \min\{a_1, \ldots, a_{K+1}\} = K, \\ \min(A - \{a_1, \ldots, a_{K+1}\}) & \text{otherwise.} \end{cases}$$

If $f_K \in G_K$, then the proper subalgebras of $\mathfrak{A}'_K$ will have cardinality at most K.

Finally, we need to guarantee that condition 2 of theorem 2.1 holds. Note, however, that we need only show that condition 2 of theorem 2.1 holds for the two-element subalgebras. Thus, for each $a, a_1, a_2 \in A$, define $\gamma_a(a_1, a_2)$ by

$$\gamma_a(a_1, a_2) = \begin{cases} a & \text{if } a_1 = a \text{ or } a_2 = a, \\ a_1 & \text{otherwise.} \end{cases}$$

Let $\gamma_a \in G_K$ for each $a \in A$. Let $\varphi \in \mathscr{I}(\mathfrak{A}'_K)$ have domain $\{a, b\}$ with $a \neq b$. Then

$$\varphi(a) = \varphi\big(\gamma_a(b, a)\big) = \gamma_a\big(\varphi(b), \varphi(a)\big).$$

By definition of γ_a, either $\gamma_a\big(\varphi(b), \varphi(a)\big) = \varphi(b)$ or $\gamma_a\big(\varphi(b), \varphi(a)\big) = a$. In the first case, $\varphi(a) = \varphi(b)$ which is impossible since $a \neq b$ and φ is 1-1. Thus $\varphi(a) = a$ and, similarly, $\varphi(b) = b$. Since $\{a, b\}$ and φ were arbitrary, condition 2 holds for $\mathfrak{A}'_K$. Thus the following theorem has been proved:

THEOREM 2.2. *Let $K \geqslant 2$ and $K+2 \leqslant |A| < \aleph_0$. Then every K-idempotent function on A can be obtained by the composition of $(K+1)$-ary K-idempotent functions on A.*

Now, let us consider the case of quasi-projections. If

$$G = \{t(x, y, z)\} \cup \{\gamma_a(x, y) \mid a \in A\},$$

then it is clear from theorem 2.1 that every quasi-projection on A is a composition of ternary quasi-projections on A. The question remains whether binary quasi-projections will do; for $|A| = 2$ we know the answer is no (see [8]). Notice that if $f\colon A^n \to A$ is a quasi-projection, then, for any $B \subseteq A$, $f(B^n) \subseteq B$. Hence, if $|A| \geqslant 3$ and every quasi-projection on A is a composition of binary quasi-projections on A, then the same would be true for every 2-element subset of A (this holds for A finite or infinite). Thus we have

THEOREM 2.3. *Let $2 \leqslant |A| < \aleph_0$. Then every quasi-projection on A can be obtained as a composition of ternary quasi-projections but not by the composition of binary quasi-projections.*

We close this section by characterizing the polynomials of $\langle A; t\rangle$, where A may be infinite. This characterization will be used in section 5. We say $f\colon B^n \to B$ *preserves* $\mathscr{I}(\mathfrak{B})$ if, for any $\varphi \in \mathscr{I}(\mathfrak{B})$ and any $a_1, \ldots, a_n$ in the domain of φ,

$$\varphi\big(f(a_1, \ldots, a_n)\big) = f\big(\varphi(a_1), \ldots, \varphi(a_n)\big).$$

Definition. Let $\mathfrak{B}$ be a finite non-trivial algebra; $\mathfrak{B}$ is *quasi-primal* if every function on B which preserves $\mathscr{S}(\mathfrak{B})$ and $\mathscr{I}(\mathfrak{B})$ is a polynomial of $\mathfrak{B}$.

