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Some combinatorial and probabilistic inequalities and their application to Banach space theory II

by

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Abstract. Some combinatorial estimates are proved. As applications they are used to study subspaces of L^1.

Introduction. This is a continuation of our previous paper [1]. In [1] we estimated

$$\operatorname*{Ave}_{\pi} \|(x_i y_{\pi(i)})_{i=1}^n\|_p$$

where the average is over all permutations π of the set $\{1, \ldots, n\}$. Here we are able to give estimates for

$$\operatorname*{Ave}_{\pi} \|(x_i y_{\pi(i)})_{i=1}^n\|_M$$

where M is an Orlicz function. We do this by considering

$$\operatorname*{Ave}_{\pi} \operatorname*{Ave}_{\sigma} \max_{1 \leqslant i \leqslant n} |x_i y_{\pi(i)} z_{\sigma(i)}|.$$

At first glance this expression looks more complicated but it turns out that it is much easier to handle. In fact, we consider a more general average which may be of some importance. Namely, we consider

$$\operatorname*{Ave}_{g} \max_{1 \leqslant i \leqslant n} |a(i, g(i))|$$

where $(a(i,j))_{n \times m}$ is a matrix and the average is over some subset of maps from $\{1, \ldots, n\}$ into $\{1, \ldots, m\}$. This more general approach makes the proofs more transparent.

Our method has been used in [5] to determine p-absolutely summing norms. In [3] Y. Raynaud and the second-named author extended the results to infinite-dimensional Banach spaces. Some of our results have been presented in [4].

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1. Combinatorial inequalities. Let I, J be finite sets with $|I| = n$, $|J| = m$. Let G be a fixed subset of the set of maps from I into J. We denote by μ the normalized counting measure on G, i.e. $\mu(A) = |A|/|G|$ for $A \subset G$. The basic assumption on G is that for each $i \in I$ the random variable $g(i)$ on G is equidistributed in J, i.e.

$$\mu(\{g\colon g(i) = j\}) = 1/m \quad \text{for all } i \in I \text{ and } j \in J.$$

If f is a function on G then we define

$$\operatorname*{Ave}_g f(g) = |G|^{-1} \sum_{g \in G} f(g) = \int_G f(g)\,\mu(dg).$$

In this section we give some estimates from above and below for the average

$$\operatorname*{Ave}_g \max_{i \in I} a(i, g(i))$$

where $(a(i,j))_{I \times J}$ is a matrix with nonnegative entries.

Let $\Delta = \{(s, k)\colon 1 \leqslant s \leqslant c, k \in K_s\}$ be a fixed set of indices and let $\mathcal{U} = (U_k^s)_{(s,k) \in \Delta}$ be a fixed family of subsets of $I \times J$ such that

(i) $U_k^s \cap U_l^s = \emptyset$ for $(s, k), (s, l) \in \Delta$ and $k \neq l$,

(ii) $\mu(\{g\colon (i, g(i)) \in U_k^s \text{ for some } i \in I\}) \leqslant r_s/m$ for all $(s, k) \in \Delta$ where r_s is an integer depending on s only.

The family $\mathcal{U}$ may be empty and we put $c = 0$ in this case. With $\mathcal{U}$ we associate another family $\mathcal{A}$ as follows:

$$\mathcal{A} = \{A \subset I \times J\colon |A| \leqslant m,\ |A \cap U_k^s| \leqslant r_s \text{ for each } (s, k) \in \Delta\}.$$

LEMMA 1.1. *Under the above notation we have for each nonnegative matrix* $(a(i,j))_{I \times J}$

$$\operatorname*{Ave}_g \max_{i \in I} a(i, g(i)) \leqslant \frac{c+2}{m} \max_{A \in \mathcal{A}} \sum_{(i,j) \in A} a(i, j).$$

Proof. Let A' be such that $A' \in \mathcal{A}$ and

$$\sum_{(i,j) \in A} a(i, j) \leqslant \sum_{(i,j) \in A'} a(i, j) \quad \text{for all } A \in \mathcal{A}.$$

Let $A'' = \{(i,j) \in I \times J \setminus A'\colon |A \cap U_k^s| < r_s \text{ for each } (s, k) \in \Delta \text{ such that } (i, j) \in U_k^s\}$, and let $A''' = I \times J \setminus (A' \cup A'')$. Also put $a' = \chi_{A'} a$, $a'' = \chi_{A''} a$ and $a''' = \chi_{A'''} a$.

If $(i,j) \in A''$ then

$$a(i, j) \leqslant m^{-1} \sum_{(u,v) \in A'} a(u, v).$$

Otherwise we could add (i, j) to the set A' if $|A'| < m$, or replace by (i,j) an element (u, v) in A' for which $a(u, v) < a(i,j)$ if $|A'| = m$, and for the new set $\hat{A}$ we would have $\hat{A} \in \mathcal{A}$ and

$$\sum_{(u,v) \in \hat{A}} a(u, v) > \sum_{(u,v) \in A'} a(u, v)$$

contrary to the definition of A'. Hence we have for all $(i,j) \in I \times J$

$$a''(i, j) \leqslant m^{-1} \sum_{(u,v) \in A'} a(u, v).$$

For the matrix a''' we have the following estimate:

$$a'''(i,j) \leqslant \sum_{(s,k) \in \Delta} \chi_{U_k^s}(i,j) \frac{1}{r_s} \sum_{(u,v) \in A' \cap U_k^s} a(u, v).$$

