J. geom. 68 (2000) 87 - 95
0047-2468/00/020087-09 $ 1.50+0.20/0

Journal of Geometry

Some Combinatorial and Geometric Characterizations of the Finite Dual Classical Generalized Hexagons

Eline Govaert and Hendrik Van Maldeghem

We characterize the dual of the generalized hexagons naturally associated to the groups $G_2(q)$ and ${}^3D_4(q)$ by looking at certain configurations, and also by considering intersections of traces. For instance, the dual of a generalized hexagon Γ of finite order (s,t) is associated to the Chevalley groups mentioned above if and only if the intersection of any two traces x^y and x^z, with some additional condition, contains at most $t/s+1$ elements.

1 Introduction

A finite *generalized polygon of order* (s,t), $1 < s,t < \infty$, is a point-line incidence geometry whose incidence graph has girth $2n$ and diameter n, for some natural number n, $n \geq 2$ (in which case we also speak about a *generalized* n*-gon*), such that there are exactly $s+1$ points incident with any line, and $t+1$ lines incident with any point. We have excluded the trivial case $s=t=1$ and the cases $t=1$ or $s=1$ which can be reduced to the case $s,t>1$ by considering an appropriate generalized $n/2$-gon. Generalized polygons were introduced by Tits [13]. For an extensive survey including most proofs, we refer to Van Maldeghem [16]. For the finite case, with emphasis on the generalized quadrangles, Thas [12] provides an overview, but mostly without proofs. For finite generalized quadrangles, see Payne & Thas [7] (including a lot of proofs).

In this paper, we are only concerned with (finite) generalized 6-gons, or hexagons. However, we would like to refer to similar situations in the theory of finite generalized quadrangles and octagons, and in order to be able to do so, we have given the general definition above. For example, some of the results we present in this paper have their roots in a recent characterization of the Ree-Tits octagons in Van Maldeghem [16]. We will explain this below. Note that by a well-known result of Feit & Higman [5] finite generalized n-gons of order (s,t), with $s,t>1$, only exist for $n=3,4,6,8$ (excluding the trivial case $n=2$).

For each of the cases $n = 4, 6, 8$, there are so-called *classical examples*, which serve as standard examples, but which are also characterized by a group-theoretical condition. For the aim of the present paper, it suffices to mention that these (finite) classical examples are the ones arising naturally from a (finite) Chevalley group of rank 2 (including the Ree groups in characteristic 2) by taking as points and lines the (left) cosets of the two respective maximal parabolic subgroups with respect to a fixed Borel subgroup (and a point and a line are incident if the corresponding cosets have nonempty intersection). The hexagons and octagons among them can be described geometrically as follows (for the type of a triality, we use the notation of TITS [13]):

(1) The *split Cayley hexagon* $\mathsf{H}(q)$ is the geometry of absolute points and lines of a triality of type I_{id} on the triality quadric $Q^+(7,q)$ (in $\mathbf{PG}(7,q)$). It has a representation on the quadric $Q(6,q)$ in $\mathbf{PG}(6,q)$. Its order is (q,q).

(2) The *twisted triality hexagon* $\mathsf{T}(q^3,q)$. This is the geometry of absolute points and lines of a triality of type I_q) on the triality quadric $Q^+(7,q^3)$. It has order (q^3,q).

(3) Finally, the *Ree-Tits octagon* $\mathsf{O}(q)$ is the geometry of absolute points and lines of any polarity in a metasymplectic space over the field $\mathbf{GF}(q)$, $q = 2^{2e+1}$. Here the order is (q,q^2).

We will call these examples *classical*. The generalized polygons obtained from these by interchanging the roles of the points and the lines, are then called *dual classical*.