By theorem 3.2 of [7], a finite non-trivial algebra $\mathfrak{B}$ is quasi-primal if and only if $t(x, y, z)$ is a polynomial of $\mathfrak{B}$. Thus, if $2 \leqslant |A| < \aleph_0$, then $\langle A; t\rangle$ is quasi-primal. Note that every bijection with domain and range contained in A is in $\mathscr{I}(\langle A; t\rangle)$. Thus, the polynomials of $\langle A; t\rangle$ are exactly those quasi-projections whose values depend only on the pattern of equalities and inequalities amongst the arguments of the functions and not on the arguments themselves; e.g. $t(x, y, z) = z$ if $x = y$ and $t(x, y, z) = x$ otherwise. Any quasi-projection with this property will be called a *pattern function*. Thus, if $2 \leqslant |A| < \aleph_0$, the polynomials of $\langle A; t\rangle$ are exactly the pattern functions on A. However, it is easily seen that this statement holds even when A is infinite.

THEOREM 2.4. *Let $|A| \geqslant 2$. The polynomials of $\langle A; t\rangle$ are exactly the pattern functions on A.*

The concept of "pattern function" is the same as that of "homogeneous quasi-trivial operation" introduced by Marczewski in [6]. Theorem 2.4 is proved in [3] for A finite.

3. Minimal generation — the idempotent case. In this section we investigate the minimal number of binary idempotent functions needed to generate all idempotent functions on A ($|A| \geqslant 3$). It is known [10] that only one binary function is needed to generate all functions on A. We now show that all idempotent functions on A can be obtained from five binary idempotent functions.

We want to find G_1 such that $\mathfrak{A}_1' = \langle A; G_1\rangle$ is semi-primal and has only trivial proper subalgebras. Moreover, G_1 is to consist of binary idempotent functions. Thus, by theorem 2.1, we have to guarantee that

(a) $t(x, y, z)$ is a polynomial of $\mathfrak{A}_1'$,

(b) $\mathfrak{A}_1'$ has no proper automorphisms,

(c) the proper subalgebras of $\mathfrak{A}_1'$ are exactly the one-element subsets.

The most difficult condition is (a). We make use of the following theorem which is implicit in theorems 3.1 and 3.2 of [7]:

THEOREM 3.1. *Let $\mathfrak{B}$ be a finite non-trivial algebra. Then $t(x, y, z)$ is a polynomial of $\mathfrak{B}$ iff*

(i) *every non-trivial subalgebra of $\mathfrak{B}$ is simple,*

(ii) *$\mathfrak{B}$ has a polynomial $p(x, y, z)$ satisfying*

$$p(x, x, y) = p(y, x, x) = y,$$

(iii) *$\mathfrak{B}$ has a polynomial $q(x, y, z)$ satisfying*

$$q(x, x, y) = q(x, y, x) = q(y, x, x) = x.$$

We are now ready to describe G_1; write $A = \{1, \ldots, n\}$. Let $\max(x, y)$ and $\min(x, y)$ be the binary maximum and minimum functions on A.

Then we have

$$q(x, y, z) = \max\big(\min(x, y), \max(\min(x, z), \min(y, z))\big).$$

We do not have to place both $\max(x, y)$ and $\min(x, y)$ in G_1 since if we can generate $p(x, y, z)$ from G_1, then

$$\min(x, y) = p\big(x, \max(x, y), y\big).$$

Thus we place only $\max(x, y)$ in G_1. Note that, with $\max(x, y) \in G_1$, $\mathfrak{A}_1'$ can have no proper automorphisms; thus (b) has been satisfied as well as (iii). We now add to G_1 the function $f_1(x, y)$ defined by (*) in section 2; thus (c) is satisfied. Since $\mathfrak{A}_1'$ is an idempotent algebra, every congruence class is a subalgebra; hence, since $f_1(x, y) \in G_1$, we infer that (i) is satisfied. Thus, it only remains to satisfy (ii); for this we turn to the concept of an idempotent latin square.

Definition. A *latin square of order n* is an $(n \times n)$-matrix with entries in $\{1, \ldots, n\}$ such that every row and every column is a permutation of $\{1, \ldots, n\}$. A latin square is *idempotent* if the (i, i)-entry is i for $i = 1, \ldots, n$.