This is true because otherwise, if for each $(s, k) \in \Delta$ such that $(i,j) \in U_k^s$ and $|A' \cap U_k^s| = r_s$ we delete from the set $A' \cap U_k^s$ an element (u, v) for which $a(u, v) < a(i, j)$ and if we replace all the deleted elements by the single (i,j) then the new set $\hat{A}$ would be in $\mathcal{A}$ and contrary to the definition of A'

$$\sum_{(u,v) \in \hat{A}} a(u, v) > \sum_{(u,v) \in A'} a(u, v).$$

Taking into account these estimates of a'' and a''' we obtain

$$\operatorname*{Ave}_g \max_{i \in I} a''(i, g(i)) \leqslant m^{-1} \sum_{(i,j) \in A'} a(i,j),$$

$$\operatorname*{Ave}_g \max_{i \in I} a'''(i, g(i)) \leqslant \sum_{(s,k) \in \Delta} \frac{1}{r_s} \Big(\sum_{(u,v) \in A' \cap U_k^s} a(u,v) \Big) \operatorname*{Ave}_g \max_{i \in I} \chi_{U_k^s}(i, g(i))$$
$$\leqslant \sum_{s=1}^{c} \sum_{k \in K_s} \frac{1}{r_s} \Big(\sum_{(u,v) \in A' \cap U_k^s} a(u,v) \Big) \frac{r_s}{m} \leqslant \frac{c}{m} \sum_{(u,v) \in A'} a(u, v).$$

The average with the matrix a' is estimated by

$$\operatorname*{Ave}_g \max_{i \in I} a'(i,j) \leqslant \sum_{(i,j) \in A'} a(i,j)\,\mu(g(i) = j) = m^{-1} \sum_{(i,j) \in A'} a(i, j).$$

The above inequalities and the obvious one

$$\operatorname*{Ave}_g \max_{i \in I} a(i, g(i)) \leqslant \operatorname*{Ave}_g \max_{i \in I} a'(i, g(i)) + \operatorname*{Ave}_g \max_{i \in I} a''(i, g(i)) + \operatorname*{Ave}_g \max_{i \in I} a'''(i, g(i))$$

conclude the proof. ∎

Remark 1.2. If $c = 0$, i.e. the family $\mathcal{U}$ is empty, then $\mathcal{A} = \{A \subset I \times J\colon |A| \leqslant m\}$. Lemma 1.1 gives then

$$\operatorname*{Ave}_g \max_{i \in I} a(i, g(i)) \leqslant \frac{2}{m} \sum_{k=1}^{m} s(k)$$

where $(s(k))_{k=1}^{nm}$ is the nonincreasing rearrangement of the numbers $a(i,j)$, $i\in I$, $j\in J$. A proof similar to the one in Theorem 1.1 of [1] allows us to reduce the constant 2 to 1 in the above inequality.

Now we will give some estimates from below. For each $0<r\leqslant 1$ and $(i,j)\in I\times J$ let

$$V_{i,j}^r=\{(k,l)\in I\times J:\ \mu(g(i)=j,g(k)=l)=r/m\}.$$

Let $t\colon (0,1]\to \mathbf{R}^+$ be a fixed function such that $t(r)\neq 0$ only for finitely many r, and let $d=\sum_r rt(r)$. With t we associate a class of subsets of $I\times J$ as follows:

$$\mathscr{B}=\{B\subset I\times J:\ |B|\leqslant m,\ |B\cap V_{i,j}^r|\leqslant t(r) \quad \text{for each } (i,j)\in I\times J \text{ and } r\in(0,1]\}.$$

LEMMA 1.3. *Under the above notation we have for each nonnegative matrix* $(a(i,j))_{I\times J}$

$$d^{-1}\max_{B\in\mathscr{B}} m^{-1}\sum_{(i,j)\in B} a(i,j)\leqslant \operatorname*{Ave}_g\max_{i\in I} a(i,g(i)).$$

Proof. Observe that the expression $\operatorname{Ave}_g \max_{i\in I}|a(i,g(i))|$ defines a norm on the space of all matrices $(a(i,j))_{I\times J}$. If a matrix $(a(i,j))_{I\times J}$ is an extreme point of the unit ball of this norm then there exist a subset $C\subset I\times J$ and a number b such that $|a(i,j)|=b\chi_C(i,j)$ for $(i,j)\in I\times J$. Therefore it is enough to prove the inequality of Lemma 1.3 for matrices $a=\chi_C$ where $C\subset I\times J$. If $B\subset I\times J$ then by the Schwarz inequality we obtain

$$\begin{aligned} m^{-1}\sum_{(i,j)\in B}\chi_C(i,j)&=\int_G\sum_{i\in I}\chi_{B\cap C}(i,g(i))\,\mu(dg)\\ &\leqslant\Big(\int_G\Big(\sum_{i\in I}\chi_{B\cap C}(i,g(i))\Big)^2\mu(dg)\Big)^{1/2}\\ &\quad\times\Big(\int_G\max_{i\in I}\chi_{B\cap C}(i,g(i))\,\mu(dg)\Big)^{1/2}.\end{aligned}$$

If $B\in\mathscr{B}$ then

$$\begin{aligned}\int_G\Big(\sum_{i\in I}\chi_{B\cap C}(i,g(i))\Big)^2\mu(dg)&=\sum_{(i,j),(k,l)\in B\cap C}\mu(g(i)=j,g(k)=l)\\ &=\sum_{(i,j)\in B\cap C}\sum_r\frac{r}{m}|B\cap C\cap V_{i,j}^r|\leqslant\sum_{(i,j)\in B\cap C}t(r)r/m\leqslant\frac{d}{m}\sum_{(i,j)\in B}\chi_C(i,j).\end{aligned}$$

Combining these inequalities with the obvious inequality

$$\int_G\max_{i\in I}\chi_{B\cap C}(i,g(i))\,\mu(dg)\leqslant\operatorname*{Ave}_g\max_{i\in I}\chi_C(i,g(i))$$

we get

$$m^{-1}\sum_{(i,j)\in B}\chi_C(i,j)\leqslant d\operatorname*{Ave}_g\max_{i\in I}\chi_C(i,g(i)).$$