Let Γ be any generalized hexagon (or polygon). For any point x, we denote by $\Gamma_i(x)$ the set of elements of Γ at distance i from x (measured in the incidence graph; we denote that distance function by δ). Dually, we will use the notation $\Gamma_j(L)$ for a line L. Two elements are called *opposite* if they are at maximal distance from each other. In particular, for hexagons, two points are opposite if they are at distance 6 from each other. For two opposite points x,y of a hexagon Γ, we denote by x^y the set $\Gamma_2(x)\cap\Gamma_4(y)$. Also, we write $\langle x,y\rangle = \Gamma_3(x)\cap\Gamma_3(y)$. The set x^y is usually called a *trace* (more precisely, a *distance-2-trace*), while the set $\langle x,y\rangle$ is sometimes called a *regulus* (or in a more systematic terminology, a *distance-3-trace*). The dual of a regulus is called a *dual regulus* and denoted by $\langle L,M\rangle$ for two opposite lines L,M at distance 3 from every point of the dual regulus. For two points y,z opposite some point x, we denote by $x^{\{y,z\}}$ the set of points $w\in x^y\cap x^z$ such that $\Gamma_1(w)\cap\Gamma_3(y)\cap\Gamma_3(z)$ is empty. If two elements u,v of a generalized polygon are not opposite, then there is a unique element incident with u and nearest to v; we denote this element by $\mathrm{proj}_u v$ and call it the *projection of v onto u*. If two elements u,v are at distance 4 in a generalized hexagon Γ, then the unique element of $\Gamma_2(u)\cap\Gamma_2(v)$ will be denoted by $u\bowtie v$. Finally, we denote the unique line incident with two distinct collinear points x and y by xy.

The following theorem is a consequence of the main result of RONAN [8].

THEOREM (Ronan [8]). *If in a finite generalized hexagon Γ of order (s,t), $s,t>1$, for all points x,y,z, with both y and z opposite x, we have $|x^y\cap x^z|\in\{0,1,t+1\}$, then Γ is isomorphic to either* $\mathsf{H}(s)$ *or* $\mathsf{T}(t^3,t)$.

Furthermore, the converse also holds. With the terminology of VAN MALDEGHEM [14], the property stated in the above theorem is called *distance-2-regularity*.

From this result, lots of other geometric and combinatorial characterizations of $\mathsf{H}(q)$ and $\mathsf{T}(q^3,q)$ were derived. We refer to results of RONAN [10], THAS [11], DE SMET & VAN

MALDEGHEM [3], VAN BON, CUYPERS & VAN MALDEGHEM [1], GOVAERT [6] and BROUNS & VAN MALDEGHEM [2]. All these results use certain point sets to characterize $\mathsf{H}(q)$ and/or $\mathsf{T}(q^3, q)$ (sometimes for restricted values of q). We know of only one characterization of the duals of $\mathsf{H}(q)$ and $\mathsf{T}(q^3, q)$ using points sets, and that is due to RONAN [10] (if we do not count the characterization of both $\mathsf{T}(q^3, q)$ and its dual in RONAN [9]). In the present paper, we present some new characterization results of the duals of $\mathsf{H}(q)$ and $\mathsf{T}(q^3, q)$ using certain point sets. In some cases, it seems to be already interesting to prove the condition we consider in the respective cases!

ACKNOWLEDGEMENT. The first author is a research assistant of the FWO, the *Fund for Scientific Research – Flanders* (Belgium) and the second author is a Research Director of the FWO.

2 STATEMENT OF THE RESULTS

Let us start by writing down some lesser-known properties of the duals of $\mathsf{H}(q)$ and $\mathsf{T}(q^3, q)$.

THEOREM 1 *Let Γ be dual to either $\mathsf{H}(q)$ or $\mathsf{T}(q^3, q)$. Let (q,t) be its order (hence $t \in \{q, q^3\}$). Then we have:*

(i) Suppose that q is not divisible by 3 when $t = q$. If the distinct points v, w, x, y, z are such that $y, z \in \Gamma_6(x)$ and $v, w \in x^y \cap x^z$, with $\mathrm{proj}_v y = \mathrm{proj}_v z$ and $\mathrm{proj}_w y \neq \mathrm{proj}_w z$, then $|x^y \cap x^z| = t/q + 1$.

(ii) Suppose $t = q$ and q is not divisible by 3. If the distinct points v, w, x, y, z are such that $y, z \in \Gamma_6(x)$ and $v, w \in x^y \cap x^z$, with $\mathrm{proj}_v y \neq \mathrm{proj}_v z$ and $\mathrm{proj}_w y \neq \mathrm{proj}_w z$, then $|x^{\{y,z\}}| = |x^y \cap x^z| = 3$.

(iii) If a point x is at distance 4 from a point y of a dual regulus R, and if all elements of $R \setminus \{y\}$ are opposite x, then all these elements are at distance 4 from $x \bowtie y$.