To each idempotent latin square L we associate a binary idempotent function f_L by defining $f_L(i, j)$ to be the (i, j)-entry of L. Consider the function

$$p(x, y, z) = h\big(f(x, y), g(y, z)\big),$$

where f, g, h are idempotent functions on A and h is associated with the idempotent latin square L. Then, for $p(x, x, y) = y$ to hold, we must have $h\big(x, g(x, y)\big) = y$. Because h is associated with L, it is possible to solve this equation uniquely for $g(x, y)$ (which will also be associated with an idempotent latin square). Similarly, the equation $h\big(f(x, y), y\big) = x$ can be uniquely solved for $f(x, y)$. Then, by adding $f(x, y)$, $g(x, y)$ and $h(x, y)$ to G_1, (ii) is satisfied and so we are done. Thus, it only remains to be seen that there exist idempotent latin squares of all orders not less than 3.

That idempotent latin squares of all orders $n \geqslant 3$ exist is a well known fact in the combinatorial theory. Let L, M be latin squares of order n; let $l_{ij}(m_{ij})$ be the (i, j)-entry of $L(M)$. Then L and M are *orthogonal* if

$$\{(l_{ij}, m_{ij}) \mid 1 \leqslant i,\ j \leqslant n\} = \{1, \ldots, n\}^2.$$

By theorems 13.2.2 and 13.4.1 of [4], for every $n > 2$ except $n = 6$, there is at least one pair of orthogonal latin squares of order n. Thus, let L, M be orthogonal latin squares of order n. Permute the columns of L to get a latin square L' such that $l'_{ii} = 1$ for $i = 1, \ldots, n$. Apply the same permutation to the columns of M to get the latin square M'; clearly, L' and M' are orthogonal. Hence

$$\{m'_{ii} \mid i = 1, \ldots, n\} = \{1, \ldots, n\}.$$

Thus, in M' make the substitution $m'_{ii} \to i$ for $i = 1, \ldots, n$ to get the latin square M''; clearly, M'' is an idempotent latin square. For $n = 6$, there are no orthogonal pairs, and so we must construct an idempotent latin square. The following is one such:

$$\begin{matrix} 1 & 3 & 5 & 2 & 6 & 4 \\ 5 & 2 & 1 & 6 & 4 & 3 \\ 4 & 6 & 3 & 5 & 2 & 1 \\ 6 & 1 & 2 & 4 & 3 & 5 \\ 3 & 4 & 6 & 1 & 5 & 2 \\ 2 & 5 & 4 & 3 & 1 & 6 \end{matrix}$$

An obvious question is how many fewer than 5 binary idempotent functions do we really need. If n is odd, then we can take $h(x, y) = 2x - y \pmod{n}$. Then it is easily calculated that $g(x, y) = h(x, y)$ while

$$f(x, y) = \frac{n+1}{2}(x+y) = \left(\frac{n+1}{2}\right)x - \left(\frac{n-1}{2}\right)y.$$

Note that $h\big(h(x, y), y\big) = 4x - 3y$, $h\Big(h\big(h(x, y, y), y\big)\Big) = 8x - 7y$ and, in general, we can get $2^k x - (2^k - 1)y$ by the composition from $h(x, y)$. If, for some k,

$$2^k \equiv \frac{n+1}{2} \pmod{n},$$

then we can obtain $f(x, y)$ from $h(x, y)$ by the composition. Since the map $x \to 2x \pmod{n}$ is a permutation on $\{1, \ldots, n\}$, there is a $k > 1$ such that $2^k \equiv 1 \pmod{n}$; therefore,

$$2^{k-1} \equiv \frac{n+1}{2} \pmod{n}.$$

Thus we do not need to add $f(x, y)$ or $g(x, y)$ to G_1. We may still need to add $f_1(x, y)$ to G_1 since, for instance, for $n = 15$, $\{1, 4, 7, 10, 13\}$ is a subalgebra of $\langle\{1, \ldots, 15\}; h(x, y)\rangle$, where

$$h(x, y) = 2x - y \pmod{15}.$$

However, the next lemma shows that we do not need $f_1(x, y)$ if n is prime.