Since B is an arbitrary set in $\mathscr{B}$ this proves the lemma. ∎

Of course the most interesting case is when the estimates from above and from below coincide up to a constant, in other words when we can find a family $\mathscr{U}$ and a function t such that the associated classes $\mathscr{A}$, $\mathscr{B}$ fulfill $\mathscr{A}\subset\mathscr{B}$. In general this requires much more assumptions on the structure of G. We will show that it is so in several cases which are shown in the following examples:

EXAMPLE 1.4. Let $I=J=\{1,\ldots,n\}$ and let G be the set of all permutations of I. The family $\mathscr{U}$ is defined to be empty and the function t is defined by $t(1)=1$, $t(1/(n-1))=n$ and $t(r)=0$ for all other r. In this case $\mathscr{A}=\mathscr{B}=\{A\in I\times I:\ |A|\leqslant n\}$, $d=2+1/(n-1)$. Lemmas 1.1 and 1.3 give

$$\Big(2+\frac{1}{n-1}\Big)^{-1}\frac{1}{n}\sum_{k=1}^n s(k)\leqslant\operatorname*{Ave}_g\max_i a(i,g(i))\leqslant\frac{2}{n}\sum_{k=1}^n s(k)$$

where $s(k)$, $k=1,\ldots,n^2$, is the nonincreasing rearrangement of $a(i,j)$, $i,j=1,\ldots,n$.

EXAMPLE 1.5. Let $I=J=\{1,\ldots,n\}$ and let G be the set of maps of I into itself which are of the form $g(i)=(i+k)\bmod n$ where k is an integer, fixed for g. The function t is given by $t(1)=1$ and $t(r)=0$ for $r\neq 1$. The family $\mathscr{U}=(U_k^1)_{k=1}^n$ is defined by $U_k^1=\{(i,j)\in I\times I:\ i-j=k \bmod n\}$. It is easy to check that $\mathscr{A}=\mathscr{B}$ in this case. However, the estimates for the average are trivial in this case.

EXAMPLE 1.6. Let $I=J=\{1,\ldots,n\}\times\{1,\ldots,n\}$ and let G be the set of all maps of the form $g(i,j)=(\pi(i),\sigma(j))$ where π and σ are permutations of $\{1,\ldots,n\}$. The family $\mathscr{U}$ is defined as follows: $\Delta=\{1,2\}\times I$, $U_{i,k}^1=\{(i,j,k,l):\ j,l=1,\ldots,n\}$ for each $(i,k)\in I$, and $U_{j,l}^2=\{(i,j,k,l):\ i,k=1,\ldots,n\}$ for $(j,l)\in I$. Moreover, let t be defined by $t(1)=1$, $t(1/(n-1)^2)=n^2$, $t(1/(n-1))=n$ and $t(r)=0$ for other r. It is easy to verify that $\mathscr{A}\subset\mathscr{B}$ in this case and

$$d=1+\frac{n}{n-1}+\Big(\frac{n}{n-1}\Big)^2.$$

Thus Lemmas 1.1 and 1.3 give

$$\begin{aligned}\frac{1}{7n^2}\max_A\sum_{(i,j,k,l)\in A}a(i,j,k,l)&\leqslant\operatorname*{Ave}_{\pi,\sigma}\max_{i,j}a(i,j,\pi(i),\sigma(i))\\ &\leqslant\frac{4}{n^2}\max_A\sum_{(i,j,k,l)\in A}a(i,j,k,l)\end{aligned}$$

where $A \subset I \times J$ with $|A| \leqslant n^2$ and $|A \cap U^1_{i,k}| \leqslant n$ and $|A \cap U^2_{j,l}| \leqslant n$ for each $(i,k),(j,l) \in I$.

The following corollary is of fundamental importance for the remaining part of this paper and is the main purpose of this section.

COROLLARY 1.7. *Let $a(i,j,k)$, $i,j,k = 1, \ldots, n$, be nonnegative numbers. Then we have*

$$\left(1+\left(\frac{n}{n-1}\right)^2\right)^{-1} \frac{1}{n^2}\sum_{k=1}^{n^2} s(k) \leqslant \operatorname*{Ave}_{\pi,\sigma} \max_{i} a(i,\pi(i),\sigma(i)) \leqslant \frac{1}{n^2}\sum_{k=1}^{n^2} s(k)$$

where the average is over all permutations π, σ of $\{1, \ldots, n\}$ and $(s(k))_{k=1}^{n^3}$ is the nonincreasing rearrangement of the numbers $a(i,j,k)$, $i,j,k = 1, \ldots, n$.

Proof. Let $I = \{1, \ldots, n\}$, $J = I \times I$ and let G be the set of all maps of I into J of the form $g(i) = (\pi(i), \sigma(i))$ where π and σ are permutations of $\{1, \ldots, n\}$. Let $\mathscr{U}$ be the empty family. The function t is defined by $t(1) = 1$, $t(1/(n-1)^2) = n^2$ and $t(r) = 0$ for the remaining r. The associated families $\mathscr{A}$ and $\mathscr{B}$ are both equal to $\{A \subset I \times J\colon |A| \leqslant n^2\}$, and $d = 1 + (n/(n-1))^2$. By Lemmas 1.1 and 1.3 we conclude the proof. The constant 1 instead of 2 on the right side of the inequality is obtained by Remark 1.2. ■