We will prove this result in the next section. We now consider slightly weaker versions of (i) and (ii) for a generalized hexagon of order (q,t):

$(i)'$ If the distinct points v, w, x, y, z are such that $y, z \in \Gamma_6(x)$ and $v, w \in x^y \cap x^z$, with $\mathrm{proj}_v y = \mathrm{proj}_v z$ and $\mathrm{proj}_w y \neq \mathrm{proj}_w z$, then $|x^y \cap x^z| \leq t/q + 1$.

$(ii)'$ Suppose $t = q$. If the distinct points v, w, x, y, z are such that $y, z \in \Gamma_6(x)$ and $v, w \in x^y \cap x^z$, with $\mathrm{proj}_v y \neq \mathrm{proj}_v z$ and $\mathrm{proj}_w y \neq \mathrm{proj}_w z$, then $|x^{\{y,z\}}| = |x^y \cap x^z| \geq 3$.

We can now state our main results:

THEOREM 2 *Let Γ be a finite generalized hexagon of order (q,t).*

(a) *If Γ satisfies* Condition $(i)'$, *then it is dual to either $\mathsf{H}(q)$, q not divisible by 3, or $\mathsf{T}(q^3, q)$.*

(b) *If Γ satisfies* Condition $(ii)'$, *then it is dual to $\mathsf{H}(q)$, q not divisible by 3.*

(c) *If q is even and Γ satisfies* Condition (iii), *then it is dual to either $\mathsf{H}(q)$ or $\mathsf{T}(q^3, q)$.*

We remark that all classical hexagons of order (s,t) with s odd satisfy Condition (iii). Hence in order to obtain the dual of some classical hexagons, the condition q even in (c) above is necessary. Also, in (a) it is not necessarily understood that t/q is an integer or that $t \geq q$.

The characterizations (a) and (b) above are partly inspired by the characterization of the Ree-Tits octagons in VAN MALDEGHEM [15]. Indeed, the Ree-Tits octagons are characterized by some intersection properties of traces x^y, from the point of view of the elements at distance 3 from x and 5 from y. This is exactly what happens in (a) and (b): given some properties of $x^y \cap x^z$ and of $\langle x,y\rangle \cap \langle x,z\rangle$, we require that we know what $x^y \cap x^z$ looks like.

As a corollary to our proof of Theorem 2(c), we obtain a common characterization of $\mathsf{T}(q^3,q)$, q odd, and the dual of $\mathsf{T}(q^3,q)$, q any prime power. See Remark 1 of Section 5.

3 Proof of Theorem 1

Let Γ be dual to either $H(q)$ or $\mathsf{T}(q^3,q)$. First, we consider Condition (i). Let v,w,x,y,z be as in (i). Let $L = \mathrm{proj}_v y = \mathrm{proj}_v z$, let $M_y = \mathrm{proj}_w y$ and let $M_z = \mathrm{proj}_w z$. Let a be the unique point of $\langle L, M_z\rangle$ at distance 4 from y. By RONAN [8](5.8), we have $x^z = x^a$. By RONAN [10], we have $|x^a \cap x^y| = t/q+1$. This proves (i).

Now we show (ii). So we assume that Γ is dual to $\mathsf{H}(q)$, q not a multiple of 3. Let v,w,x,y,z be as in (ii), and put $L_y = \mathrm{proj}_v y$, $L_z = \mathrm{proj}_v z$, $M_y = \mathrm{proj}_w y$ and $M_z = \mathrm{proj}_w z$. From the explicit form of generalized homologies in DE SMET & VAN MALDEGHEM [4], we deduce that the stabilizer in the automorphism group of Γ of $\{x,v,w,z,M_y\}$ acts transitively on the set $\Gamma_1(v)\setminus\{vx,L_z\}$. For any line $L \in \Gamma_1(v)\setminus\{vx\}$, let u_L be the unique point of $\langle L, M_y\rangle$ closest to y. The q traces x^{u_L} thus obtained meet pairwise in $\{v,w\}$, by (i). Hence their union meets x^z in $q+1$ points. By the transitivity just mentioned, the $q-1$ traces x^{u_L}, with $L \in \Gamma_1(v)\setminus\{vx,L_z\}$ meet x^z in a constant number of points, while $x^{u_{L_z}}$ meets x^z in just $\{v,w\}$. It now follows that $|x^z \cap x^{u_L}| = 3$, for all $L \in \Gamma_1(v)\setminus\{vx,L_z\}$. The other assertion follows readily from (i).