LEMMA 3.2. *If* $|A| = n \geqslant 3$ *is prime, then* $\langle A; h\rangle$ *has no non-trivial proper subalgebras* (*where* $h(x, y) = 2x - y \pmod{n}$).

Proof. All arithmetic will be done modulo n.

First, note that $\langle A; h\rangle$ has no two-element subalgebras since, for $i, j \in A$ with $i \neq j$, $2i - j = i$ implies $i = j$ while $2i - j = j$ implies $2i = 2j$, and since $x \to 2x$ is a permutation of A, this means $i = j$; similarly, $2j - i \notin \{i, j\}$. Thus, if $n = 3$, we are done.

If $n > 3$, then $x \to 3x$ is a permutation of A; hence, $2i-j = 2j-i$ implies $3i = 3j$ which implies $i = j$. Thus, $\langle A; h\rangle$ has no three-element subalgebras if $n > 3$.

Assume, inductively, that A has no non-trivial subalgebras with less than $k \geqslant 4$ elements. Let $\{a_1, \ldots, a_k\} = A_1 \subset A$ be a k-element subalgebra of $\langle A; h\rangle$ and suppose $k < n$. Then, for $i = 2, \ldots, n$,

$$A_i = \{a_1+(i-1), \ldots, a_k+(i-1)\}$$

is a subalgebra of $\langle A; h\rangle$. Since $k < n$ and n is prime, $A_i \neq A_j$ for $i \neq j$. Thus, if $i \neq j$, then $|A_i \cap A_j| < k$ and so $|A_i \cap A_j| \leqslant 1$. Without loss of generality we may assume that $n = a_1 \in A_1$. Then, for $i = 2, \ldots, k$, $a_i \in A_1$ implies $2n - a_i = -a_i \in A_1$. Suppose that

$$a_2 < a_3 \leqslant \frac{n-1}{2}$$

(we assume a_2 and a_3 are the two smallest members of A_1). Then we may assume that $a_4 = -a_2$ and $a_5 = -a_3$. Thus

$$\frac{n+1}{2} \leqslant a_5 < a_4 \quad \text{and} \quad a_3 - a_2 = a_4 - a_5.$$

Hence $\{a_4, a_5\} \subseteq A_{1+n-2a_3}$, so that $|A_1 \cap A_{1+n-2a_3}| > 1$. As $1 \neq 1+ +n-2a_3$, this contradicts the inductive assumption. But then we must have $|A_1| \leqslant 3$ again contrary to the assumption. Thus $\langle A; h\rangle$ has no subalgebras of cardinality k and so the proof of the lemma is complete.

THEOREM 3.3. *Let $|A| = n \geqslant 3$ be odd. If n is prime, then there are two binary idempotent functions such that every idempotent function on A is a composition of them; otherwise, there are three binary idempotent functions such that every idempotent function on A is a composition of them.*

4. Minimal generation — the general case. For K-idempotent functions with $K \geqslant 2$ and for quasi-projections, the minimal number of functions needed depends on the size of the set A. This is due to the presence of numerous two-element subalgebras and the necessity of killing isomorphisms between them. Only the proof for the case of quasi-projections will be given since the proof for the case of K-idempotent functions is quite similar.

Thus let $|A| = n$; we wish to find the least number of ternary quasi-projections of which G can consist so that $\mathfrak{A}' = \langle A; G\rangle$ is semi-primal. By theorem 2.1 and the remarks which follow it, we must show that

(a) $t(x, y, z)$ is a polynomial of $\mathfrak{A}'$,

(b) there are no isomorphisms between two-element subalgebras other than restrictions of the identity map on A.