2. Orlicz spaces and averages over permutations. The following are the main results of this section.

THEOREM 2.1. *Let $a, b \in \mathbf{R}^n$ with $\|a\|_1 = \|b\|_1 = n$ and $a_1 \geqslant \ldots \geqslant a_n > 0$, $b_1 \geqslant \ldots \geqslant b_n > 0$, and let M be an Orlicz function such that the conjugate function M^* satisfies*

$$M^*\left(n^{-2}\sum_{k=1}^{l} s(k)\right) = l/n^2, \quad l = 1, \ldots, n^2,$$

where $s(k)$, $k = 1, \ldots, n^2$, is the nonincreasing rearrangement of $|a_i b_j|$, $i,j = 1, \ldots, n$. Then for all $x \in \mathbf{R}^n$ we have

$$\frac{1}{2}\left(\frac{1}{2}-\frac{1}{n-1}\right)\|x\|_M \leqslant \operatorname*{Ave}_{\pi,\sigma} \max_{i} |x_i a_{\pi(i)} b_{\sigma(i)}| \leqslant 2\|x\|_M.$$

THEOREM 2.2. *Let $a \in \mathbf{R}^n$ with $\|a\|_1 = n$ and $a_1 \geqslant \ldots \geqslant a_n > 0$, and let M be an Orlicz function. Let N be an Orlicz function satisfying*

$$N^*\left(n^{-2}\sum_{k=1}^{l} s(k)\right) = l/n^2, \quad l = 1, \ldots, n^2,$$

where $s(k)$, $k = 1, \ldots, n^2$, is the nonincreasing rearrangement of

$$\left|a_i n\left(M^{*-1}\left(\frac{j}{n}\right) - M^{*-1}\left(\frac{j-1}{n}\right)\right)\right|, \quad i,j = 1, \ldots, n.$$

Then we have

$$\frac{1}{4}\left(\frac{1}{2}-\frac{1}{n-1}\right)\|x\|_N \leqslant \operatorname*{Ave}_{\pi} \|(x_i a_{\pi(i)})_{i=1}^n\|_M \leqslant 8\left(1+\frac{2}{n-3}\right)\|x\|_N.$$

COROLLARY 2.3. (i) *Let $1 \leqslant p \leqslant \infty$, $a \in \mathbf{R}^n$ with $\|a\|_1 = n$ and $a_1 \geqslant \ldots \geqslant a_n > 0$. Then there is an Orlicz function N with*

$$\frac{1}{5}N^{*-1}\left(\frac{l}{n}\right) \leqslant \frac{1}{n}\left\{\sum_{i=1}^{l} a_i + l^{1/p'}\left(\sum_{i=l+1}^{n} a_i^p\right)^{1/p}\right\} \leqslant 2N^{*-1}\left(\frac{l+1}{n}\right), \quad l = 1, \ldots, n,$$

such that for all $x \in \mathbf{R}^n$

$$\frac{1}{4}\left(\frac{1}{2}-\frac{1}{n-1}\right)\|x\|_N \leqslant \operatorname*{Ave}_{\pi} \|(x_i a_{\pi(i)})_{i=1}^n\|_p \leqslant 8\left(1+\frac{2}{n-1}\right)\|x\|_N.$$

(ii) *For each $1 \leqslant p \leqslant \infty$ and all $x, a \in \mathbf{R}^n$ we have*

$$\frac{1}{5}n^{-1}\sum_{k=1}^{n} s(k) + \left(n^{-1}\sum_{k=n+1}^{n^2} s(k)^p\right)^{1/p} \leqslant \operatorname*{Ave}_{\pi} \|(x_i a_{\pi(i)})_{i=1}^n\|_p$$
$$\leqslant n^{-1}\sum_{k=1}^{n} s(k) + \left(n^{-1}\sum_{k=n+1}^{n^2} s(k)^p\right)^{1/p}$$

where $s(k)$, $k = 1, \ldots, n^2$, is the nonincreasing rearrangement of $|x_i a_j|$, $i,j = 1, \ldots, n$.

We start the proofs of these theorems by the following lemma.

LEMMA 2.4. *Let $a_1 \geqslant \ldots \geqslant a_s > 0$ with $\sum_{i=1}^{s} a_i = 1$, $n \leqslant s$ and*

$$\|x\|_E = \max_{\sum k_j = s} \sum_{i=1}^{n}\left(\sum_{j=1}^{k_i} a_j\right)|x_i|.$$

If M is an Orlicz function such that $M^*(\sum_{j=1}^{l} a_j) = l/s$, $l = 1, \ldots, s$, then

$$\tfrac{1}{2}\|x\|_E \leqslant \|x\|_M \leqslant 2\|x\|_E \quad \textit{for all } x \in \mathbf{R}^n.$$

Proof. Let $|||\ |||$ denote the norm dual to $\|\ \|_{M^*}$. We have

$$\|x\|_M \leqslant |||x||| \leqslant 2\|x\|_M \quad \text{(cf. [2], p. 147).}$$

We show the right-hand inequality of the lemma. Suppose that $x \in \mathbf{R}^n$ is such that $x_1 \geqslant \ldots \geqslant x_n > 0$ and $\sum_{i=1}^{n} M^*(x_i) = 1$. Then there are nonnegative integers k_i such that

$$\sum_{j=1}^{k_i} a_j \leqslant x_i \leqslant \sum_{j=1}^{k_i+1} a_j \quad \text{for } i = 1, \ldots, n.$$

We get

$$1=\sum_{i=1}^{n} M^*(x_i)\geqslant \sum_{i=1}^{n} M^*\Big(\sum_{j=1}^{k_i} a_j\Big)=\sum_{i=1}^{n} k_i/s$$

so that $\sum_{i=1}^{n} k_i\leqslant s$. Thus $x_i\leqslant a_1+\sum_{j=1}^{k_i}a_j$ for $i=1,\dots,n$. We have $(a_1,a_1,\dots,a_1)\in B_{E^*}$ because $n\leqslant s$, and also $(\sum_{j=1}^{k_i}a_j)_{i=1}^{n}$ is in B_{E^*} because $\sum k_i\leqslant s$. Hence $x\in 2B_{E^*}$ or $B_{M^*}\subset 2B_{E^*}$. Thus $\|x\|_E\geqslant \frac{1}{2}|||x|||\geqslant \frac{1}{2}\|x\|_M$. ■