Finally, we show (iii). So let x be at distance 4 from a point y, suppose that y belongs to a dual regulus R and let x be opposite every element of $R\setminus\{y\}$. Put $L = \mathrm{proj}_x y$ and $R = \langle M,N\rangle$. Note that, by RONAN [8](5.8), M and N can be chosen arbitrarily in $\langle y,z\rangle$, for any $z \in R\setminus\{y\}$. In particular, we may choose M at distance ≤ 4 from L. Now suppose $\delta(L,M) = 4$. Then the point $x \bowtie y$ does not belong to M; we deduce that $x \bowtie y$ is opposite every element of $R\setminus\{y\}$. Since $|R| = q+1$, and since also x is opposite every element of $R\setminus\{y\}$, there must be some point w incident with L which is at distance 4 from at least two points $z_1, z_2 \in R\setminus\{y\}$. Now we may choose N in such a way that it meets $\mathrm{proj}_{z_1} w$ (and clearly this is allowed since $\mathrm{proj}_{z_1} w$ does not meet M). Note that $\delta(z_1,M) = 3$. With dual notation, we thus obtain $\{\mathrm{proj}_y N, \mathrm{proj}_{z_1} N, \mathrm{proj}_{z_2} N\} \subseteq N^M$, and $\{\mathrm{proj}_y N, \mathrm{proj}_{z_1} N\} \subseteq N^L$. By the dual of the distance-2-regularity, we must have $\delta(L, \mathrm{proj}_{z_2} N) = 4$. But then, there are two shortest paths from w to N, a contradiction. Hence $\delta(L,M) = 2$ and consequently $x \bowtie y$ is incident with M. Clearly $x \bowtie y$ is now at distance 4 from every element of $R\setminus\{y\}$.

Theorem 1 is completely proved.

Note that an alternative proof with coordinates in the sense of De Smet & Van Maldeghem [4] would be very easy and short here, but less beautiful than the geometric arguments given above.

4 Proof of Theorem 2

We now prove the characterizations given in (a), (b) and (c) of Theorem 2. First we introduce a little bit of terminology. Let Γ be a generalized hexagon. We say that Γ is *distance-3-regular* if for every pair of opposite lines L, M we have $\langle x, y\rangle = \langle a, b\rangle$ for all $x, y, a, b \in \langle L, M\rangle$, with $x \neq y$ and $a \neq b$. This is called the *regulus condition* in Ronan [8].

4.1 Proof of (a)

Let x, y, v, w be as in $(i)'$ and put $u = v \bowtie y$. We first count the number of pairs (a, b), with $a \in \Gamma_2(x) \setminus (\Gamma_1(vx) \cup \Gamma_1(wx))$, $b \in u^w \setminus \{v\}$ and with $\delta(a, b) = 4$. Let k be the number of possible choices for a. When we fix a, then the set of points b is exactly $(u^a \cap u^w) \setminus \{v\}$. By Condition $(i)'$, there are at most t/q such points.

Now we first fix b and we easily obtain that there are exactly $t(t-1)$ such pairs. Hence we have $k \cdot \frac{t}{q} \geq t(t-1)$. Since clearly $k \leq q(t-1)$, we need the equality everywhere. This shows that every element of $\Gamma_2(x) \setminus (\Gamma_1(vx) \cup \Gamma_1(wx))$ is the projection of exactly t/q points of $u^w \setminus \{v\}$.

Now we prove that Γ is distance-3-regular. Let $z \in \langle uv, \mathrm{proj}_w y\rangle \setminus \{x, y\}$ be arbitrary. We must show that $\delta(z, L) = 3$, for all $L \in \langle x, y\rangle$. We count the number of pairs (a, b) with $a \in x^z \setminus \{v, w\}$, $b \in u^w \setminus \{v\}$ and with $\delta(a, b) = 4$. By the previous paragraph, this number is equal to $(t-1) \cdot \frac{t}{q}$. On the other hand, if $b \neq y$, then there at most $\frac{t}{q} - 1$ choices for a, by Condition $(i)'$. Putting $\ell = |x^y \cap x^z \setminus \{v, w\}|$, we hence obtain

$$(t-1) \cdot \frac{t}{q} \leq (t-1)(\frac{t}{q} - 1) + \ell,$$

implying $\ell = t-1$, which means that $x^y = x^z$. It is now easily seen that $\mathrm{proj}_r y = \mathrm{proj}_r z$ for all $r \in x^y \cap x^z$, for otherwise Condition $(i)'$ would imply that $|x^y \cap x^z| \leq \frac{t}{q} + 1 < t + 1$.