Condition (a) is most easily satisfied by placing $t(x, y, z)$ in G. To handle condition (b) let $f(x, y, z)$ be a ternary quasi-projection on A and let $a, b, c, d \in A$. Let $\varphi(a) = c$ and $\varphi(b) = d$; in order that $\varphi \notin \mathscr{I}(\langle A; f\rangle)$ it is necessary that $f(a, a, b) = a$ and $f(c, c, d) = d$ or some such similar occurrence.

Consider the ordered pair (a, b) with $a \neq b$; there are three ordered triples in which both a and b appear and in which a appears first: (a, a, b), (a, b, a), (a, b, b). Since f can take on the values a or b at any of these triples, there are 8 choices for f. Similarly, for the triples (b, a, a), (b, a, b), (b, b, a), there are also 8 choices for f. However, we have to insure that $\varphi \notin \mathscr{I}(\langle A; f\rangle)$, where $\varphi(a) = b$ and $\varphi(b) = a$. Thus there are $8 \cdot 7 = 56$ admissible choices or patterns for f. Given two ordered distinct pairs (a, b) and (c, d) and the mapping $\varphi(a) = c$, $\varphi(b) = d$, $\varphi \in \mathscr{I}(\langle A; f\rangle)$ iff (a, b) and (c, d) are associated with the same f-pattern. If $|A| = n$, then A has $n(n-1)$ ordered distinct pairs; thus, if $|A| > 8$, f cannot kill all nontrivial isomorphisms between two-element subalgebras of $\langle A; f\rangle$, while if $|A| \leqslant 8$, f can be chosen to do so.

If we take 2 ternary quasi-projections f' and f'', then it is easily seen that there are $2^6(2^6-1) = 64 \cdot 63$ admissible (f', f'')-patterns, and so we can handle sets up to size 64 with 2 ternary quasi-projections. In general, we can handle sets up to size 2^{3k} with k suitably chosen ternary quasi-projections. Recalling that we have to add in $t(x, y, z)$ unless we are able to generate it from the other functions, the following is seen to hold ($[x]$ is the largest integer not greater than x):

THEOREM 4.1. *Let* $|A| = n \geqslant 2$ *and let* $p = [\log_8(n-1)]$. *Then there are* $p+2$ *ternary quasi-projections such that every quasi-projection on* A *is a composition of them; given any set of* p *ternary quasi-projections, not every quasi-projection can be obtained by a composition from them.*

Turning to K-idempotent functions, we note that G_K must include three kinds of functions: $f_K(X_1, \ldots, X_{K+1})$ as defined by (*) in section 2, $t(x, y, z)$, and the functions which kill the potential isomorphisms between two-element subalgebras. For the latter we are only concerned with the values of the functions when the arguments come from two-element subsets of A; thus, we can modify the definition of f_K so that it is also one of the latter functions. Hence, we have the following result:

THEOREM 4.2. *Let* $|A| = n \geqslant K+2$ *and let* $K \geqslant 2$. *Let*

$$q = 2^{(2^K-1)} \quad \textit{and} \quad p = [\log_q(n-1)].$$

Then there are $p+2$ $(K+1)$*-ary* K*-idempotent functions such that every* K*-idempotent function on* A *is a composition of them; given any* p $(K+1)$*-ary* K*-idempotent functions, not every* K*-idempotent function can be obtained by a composition from them.*

5. The infinite case. In this section we assume that A is infinite; we are concerned with the following questions:

(1) Is every K-idempotent function on A a composition of $(K+1)$-ary K-idempotent functions?

(2) Is every quasi-projection on A a composition of ternary quasi-projections?

It will be shown that the answer to question (1) is yes, while the answer to question (2) is no. We will prove, in detail, that an arbitrary 4-ary 2-idempotent function on A can be obtained as a composition of ternary 2-idempotent functions. The general case of obtaining an n-ary K-idempotent function by a composition from $(K+1)$-ary K-idempotent functions is similar and its proof is omitted.