Proof of Theorem 2.1. We apply Corollary 1.7 and Lemma 2.4. We have

$$n^{-2}\sum_{k=1}^{n^2} s(k)=n^{-2}\sum_{i=1}^{n}|x_i|\sum_{(j,k)\in M_i}|a_j b_k|$$

where $\sum_{i=1}^{n}|M_i|=n^2$ and the sets M_i, $i=1,\dots,n$, are chosen in such a way that the sum is maximal, i.e.

$$\sum_{(j,k)\in M_i}|a_j b_k|=\sum_{k=1}^{|M_i|} s(k).\ ■$$

Lemma 2.5. *Let M be an Orlicz function. Then we have for all $x\in \mathbf{R}^n$*

$$\frac{1}{2}\Big(\frac{1}{2}-\frac{1}{n-1}\Big)\|x\|_M\leqslant \operatorname*{Ave}_{\pi}\max_{1\leqslant i\leqslant n}\Big|x_i n\Big(M^{*-1}\Big(\frac{\pi(i)}{n}\Big)-M^{*-1}\Big(\frac{\pi(i)-1}{n}\Big)\Big)\Big|\leqslant 2\|x\|_M.$$

Proof. This is a consequence of Theorem 2.1. We put

$$b=(1,\dots,1),\qquad a_i=n\Big(M^{*-1}\Big(\frac{i}{n}\Big)-M^{*-1}\Big(\frac{i-1}{n}\Big)\Big),\qquad i=1,\dots,n.\ ■$$

Proof of Theorem 2.2. We apply Theorem 2.1 and Lemma 2.5. By Lemma 2.5 we have for all $x\in \mathbf{R}^n$

$$\frac{1}{2}\Big(\frac{1}{2}-\frac{1}{n-1}\Big)\operatorname*{Ave}_{\pi}\|(x_i a_{\pi(i)})_{i=1}^{n}\|_M$$
$$\leqslant \operatorname*{Ave}_{\pi,\sigma}\max_{1\leqslant i\leqslant n}\Big|x_i a_{\pi(i)} n\Big(M^{*-1}\Big(\frac{\sigma(i)}{n}\Big)-M^{*-1}\Big(\frac{\sigma(i)-1}{n}\Big)\Big)\Big|$$
$$\leqslant 2\operatorname*{Ave}_{\pi}\|(x_i a_{\pi(i)})_{i=1}^{n}\|_M.$$

Now it is enough to apply Theorem 2.1 to conclude the proof. ■

Proof of Corollary 2.3. We apply Theorem 2.2 to $M(t)=t^p$. We have

$$M^*(t)=\Big|\frac{t}{p^{1/p}(p')^{1/p'}}\Big|^{p'}\quad\text{or equivalently}\quad M^{*-1}(t)=p^{1/p}(p')^{1/p'}t^{1/p'}.$$

We have to estimate

$$n^{-2}\sum_{k=1}^{l} s(k)=\max_{k_i} n^{-2}\sum_{i=1}^{n} a_i\sum_{j=1}^{k_i} n\Big(M^{*-1}\Big(\frac{j}{n}\Big)-M^{*-1}\Big(\frac{j-1}{n}\Big)\Big)$$
$$=\max_{k_i} n^{-1}\sum_{i=1}^{n} a_i M^{*-1}(k_i/n)=\max_{k_i} n^{-1}p^{1/p}(p')^{1/p'}\sum_{i=1}^{n} a_i(k_i/n)^{1/p'}.$$

Here and below, $\max_{k_i}$ denotes the maximum over all sequences (k_i) with $k_i\leqslant n$ and $\Sigma k_i=l$. Therefore we get

$$n^{-2}\sum_{k=1}^{l} s(k)\leqslant 2n^{-1}\Big\{\sum_{i\leqslant l/n} a_i+\max_{k_i}\sum_{i>l/n} a_i(k_i/n)^{1/p'}\Big\}.$$

Hence by the Hölder inequality we deduce that the last expression is less than or equal to

$$2n^{-1}\Big\{\sum_{i\leqslant l/n} a_i+(l/n)^{1/p'}\Big(\sum_{i>l/n} a_i^p\Big)^{1/p}\Big\}.$$

On the other hand, choosing $k_i=n$ for $i\leqslant l/n$ we get

$$n^{-2}\sum_{k=1}^{l} s(k)\geqslant n^{-1}\sum_{i\leqslant l/n} a_i.$$

Now we consider instead of a the vector $\tilde a$ with $\tilde a_1=\ldots=\tilde a_{[l/n]+1}=a_{[l/n]+1}$ and $\tilde a_i=a_i$ for $i>[l/n]+1$. We have

$$n^{-2}\sum_{k=1}^{l} s(k)\geqslant n^{-1}\max_{k_i}\sum_{i=1}^{n}\tilde a_i(k_i/n)^{1/p'}.$$

Since $\tilde a_1=\ldots=\tilde a_{[l/n]+1}$ and $(\tilde a_i)$ is decreasing, we know that the maximum $n^{-1}\max_{\Sigma k_i=l}\sum_{i=1}^{n}\tilde a_i(k_i/n)^{1/p'}$ is attained for a sequence (k_i) which is decreasing and $k_1-1\leqslant k_{[l/n]+1}$. This implies that $k_i\leqslant n$ for all $i=1,\dots,n$. Therefore we may write

$$n^{-2}\sum_{k=1}^{l} s(k)\geqslant n^{-1}\max_{\Sigma k_i=l}\sum_{i=1}^{n}\tilde a_i(k_i/n)^{1/p'}.$$

Now if we apply Lemma 2.4 to the right-hand expression we obtain for $n\leqslant l\leqslant n^2$

$$n^{-2}\sum_{k=1}^{l} s(k)\geqslant \tfrac{1}{4}n^{-1}(l/n)^{1/p'}\|\tilde a\|_p\geqslant \tfrac{1}{4}n^{-1}(l/n)^{1/p'}\Big(\sum_{i>l/n} a_i^p\Big)^{1/p}.$$

Altogether we obtain

$$\tfrac{1}{2}n^{-2}\sum_{k=1}^{l} s(k)\leqslant n^{-1}\Big\{\sum_{i\leqslant l/n} a_i+(l/n)^{1/p'}\Big(\sum_{i>l/n} a_i^p\Big)^{1/p}\Big\}\leqslant 5n^{-2}\sum_{k=1}^{l} s(k),$$

for $n \leq l \leq n^2$. Now it is enough to substitute ln instead of l in the above inequalities to conclude that the Orlicz function N from Theorem 2.2 satisfies the required condition.