The result now follows directly from Ronan [10].

4.2 Proof of (b)

Our aim is to show that Condition (i) is satisfied for $q = t$. The result will then follow from (a).

So let v, w, x, y, z be as in (i). Put $u = v \bowtie y$ and $u' = w \bowtie y$. If z is collinear with u, then it is clear that $\mathrm{proj}_r y \neq \mathrm{proj}_r z$ for every $r \in x^y \cap x^z \setminus \{v\}$. Hence, since $\mathrm{proj}_v y = \mathrm{proj}_v z$, no such point r different from w exists by Condition $(ii)'$. So in this case $|x^y \cap x^z| = 2$.

Suppose now that $\delta(u,z)=4$. Let $\mathcal{T}=\{x^r | r\in\Gamma_2(u')\cap\Gamma_4(v)\cap\Gamma_6(x)\}$. By the previous paragraph, every two elements of $\mathcal{T}$ meet in just $\{v,w\}$. Also, $T\cap x^z$ contains at least one element different from v,w, for all $T\in\mathcal{T}\setminus\{x^y\}$. Since this gives rise to at least $q-1$ elements of $x^z\setminus\{v,w\}$, there is no room anymore in x^z for elements of $x^y\setminus\{v,w\}$. We have shown $x^y\cap x^z=\{v,w\}$ and so, (a) applies. The result follows.

4.3 Proof of (c)

We show (c) in three steps. In the first two steps, we do not use the fact that q is even.

(I) We first show that *(iii)* implies distance-3-regularity. Let x,y,z be three distinct points of a dual regulus $\langle L,M\rangle$ and let $N\in\langle x,y\rangle$. We have to show that $\delta(z,N)=3$. Let $v=\mathrm{proj}_N x$ and let $w=\mathrm{proj}_N y$.

Suppose that there exists a line $N'\in\Gamma_1(v)\setminus\{vx\}$ at distance 5 from every element of $\langle L,M\rangle\setminus\{x\}$. Consider the projection of $\langle L,M\rangle$ onto N'. Since v is the image of at least two points (x and y), this mapping is not injective, hence neither is it surjective. So there is a point incident with N' opposite every element of $\langle L,M\rangle\setminus\{x\}$. Condition *(iii)* implies now that every point of $\langle L,M\rangle\setminus\{x\}$ is at distance 4 from v. As a consequence, every point z' of $\langle L,M\rangle\setminus\{y\}$ is at distance 5 from every element N'' of $\Gamma_1(w)\setminus\{N\}$ (otherwise there arises a circuit of length $\delta(z',v)+\delta(v,N'')+\delta(N'',z')<4+3+5$). Hence, interchanging the roles of v and w, we see that $\delta(z,v)=\delta(z,w)=4$, which is only possible if $\delta(z,N)=3$.

Now suppose that for every line $N'\neq vx$ through v there exists a point of $\langle L,M\rangle\setminus\{x\}$ at distance 3 from N'. By an argument in the previous paragraph, this is also true mutatis mutandis for every line $N''\neq wy$ through w. Put $z'=\mathrm{proj}_L z$ and $z''=\mathrm{proj}_M z$. Note that at least one of the lines zz' and zz'' is opposite N, otherwise $\delta(z,N)=3$ and we are done. Suppose $\delta(zz'',N)=6$. Similarly, z is opposite at least one of v,w. Suppose $\delta(v,z)=6$. Let r be the projection of v onto zz'' (then $z\neq r\neq z''$). Put $R_1=\mathrm{proj}_v r$ and $R_2=\mathrm{proj}_r v$ (then $R_1\neq N$). By assumption, there exists a point $y'\in\langle L,M\rangle\setminus\{x\}$ at distance 3 from R_1 (and note that $\mathrm{proj}_{R_1}y'\neq r\bowtie v$ because otherwise $y',\mathrm{proj}_M y',z'',r,r\bowtie v$ defines a pentagon). We consider the projection of $\langle L,M\rangle$ onto the line R_2. Since x and y' are both mapped on $r\bowtie v$, this projection is, as above, not surjective, and hence, again as before, r must be at distance 4 from every element of $\langle L,M\rangle\setminus\{z\}$, a final contradiction as, for example, $\delta(r,x)=6$.

We have shown the distance-3-regularity.