Thus, let $f(x, y, u, v)$ be an arbitrary 4-ary 2-idempotent function. For $1 \leqslant i \leqslant 6$, we define the ternary 2-idempotent function $h_i(x, y, z)$ by the following:

$$h_1(a, b, c) = f(a, a, b, c), \quad h_2(a, b, c) = f(a, b, a, c),$$

$$h_3(a, b, c) = f(a, b, c, a), \quad h_4(a, b, c) = f(a, b, b, c),$$

$$h_5(a, b, c) = f(a, b, c, b), \quad h_6(a, b, c) = f(a, b, c, c).$$

Next, we come to the case where $a, b, c, d \in A$ with $|\{a, b, c, d\}| = 4$. First, partition A into countably many disjoint subsets $A_1, A_2, \ldots$ such that $|A| = |A_i|$ for all i. For any i, j, k, l, let $\varphi[i, j, k, l]$ be a bijection from $A_i \times A_j \times A_k$ onto A_l. Let $a_1 \in A_i$, $a_2 \in A_j$, $a_3 \in A_k$; write

$$h_7(a_1, a_2, a_3) = \begin{cases} a_1 & \text{if } |\{a_1, a_2, a_3\}| \leqslant 2, \\ \varphi[i, j, k, 2^i \cdot 3^j \cdot 5^k](a_1, a_2, a_3) & \text{otherwise.} \end{cases}$$

If $|\{a, b, c, d\}| = 4$, then a, $h_7(a, b, c)$, and $h_7(a, b, d)$ are all distinct and, moreover, (a, b, c, d) is the only distinct 4-tuple in A^4 which yields a, $h_7(a, b, c)$, $h_7(a, b, d)$. Hence, we may write

$$h_8\big(a, h_7(a, b, c), h_7(a, b, d)\big) = f(a, b, c, d)$$

and in all other cases

$$h_8(a_1, a_2, a_3) = a_1.$$

Thus

$$h_9(a, b, c, d) = h_8\big(a, h_7(a, b, c), h_7(a, b, d)\big) = f(a, b, c, d) \qquad \text{if } |\{a, b, c, d\}| = 4.$$

Now we define the pattern function $r(x_1, \ldots, x_{11})$ by

$$r(x_1, \ldots, x_{11}) = \begin{cases} x_5 & \text{if } |\{x_1, x_2, x_3, x_4\}| = 4, \\ x_6 & \text{if } x_1 = x_2, \\ x_7 & \text{if } x_1 = x_3, \\ x_8 & \text{if } x_1 = x_4, \\ x_9 & \text{if } x_2 = x_3, \\ x_{10} & \text{if } x_2 = x_4, \\ x_{11} & \text{if } x_3 = x_4. \end{cases}$$

Since r is a pattern function, theorem 2.4 says that r is a polynomial in $t(x, y, z)$ and so is a composition of ternary 2-idempotent functions. Thus we obtain $f(x, y, u, v)$ as a composition of ternary 2-idempotent functions *via*

$$f(x, y, u, v) = r\big(x, y, u, v, h_9(x, y, u, v), h_1(x, u, v), h_2(x, y, v), \\ h_3(x, y, u), h_4(x, y, v), h_5(x, y, u), h_6(x, y, u)\big).$$

By imitating this proof, one can prove that every n-ary 2-idempotent function is a composition of ternary 2-idempotent functions. In the case of K-idempotent functions with $K \geqslant 3$ one proceeds in a similar manner. Thus we arrive at the following result:

THEOREM 5.1. *Let A be an infinite set. Then every n-ary K-idempotent function on A is a composition of $(K+1)$-ary K-idempotent functions on A.*

