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Some Comments on a Stability of the $\tilde{S}_2 \otimes S_2$-Type Space-Time

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In terms of the Abbott-Deser energy method, the stability for a small perturbation of Nariai's $\tilde{S}_2 \otimes S_2$-type solution of the vacuum Einstein equations with non-vanishing cosmological constant is studied. It is shown that the space-time is stable for some perturbations which arise only inside its event horizon.

Recently, an effective cosmological term (Λ-term) has been considered by many authors in relation to the phase transition in an early stage of the big-bang universe. The de Sitter space-time is famous as an exact solution of the vacuum Einstein equations with a positive cosmological constant Λ:

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=0\,. \tag{1}$$

In addition to the above maximally symmetric solution, there is another solution of Eq. (1) with the topology $\tilde{S}_2 \otimes S_2$, found by Nariai:[1)]

$$ds^2=-d\tau^2+e^{2\tau/a}d\chi^2+a^2(d\theta^2+\sin^2\theta d\varphi^2).$$
$$(a\equiv 1\sqrt{\Lambda}) \tag{2}$$

After that, Bertotti and Robinson[2)] dealt with the same type space-time with a constant electromagnetic field.

We can easily extend a solution of this type to a higher dimensional space-time,[3)] and there are candidates for the base space of an extended Kaluza-Klein theory.[4)] Of course, though we may consider M (external 4-dim. space-time)$\otimes$ S (compact internal space)-type solutions with or without Λ-term and the Bertotti-Robinson-type source, we shall restrict ourselves to the following case:

$$ds^2=-d\tau^2+e^{2\tau/A}\{d\chi^2+\chi^2(d\theta^2+\sin^2\theta d\varphi^2)\}$$
$$+\gamma_{\alpha\beta}dx^\alpha dx^\beta\,, \tag{3}$$

where $A\equiv(3/\Lambda)^{1/2}$, and the part $\gamma_{\alpha\beta}dx^\alpha dx^\beta$ stands for an n-dimensional sphere S_n (or an m-product of 2-dimensional spheres $S_2\otimes S_2\otimes\ldots\ldots\otimes S_2$). One of the most important problems is to decide the total number of dimensions and the topology of these higher dimensional space-times, but we have at present no good idea to answer them. However, there is another problem, i.e., the question whether the space-time is stable or not for its small perturbation. As regards a radial perturbation of the classical solutions in an extended Kaluza-Klein theories, their instability has been studied.[5)] Contrary to this, when the Bertotti-Robinson-type source is endowed, a higher dimensional space-time with the topology M_4 (Minkowski's 4-space)$\otimes S_2$ may be stable.[6)]

At any rate, radial perturbation is related to a compactification of internal space, but stability for non-radial perturbation will also be important. The purpose of this letter is to look into the behavior of a small perturbation in Nariai's $\tilde{S}_2\otimes S_2$ space-time, as the first step to deal with a more general case. Since Abbott and Deser's energy method[7)] is successfully applied to the examination of stability of de Sitter and anti-de Sitter space-times, we shall again apply their method to our $\tilde{S}_2\otimes S_2$ space-time.

To do this, we shall make use of the following gravitational action:[8)]

$$I=\int d^4x\,[\pi^{ij}\partial_t g_{ij}+Ng^{-1/2}\{g(^3R-2\Lambda)$$
$$+\pi^2/2-\pi_{ij}\pi^{ij}\}+2N_i\pi^{ij}{}_{|j}], \tag{4}$$

where

$$\begin{cases} N\equiv(-{}^4g^{00})^{-1/2}, \quad N_i\equiv{}^4g_{0i}, \\ g_{ij}\equiv{}^4g_{ij}, \quad g\equiv\det(g_{ij}), \\ \pi^{ij}\equiv Ng^{1/2}({}^4\Gamma^0_{kl}-g_{kl}g^{mn\,4}\Gamma^0_{mn})g^{ik}g^{jl}, \\ \pi\equiv g_{ij}\pi^{ij}\,. \end{cases}$$