(II) We now show that, if a point x is at distance at most 4 from at least three points y_1,y_2,y_3 of a dual regulus R, then it is at distance 2 from a unique element of R and at distance 4 from all other elements of R. It is easily seen that this follows immediately from the distance-3-regularity shown in (I) whenever $\mathrm{proj}_x y_1=\mathrm{proj}_x y_2$. Hence we may suppose that the projections of y_1,y_2,y_3 onto x are three distinct lines. It suffices to find a contradiction. Put $N=\mathrm{proj}_x y_1$. If N is at distance 3 from an element of $R\setminus\{y_1\}$, then again, by the distance-3-regularity, the result follows. So we may assume that every point of $R\setminus\{y_1\}$ is at distance 5 from N. We then consider the projection of R onto N. Since y_2 and y_3 have the same image, there is some point incident with N and distinct from $x\bowtie y_1$ opposite every element of $R\setminus\{y_1\}$. Condition *(iii)* implies that $\delta(y_2,x\bowtie y_1)=4$, a contradiction.

(III) Finally we suppose that q is even and we show that every line L of Γ is at distance at most 4 from at least one line of any regulus $\langle x, y\rangle$. Let R be the dual regulus $\langle M, N\rangle$, with $M, N \in \langle x, y\rangle$ and $M \neq N$. If L is at distance 3 from some point of R, then the distance-3-regularity immediately implies the result. So we may assume that L is at distance 5 from every point of R. Hence, by (II), no point incident with L is the projection of at least 3 elements of R. Since $q+1$ is odd, there is a point v on L which is the projection of exactly one point z of R. Condition (*iii*) now implies that $v \bowtie z$ is at distance 4 from every point of $R \setminus \{z\}$. By the distance-3-regularity, the projection of any element of $R \setminus \{z\}$ is a (unique) line of $\langle x, y\rangle$, which is hence at distance 4 from L. We have shown the result.

Now (c) follows from (III) and Theorem 1 of GOVAERT [6], which says that a finite generalized hexagon Γ is isomorphic to $\mathsf{H}(q)$ or to $\mathsf{T}(q^3, q)$, both with q even, if and only if for any point x and any pair of lines L, M, there exists at least one point not opposite x at distance ≤ 3 from both L and M.

5 Some Remarks

REMARK 1

Following VAN MALDEGHEM [16], let us call a finite generalized hexagon of order (s,t) with $s = t^3$ or $t = s^3$ an *extremal* hexagon. From (I) of the proof of Theorem 2(c) above and the main result of RONAN [9], we deduce:

COROLLARY 1 *If Γ is an extremal hexagon satisfying* Condition (*iii*), *then it is isomorphic or dual to* $\mathsf{T}(q^3, q)$.

If Γ is isomorphic to $\mathsf{T}(q^3, q)$, then it satisfies Condition (*iii*) if and only if q is odd. This can easily be proved using explicit coordinates, as for instance in DE SMET & VAN MALDEGHEM [4].

REMARK 2

Probably, the property mentioned in (*ii*) above is the best analogue for hexagons of *anti-regularity* for generalized quadrangles (defined in case of order (s,s)). Indeed, one easily deduces from PAYNE & THAS [7] that a finite generalized quadrangle Γ is *anti-regular* if for any three points x, y, z, with z, y both opposite x, such that $|\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_2(z)| \geq 1$ and $|\Gamma_1(w) \cap \Gamma_1(y) \cap \Gamma_1(z)| = 0$, for at least one element w of $\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_2(z)$, we have that $|\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_2(z)| = 2$ and $|\Gamma_1(w) \cap \Gamma_1(y) \cap \Gamma_1(z)| = 0$, for all elements w of $\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_2(z)$. Hence, one can generalize this property to finite generalized $2n$-gons (of order (s,s)) in such a way that for hexagons we obtain (*ii*), as follows:

For any three points x, y, z, with z, y both opposite x, such that $|\Gamma_2(x) \cap \Gamma_{2n-2}(y) \cap \Gamma_{2n-2}(z)| \geq n-1$ and $|\Gamma_1(w) \cap \Gamma_{2n-3}(y) \cap \Gamma_{2n-3}(z)| = 0$, for at least $n-1$ elements w of $\Gamma_2(x) \cap \Gamma_{2n-2}(y) \cap$

$\Gamma_{2n-2}(z)$, *we have that* $|\Gamma_2(x) \cap \Gamma_{2n-2}(y) \cap \Gamma_{2n-2}(z)| = n$ *and* $|\Gamma_1(w) \cap \Gamma_{2n-3}(y) \cap \Gamma_{2n-3}(z)| = 0$, *for all elements* w *of* $\Gamma_2(x) \cap \Gamma_{2n-2}(y) \cap \Gamma_{2n-2}(z)$.