(ii) can be derived from the previous estimates as well. Since we already obtained these inequalities in [1] we refrain from doing it. ∎

3. Matrix subspaces of L^1. The following theorem is useful for showing that some spaces are not isomorphic to subspaces of L^1.

THEOREM 3.1. *Let $\{e_{i,j}\}_{i,j}^n$ be a 1-unconditional basis of E. Then for each n^2-dimensional subspace G of L^1 we have*

$$d(E,G) \geq \frac{1}{5\sqrt{2}} \sup_{\substack{a\neq 0\\ a\in R^n}} \frac{\operatorname{Ave}_i \left\| \sum_{j=1}^n a_j e_{i,j} \right\|}{\operatorname{Ave}_\pi \left\| \sum_{j=1}^n a_j e_{\pi(j),j} \right\|}.$$

Here it is crucial that the average over permutations is applied to the first index of basis vectors.

LEMMA 3.2. *Let $(a_{i,j})_{n\times n}$ be a matrix and $s(k)$, $k = 1, \ldots, n^2$, the nonincreasing rearrangement of $|a_{i,j}|$, $i, j = 1, \ldots, n$. Then we have*

$$\operatorname{Ave}_i \left(\sum_{j=1}^n a_{i,j}^2 \right)^{1/2} \leq n^{-1} \sum_{k=1}^n s(k) + \left(n^{-1} \sum_{k=n+1}^{n^2} s(k)^2 \right)^{1/2}.$$

Proof. By the Schwarz inequality we get

$$\operatorname{Ave}_i \left(\sum_{j=1}^n a_{i,j}'^2 \right)^{1/2} \leq \left(\operatorname{Ave}_i \sum_{j=1}^n a_{i,j}'^2 \right)^{1/2} = \left(n^{-1} \sum_{k=n+1}^{n^2} s(k)^2 \right)^{1/2}$$

where $a'_{i,j} = a_{i,j}$ if $|a_{i,j}|$ in the nonincreasing rearrangement $s(k)$, $k = 1, \ldots, n^2$, has index greater than n, and $a'_{i,j} = 0$ otherwise. Also we have

$$\operatorname{Ave}_i \left(\sum_{j=1}^n a_{i,j}''^2 \right)^{1/2} \leq \operatorname{Ave}_i \sum_{j=1}^n |a''_{i,j}| = n^{-1} \sum_{k=1}^n s(k)$$

where $a''_{i,j} = a_{i,j}$ if in the nonincreasing rearrangement $s(k)$, $k = 1, \ldots, n^2$, $|a_{ij}|$ has index less than or equal to n, and $a''_{ij} = 0$ otherwise. Now the proof follows from the triangle inequality

$$\operatorname{Ave}_i \left(\sum_{j=1}^n a_{i,j}^2 \right)^{1/2} \leq \operatorname{Ave}_i \left(\sum_{j=1}^n a_{i,j}'^2 \right)^{1/2} + \operatorname{Ave}_i \left(\sum_{j=1}^n a_{i,j}''^2 \right)^{1/2}. \ ∎$$

Proof of Theorem 3.1. Let $S \in L(E, G)$, $G \subset l^1$. Then we have for $S(e_{i,j}) = x_{i,j}$, $i, j = 1, \ldots, n$,

$$\|S\| \operatorname{Ave}_\pi \left\| \sum_{j=1}^n a_j e_{\pi(j),j} \right\| = \|S\| \operatorname{Ave}_\pi \operatorname{Ave}_\varepsilon \left\| \sum_{j=1}^n a_j \varepsilon_j e_{\pi(j),j} \right\|$$

$$\geq \operatorname{Ave}_\pi \operatorname{Ave}_\varepsilon \sum_{k=1}^\infty \left| \sum_{j=1}^n a_j \varepsilon_j x_{\pi(j),j}(k) \right|$$

where the average over ε is extended over all sequences $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ of ± 1. By the Khinchin inequality we estimate the last expression by

$$\frac{1}{\sqrt{2}} \sum_{k=1}^\infty \operatorname{Ave}_\pi \left(\sum_{j=1}^n |a_j x_{\pi(j),j}(k)|^2 \right)^{1/2}.$$