To answer the second question we first make an analysis of the number of n-ary quasi-projections on a finite set. Let $N(n, p)$ be the number of n-ary quasi-projections on a p-element set. On a p-element set with $p > n$ the number of n-tuples with no repeated components is bounded below by $(p-n)^n$; on the other hand, the total number of n-tuples is p^n. Hence,

$$n^{(p-n)^n} \leqslant N(n, p) \leqslant n^{p^n} \quad \text{for } p > n.$$

Let f be an n-ary quasi-projection on a p-element set such that f is a composition of $(n-1)$-ary quasi-projections. The *length* of f is the minimal number of occurrences of $(n-1)$-ary quasi-projections in any composition of $(n-1)$-ary quasi-projections equal to f. Let $N(n, p, m)$ be the number of n-ary quasi-projections on a p-element set which have the length not greater than m. Since there are at most $(n-1)^{p^{n-1}}$ $(n-1)$-ary quasi-projections on a p-element set, it is easily seen that

$$N(n, p, m) \leqslant C_{n,m}(n-1)^{mp^{n-1}},$$

where $C_{n,m}$ is independent of p.

Thus we see that, for $p > n$,

$$\frac{N(n,p,m)}{N(n,p)} \leqslant \frac{C_{n,m}(n-1)^{mp^{n-1}}}{n^{(p-n)^n}} \leqslant C_{n,m}(n-1)^{[mp^{n-1}-(p-n)^n]} \to 0 \quad \text{as } p\to\infty.$$

Hence, for every $n \geqslant 4$ and $m \geqslant 1$, there is an integer $p(n, m)$ and an n-ary quasi-projection on a set of $p(n, m)$ elements which is a composition of $(n-1)$-ary quasi-projections and has the length at least m. Now we are ready to answer question (2).

THEOREM 5.2. *Let A be an infinite set. For every $n \geqslant 4$, there is an n-ary quasi-projection on A which is not a composition of $(n-1)$-ary quasi-projections.*

Proof. Partition A into countably many subsets $A_0, A_1, \ldots$ such that $|A_i| = p(n, i)$ for $i \geqslant 1$. Let $f_{n,i}$ be an n-ary quasi-projection on A_i which is a composition of $(n-1)$-ary quasi-projections and has the length not less than i. Let f be an n-ary quasi-projection on A such that $f|_{A_i} = f_{n,i}$ for $i \geqslant 1$; clearly, such an f exists. If f is a composition of $(n-1)$-ary quasi-projections, then it has a length, say, m. But then $f|_{A_i} = f_{n,i}$ is a composition of $(n-1)$-ary quasi-projections on A_i and has the length not greater than m. Clearly, this is impossible for $i > m$. Thus, f is not a composition of $(n-1)$-ary quasi-projections.

REFERENCES

[1] A. L. Foster and A. F. Pixley, *Semi-categorical algebras. I. Semi-primal algebras*, Mathematische Zeitschrift 83 (1964), p. 147-169.

[2] — *Semi-categorical algebras. II*, ibidem 85 (1964), p. 169-184.

[3] B. Ganter, J. Płonka and H. Werner, *Homogeneous algebras are simple*, Fundamenta Mathematicae 79 (1973), p. 217-220.

[4] M. Hall, Jr., *Combinatorial theory*, Waltham, Mass., 1967.

[5] R. A. Knoebel, *A simplification of the functional completeness proofs of Quackenbush and Sierpiński*, preprint.

[6] E. Marczewski, *Homogeneous operations and homogeneous algebras*, Fundamenta Mathematicae 56 (1964), p. 81-103.

[7] A. F. Pixley, *The ternary discriminator function in universal algebra*, Mathematische Annalen 191 (1971), p. 167-180.

[8] R. Quackenbush, *On the composition of idempotent functions*, Algebra Universalis 1 (1971), p. 7-12.

[9] W. Sierpiński, *Sur les fonctions de plusieurs variables*, Fundamenta Mathematicae 33 (1945), p. 169-173.

[10] D. Webb, *Definition of Post's generalized negative and maximum in terms of one binary operation*, American Journal of Mathematics 58 (1936), p. 193-194.

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