If we call this property *anti-regularity*, then we might rephrase Theorem 2(b) as follows:

COROLLARY 2 *A finite generalized hexagon of order* (s, s) *is anti-regular if and only if it is dual to the classical hexagon* $\mathsf{H}(s)$, *with* s *not a power of* 3.

In particular, for an anti-regular finite generalized hexagon of order (s, s) one automatically has that s is not a power of 3.

The finite anti-regular quadrangles are not all classified, although it has been conjectured that they should be dual to the quadrangle $\mathsf{W}(s)$ arising from a symplectic polarity in $\mathbf{PG}(3, s)$, s not a power of 2. This once again shows the similarity between the symplectic quadrangle $\mathsf{W}(s)$ and the split Cayley hexagon $\mathsf{H}(s)$, see various places in VAN MALDEGHEM [16].

REFERENCES

[1] VAN BON, J., H. CUYPERS and H. VAN MALDEGHEM, Hyperbolic lines in generalized polygons, *Forum Math.* **8** (1994), 362.

[2] BROUNS, L. and H. VAN MALDEGHEM, Characterizations for classical finite hexagons, *Bull. Belg. Math. Soc. Simon Stevin* **5** (1998), 163 - 176.

[3] DE SMET, V. and H. VAN MALDEGHEM, Ovoids and windows in finite generalized hexagons, in *Finite Geometry and Combinatorics*, Proceedings Deinze 1992 (ed. F. De Clerck *et al.*), Cambridge University Press, *London Math. Soc. Lecture Note Ser.* **191** (1993), 131 - 138.

[4] DE SMET, V. and H. VAN MALDEGHEM, The finite Moufang hexagons coordinatized, *Beiträge Algebra Geom.* **34** (1993), 217 - 232.

[5] FEIT, W. and G. HIGMAN, The nonexistence of certain generalized polygons, *J. Algebra* **1** (1964), 114 - 131.

[6] GOVAERT, E., A combinatorial characterization of some finite classical generalized hexagons, *J. Combin. Theory Ser.* A. **80** (1997), 339 - 346.

[7] PAYNE, S. E. and J. A. THAS, *Finite Generalized Quadrangles*, Pitman, Boston, London, Melbourne, 1984.

[8] RONAN, M. A., A geometric characterization of Moufang hexagons, *Invent. Math.* **57** (1980), 227 - 262.

[9] RONAN, M. A., A note on the ${}^3D_4(q)$ generalised hexagons, *J. Combin. Theory Ser.* A **29** (1980), 249 - 250.

[10] RONAN, M. A., A combinatorial characterization of dual Moufang hexagons, *Geom. Dedicata* **11** (1981), 61 - 67.

[11] THAS, J. A., A remark on the theorem of Yanushka and Ronan characterizing the generalized hexagon $H(q)$ arising from the group $G_2(q)$, *J. Combin. Theory Ser.* A **29** (1980), 361 - 362.

[12] THAS, J. A., Generalized polygons, **in** *Handbook of Incidence Geometry, Buildings and Foundations*, (ed. F. Buekenhout), Chapter 9, North-Holland (1995), 383 - 431.

[13] TITS, J., Sur la trialité et certains groupes qui s'en déduisent,*Inst. Hautes Etudes Sci. Publ. Math.* **2** (1959), 14 - 60.

[14] VAN MALDEGHEM, H., The nonexistence of certain regular generalized polygons, *Arch. Math.* **64** (1995), 86 - 96.

[15] VAN MALDEGHEM, H., A geometric characterization of the perfect Ree-Tits octagons, *Proc. London Math. Soc.* (3) **76**, no. 1 (1998), 203 - 256.

[16] VAN MALDEGHEM, H., *Generalized Polygons*, Birkhäuser, Basel, *Monographs in Mathematics* **93** (1998).

University of Ghent
Department of Pure Mathematics
and Computer Algebra
Galglaan 2,
9000 Gent
Belgium

Eingegangen am 5. Dezember 1997; in revidierter Fassung am 24. August 1999