Here ^3R is the 3-dimensional curvature scalar and $\pi^{ij}{}_{|j}$ stands for a covariant divergence of the 3-tensor density π^{ij}. We shall further decompose the above quantities into background ones (denoted by the barred symbols) and their small perturbations, i.e.,

$$\begin{cases} g_{ij}=\bar{g}_{ij}+h_{ij}, \quad \pi^{ij}=\bar{\pi}^{ij}+p^{ij}, \\ N=\bar{N}+n, \qquad N_i=\bar{N}_i+h_{0i}\,. \end{cases} \tag{5}$$

On the second order approximation to the action given by Eq. (4), we obtain the equations of motion and constraints for first-order variables. Let us impose the gauge conditions:

$$h^{ij}{}_{|j}=0\,,\quad p^i{}_i=0\,, \tag{6}$$

where the symbol $_{|j}$ stands for a 3-dimensional background covariant derivative. Then the constraint equations are of the form

$$\bar{\Delta}h^i{}_i+\bar{R}_{ij}h^{ij}=0\,,\quad p^{ij}{}_{|j}=0\,, \tag{7}$$

where $\bar{\Delta}\equiv{}^{|j}{}_{|j}$. Similarly, the Hamiltonian density in the sense of Abbott-Deser[7] is given by

$$\begin{aligned}\mathcal{H}&=-T^0{}_0\\ &=-\bar{N}\bar{g}^{-1/2}\{g({}^3R-2\Lambda)+\pi^2/2-\pi_{ij}\pi^{ij}\}.\end{aligned} \tag{8}$$

For convenience, we shall use the static form of the metric

$$ds^2=-\cos^2 r dt^2+a^2\{dr^2+(d\theta^2+\sin^2\theta d\varphi^2)\}, \tag{9}$$

which is connected with Eq. (2) by the coordinate transformation $e^{\tau/a}\chi=a\sin r$, $\tau=t+a\ln(\cos r)$. It is useful only inside its event horizon in the sense of Rindler $l=a$.[9] Similarly to the situation in the de Sitter universe,[7] the conserved "energy" can be defined only inside the event horizon, so we shall restrict ourselves to discussing solely particular perturbations inside the horizon. Then, by making use of Eqs. (5)~(9), we can reduce $H\equiv\int\mathcal{H}d^3x$ to

$$\begin{aligned}H=\int d^3x\bar{N}\Big[\sqrt{g}\Big\{&\frac{1}{4}h^{ij|k}h_{ij|k}\\ &+\frac{1}{2}a^{-2}\Big(\frac{3}{4}h^i{}_ih^j{}_j+h^i{}_jh^j{}_i\Big)\\ &+\frac{1}{4}a^{-2}(h^1{}_1+h^2{}_2+h^3{}_3)(h^2{}_2+h^3{}_3)\Big\}\\ &+\frac{1}{\sqrt{g}}p_{ij}p^{ij}\Big].\end{aligned} \tag{10}$$

If we consider the case $h^1{}_1=0$, then Eq. (7) means

$$(\bar{\Delta}+a^{-2})(h^2{}_2+h^3{}_3)=0\,. \tag{11}$$

In this case, the Hamiltonian is non-negative or $H\geq 0$. We can easily see that other cases which make the Hamiltonian non-negative ($h^i{}_i=0$ or $h^2{}_2+h^3{}_3=0$) are reduced to the above one. In other words, so far as these perturbations (including the non-radial perturbation $h^1{}_1=h^2{}_2+h^3{}_3=0$) are concerned, our space-time with the topology $\tilde{S}_2\otimes S_2$ is stable. (If $h^1{}_1\neq 0$, we must introduce some stabilizer such as the Bertotti-Robinsons-type source, etc., into the system in such a way that the Hamiltonian thus modified may be non-negative.)

In the above derivation, we have assumed that the perturbation arises only inside the event horizon $l=a$ (just like the situation in the de Sitter space-time whose event horizon in $l=A$), but there may be some doubt for this procedure. Moreover, is there any mean to decide the value of n in a general $(4+n)$- dimensional space-time satisfying $R_{MN}=\Lambda g_{MN}(M,N=1,2,\ldots,4+n)$? As regards the stabilization of these higher dimensional space-time in the radial direction, one may consider not only the Bertotti-Robinson-type source, but also the effects of particle creation and vacuum polarization (if quantized fields in our space-time are called for). These problems will be attacked in the future.

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