Now we apply Corollary 2.3(ii) and Lemma 3.2 to the matrices $(a_j x_{i,j}(k))_{i,j}^n$, $k = 1, 2, \ldots$, and we obtain

$$\begin{aligned}
\|S\| \operatorname{Ave}_\pi \left\| \sum_{j=1}^n a_j e_{\pi(j),j} \right\| &\geq \frac{1}{5\sqrt{2}} \sum_{k=1}^\infty \operatorname{Ave}_i \left(\sum_{j=1}^n |a_j x_{i,j}(k)|^2 \right)^{1/2} \\
&\geq \frac{1}{5\sqrt{2}} \sum_{k=1}^\infty \operatorname{Ave}_i \operatorname{Ave}_\varepsilon \left| \sum_{j=1}^n a_j \varepsilon_j x_{i,j}(k) \right| \\
&= \frac{1}{5\sqrt{2}} \operatorname{Ave}_i \operatorname{Ave}_\varepsilon \left\| \sum_{j=1}^n a_j \varepsilon_j x_{i,j} \right\| \\
&\geq \frac{1}{\|S^{-1}\|} \frac{1}{5\sqrt{2}} \operatorname{Ave}_i \operatorname{Ave}_\varepsilon \left\| \sum_{j=1}^n a_j \varepsilon_j e_{i,j} \right\| \\
&= \frac{1}{5\sqrt{2}} \frac{1}{\|S^{-1}\|} \operatorname{Ave}_i \left\| \sum_{j=1}^n a_j e_{i,j} \right\|
\end{aligned}$$

and this proves the theorem. ∎

Let $\{e_i\}_{i=1}^n$ and $\{f_j\}_{j=1}^n$ be 1-unconditional bases of E and F respectively. Then we define the space $E(F)$ to be the space of all matrices $(a_{i,j})_{n\times n}$ with the norm

$$\left\| \sum_{i,j=1}^n a_{i,j} e_i \otimes f_j \right\| = \left\| \sum_{i=1}^n \left\| \sum_{j=1}^n a_{ij} f_j \right\| e_i \right\|.$$

As an easy consequence of Theorem 3.1 we obtain the following

COROLLARY 3.3. *Let $\{e_i\}_{i=1}^n$ be a 1-symmetric basis of E and $\{f_j\}_{j=1}^n$ a 1-unconditional basis of F. Let $\mathrm{Id} \in L(E, F)$ be the natural identity map, i.e. $\mathrm{Id}(\sum_{i=1}^n a_i e_i) = \sum_{j=1}^n a_j f_j$. Then for each n^2-dimensional subspace G of L^1 we have*

$$\frac{1}{5\sqrt{2}} \|\mathrm{Id}\| \leq d(E(F), G).$$

COROLLARY 3.4. *Let $1 \leq p < r \leq 2$, $n \in N$.*

(i) *If G is an n^2-dimensional subspace of L^1 then we have*

$$d\big(l_n^r(l_n^p), G\big) \geqslant \frac{1}{5\sqrt{2}} n^{1/p-1/r}.$$

(ii) $l_n^p(l_n^r)$ *is isometric to a subspace of* L^1.

4. Symmetric subspaces of L^1. We give here a necessary condition for a space with a symmetric basis to be a subspace of L^1. First, we introduce some notation. Suppose $\{e_i\}_{i=1}^{\infty}$ is a 1-symmetric basis of a space E. Then there is an Orlicz function M_E such that

$$\Big\|\sum_{i=1}^{k} e_i\Big\|_E \leqslant \Big\|\sum_{i=1}^{k} e_i\Big\|_{M_E} \leqslant 2\Big\|\sum_{i=1}^{k} e_i\Big\|_E, \quad k = 1, 2, \ldots$$

We say that M_E is *associated to E*. M_E is certainly not unique but norms given by different associated functions are equivalent. In order to see that such a function exists we use a result of Zippin (cf. [2], Proposition 3.a.7).

LEMMA 4.1. *Let B be a Banach space with a 1-symmetric basis $\{e_i\}_{i=1}^{\infty}$. Then there exists a new norm $\|\ \|_0$ on E such that:*

(i) $\|x\| \leqslant \|x\|_0 \leqslant 2\|x\|$ *for all* $x \in E$.

(ii) *The symmetric constant of $\{e_i\}_{i=1}^{\infty}$ with respect to $\|\ \|_0$ is equal to* 1.

(iii) *If we put $\lambda_0(k) = \|\sum_{i=1}^{k} e_i\|_0$, $k = 1, 2\ldots$ then the sequence $(\lambda_0(k+1) - \lambda_0(k))_{k=1}^{\infty}$ is nonincreasing, i.e. $\lambda_0(\cdot)$ is a concave function on the positive integers.*

Now let us define an Orlicz function M as follows: $M\big(1/\lambda_0(k)\big) = 1/k$, $k = 1, 2, \ldots$, and for the other values the function is extended linearly. Then by (iii), M is convex and $\|\sum_{i=1}^{k} e_i\|_0 = \|\sum_{i=1}^{k} e_i\|_M$.

THEOREM 4.2. *Let E be a subspace of L^1 with 1-symmetric basis $\{e_i\}_{i=1}^{n}$ and let M_E be an Orlicz function associated to E. Then for all $x \in \boldsymbol{R}^n$ we have*

$$\Big\|\sum_{i=1}^{n} x_i e_i\Big\|_{M_E} \leqslant C\Big\|\sum_{i=1}^{n} x_i e_i\Big\|_E$$

where C is a universal constant.

Since $M_{l^{p,q}}(t) \sim t^p$, by Theorem 4.2 we immediately obtain

COROLLARY 4.3. *Let $1 \leqslant p \leqslant 2$ and $p < q$. Then the Lorentz space $l^{p,q}$ does not embed into L^1.*

LEMMA 4.4. *Let E be a subspace of L^1 with 1-symmetric basis $\{e_i\}_{i=1}^{n}$. Then there are numbers $a_j > 0$ and Orlicz functions M_j, $j = 1, \ldots, N$, with $\sum_{j=1}^{N} a_j = 1$ such that for all $x \in \boldsymbol{R}^n$*

$$C^{-1}\Big\|\sum_{i=1}^{n} x_i e_i\Big\|_E \leqslant \sum_{j=1}^{N} a_j \|x\|_{M_j} \leqslant C\Big\|\sum_{i=1}^{n} x_i e_i\Big\|_E$$

where C is a universal constant.

This result was proved in [1] but it is also an easy consequence of Theorem 2.2.

The following lemma follows from the definition of norms in Orlicz spaces and from the definition of the conjugate Orlicz function.

LEMMA 4.5. *Let M be an Orlicz function and M^* its dual. Then we have*

(i) $M^{-1}(1/k) = \Big\|\sum_{i=1}^{k} e_i\Big\|_M^{-1}$, $\quad k = 1, 2, \ldots,$

(ii) $\frac{1}{2} k M^{*-1}(1/k) \leqslant 1/M^{-1}(1/k) \leqslant k M^{*-1}(1/k)$, $\quad k = 1, 2, \ldots$

Proof of Theorem 4.2. By Lemma 4.4 it is enough to consider spaces with the norm given by $\|x\|_E = \sum_{j=1}^{N} a_j \|x\|_{M_j}$. By Lemma 4.5 we get for M_E

$$\tfrac{1}{2} k M_E^{*-1}(1/k) \leqslant 1/M_E^{-1}(1/k) \leqslant 2\sum_{j=1}^{N} a_j/M_j^{-1}(1/k)$$

$$\leqslant 2\sum_{j=1}^{N} a_j k M_j^{*-1}(1/k) \leqslant 4\sum_{j=1}^{N} a_j/M_j^{-1}(1/k)\ 4M_E^{-1}(1/k) \leqslant 4kM_E^{*-1}(1/k).$$

Thus we have

$$\tfrac{1}{4} M_E^{*-1}(1/k) \leqslant \sum_{j=1}^{N} a_j M_j^{*-1}(1/k) \leqslant 2M_E^{*-1}(1/k), \quad k = 1, 2, \ldots$$

Now we show that $B_{M_E^*} \subset 32 B_{E^*}$ and this is enough to prove the theorem. Clearly it is sufficient to consider vectors with positive coordinates. Every such vector in $B_{M_E^*}$ may be written as $(M^{*-1}(z_i))_{i=1}^{n}$ with $z_i > 0$ and $\sum_{i=1}^{n} z_i \leqslant 1$. There are positive integers k_i such that $2 \leqslant k_i \leqslant n$ and $1/k_i < z_i \leqslant 2/k_i$ or $z_i \leqslant 1/k_i = 1/n$, $i = 1, \ldots, n$. Because $\|(M^{*-1}(z_i))_{i=1}^{n}\|_M \leqslant 1$ we deduce that $\sum_{i=1}^{n} 1/k_i \leqslant 2$. By the above inequality this yields

$$M_E^{*-1}(z_i) \leqslant 4\sum_{j=1}^{N} a_j M_j^{*-1}(2/k_i).$$

Now we conclude the proof as follows:

$$\|(M_E^{*-1}(z_i))_{i=1}^{n}\|_{E^*} = \sup_{\|x\|_E=1} |\langle x, (M_E^{*-1}(z_i))_{i=1}^{n}\rangle|$$

$$\leqslant 4\sup_{\|x\|_E=1} \Big|\Big\langle x, \Big(\sum_{j=1}^{N} a_j M_j^{*-1}(2/k_i)\Big)_{i=1}^{n}\Big\rangle\Big|$$

$$= 4 \sup_{\|x\|_E = 1} \sum_{j=1}^{N} a_j |\langle x, (M_j^{*-1}(2/k_i))_{i=1}^n \rangle|$$

$$\leqslant 8 \sup_{\|x\|_E = 1} \sum_{j=1}^{N} a_j \|x\|_{M_j} \|(M_j^{*-1}(2/k_i))_{i=1}^n\|_{M_j^*} \leqslant 32.$$

The last inequality follows from the inequality $\sum_{i=1}^{n} 1/k_i \leqslant 2$. ■

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Functional calculi for pseudodifferential operators, III

by

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Abstract. We construct a selfadjoint algebra of L^p-bounded pseudodifferential operators of nonpositive order, acting on functions defined on a compact manifold without boundary. Then, using the Weyl formula, we give a meaning to $f(A_1, \ldots, A_r)$ for functions f with a prescribed finite number of derivatives, when $(A_1, \ldots, A_r)$ is an r-tuple of selfadjoint, commuting or noncommuting operators in the algebra.

§0. Introduction. This is the last paper in a series (cf. [1]–[4]) that studies functions of several commuting and noncommuting selfadjoint pseudodifferential operators of nonpositive order, by means of the Hermann Weyl formula (cf. [14], [11], [12], [5])

$$f(A_1, \ldots, A_r) = \int_{R^r} e^{-2\pi i(t_1 A_1 + \ldots + t_r A_r)} \hat{f}(t)\, dt. \tag{0.1}$$

The pseudodifferential operators we consider depend on symbols with a finite number of derivatives. No assumption of homogeneity is made. These operators act on functions defined on a space that is either the euclidean space R^n (cf. [1], [3]), or a compact manifold without boundary of class C^M, for some $M < \infty$ (cf. [2]). We give in each case sufficient differentiability conditions on a function f so that (0.1) defines an operator in the same class. In fact, we prove that this class is a selfadjoint Banach algebra.

The aim of this paper, as stated in [1] and [2], is to extend to L^p-bounded operators, $1 < p < \infty$, the results proved in [2] for $p = 2$.

The key point in dealing with the formula (0.1) is to obtain a "good" estimate of the norm

$$\|e^{-2\pi i t \cdot A}\| \tag{0.2}$$

in terms of $|t|$. Typically, a good estimate is expected to be a polynomial one. In [1] and [2], a roundabout argument is used, by introducing a suitable version of the characteristic operators defined by A. P. Calderón in [6]. Essentially, we make use of the same machinery here, plus a "self-improving" argument, where the Sobolev immersion theorem and the fact that $L^p(X)$ is continuously included in $L^q(X)$ if $p \geqslant q$ play a crucial role (cf. Theorem (3.1) below).

An announcement of these results appeared in Proceedings of the Seminar on Fourier Analysis held at El Escorial, North-Holland, 1985, 3–11.