Some completeness theorems in the Menger probabilistic metric space

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Abstract. In this article, some new completeness theorems in probabilistic normed space are proved. Moreover, the existence of a constrictive Menger probabilistic normed space is shown.

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1 Introduction

It is well known that the theory of probabilistic metric space is a new frontier branch between probability theory and functional analysis and has an important background, which contains the common metric space as a special case. One can study the completeness theory in the probabilistic metric space. This study has an important applications, for example, in fixed point theory and etc. Due to do this and for the sake of convenience, some definitions and notations are recalled from [4], [1] and [5].

Definition 1.1. A mapping $F : R \to R^+$(non-negative real numbers) is called a *distribution function* if it is nondecreasing and left-continuous and it has the following properties:

$$(i) \quad \inf_{t\in R} F(t) = 0,$$
$$(ii) \quad \sup_{t\in R} F(t) = 1.$$

Let D^+ be the set of all distribution functions F such that $F(0) = 0$. Also denote by H the distribution function

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Definition 1.2. A *probabilistic metric space* (briefly, *PM-space*) is an ordered pair (S, F) where S is a nonempty set and $F : S \times S \to D^+ \Big(F(p, q)$ is denoted by $F_{p,q}$ for every $(p, q) \in S \times S\Big)$ satisfies the following conditions:

1. $F_{p,q}(t) = 1$ for all $t > 0$ if and only if $p = q$ $(p, q \in S)$.
2. $F_{p,q}(t) = F_{q,p}(t)$ for all $p, q \in S$ and $t \in R$.
3. If $F_{p,q}(t_1) = 1$ and $F_{q,r}(t_2) = 1$ then $F_{p,r}(t_1 + t_2) = 1$ for $p, q, r \in S$ and $t_1, t_2 \in R^+$.

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Definition 1.3. a mapping $* : [0,1] \times [0,1] \to [0,1]$ is called a *triangular norm* (abbreviated, *t-norm*) if the following conditions are satisfied:

(i) $a * 1 = a$ for every $a \in [0,1]$,
(ii) $a * b = b * a$ for every $a, b \in [0,1]$,
(iii) $a \geq b,\ c \geq d \to a * c \geq b * d$ $(a, b, c, d \in [0,1])$,
(iv) $a * (b * c) = (a * b) * c$ $(a, b, c \in [0,1])$.

The rest of the paper is organized as follows: in Section 2, the definition of Menger probabilistic metric and Menger probabilistic normed spaces are recalled and then a norm is defined and it is shown the existence of a constrictive Menger probabilistic normed space. Section 3 is devoted to some new results about completeness theory.

2 Some PN-spaces

In this section, first we recall the definition of Menger probabilistic metric and Menger probabilistic normed spaces are recalled from [1] and [4].

Definition 2.1. A *Menger probabilistic metric space* (briefly, *Menger PM-space*) is a triple $(S, F, *)$, where (S, F) is a probabilistic metric space, $*$ is a t-norm and the following inequality holds:

$$F_{p,q}(t_1 + t_2) \geq F_{p,r}(t_1) * F_{r,q}(t_2), \tag{2.1}$$

for all $p, q, r \in S$ and every $t_1 > 0,\ t_2 > 0$.

Definition 2.2. A triple $(S, F, *)$ is called a *Menger probabilistic normed space* (briefly, *Menger PN-space*) if S is a real vector space, F is a mapping from S into D (for $x \in S$, the distribution function $F(x)$ is denoted by F_x and $F_x(t)$ is the value of F_x at $t \in R$) and $*$ is a t-norm satisfying the following conditions:

(i) $F_x(0) = 0$,
(ii) $F_x(t) = H(t)$ for all $t > 0$ if and only if $x = 0$,
(iii) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$ for all $\alpha \in R, \alpha \neq 0$,
(iv) $F_{x+y}(t_1 + t_2) \geq F_x(t_1) * F_y(t_2)$ for all $x, y \in E$ and $t_1, t_2 \in R^+$.

Remark 2.3. *Let (R, F, Δ) be a Menger PN-space, then $(S, \overline{F}, \Delta)$ is a Menger PM-space, where*

$$\overline{F}_{x,y}(t) = F_{x-y}(t)$$

Schweizer, Sklar and Thorp [5] proved that if $(S, F, *)$ is a Menger PM-space with $\sup_{0<t<1} t * t = 1$, then $(S, F, *)$ is a Hausdorff topological space in the topology τ induced by the family of $(\epsilon, \lambda)-$neighborhoods

$$\{U_p(\epsilon, \lambda) : p \in S, \epsilon > 0, \lambda > 0\},$$

where

$$U_p(\epsilon, \lambda) = \{u \in S : F_{u,p}(\epsilon) > 1 - \lambda\}.$$

Definition 2.4. Let $(S, F, *)$ be a Menger PM-space with $\sup_{0<t<1} *(t,t) = 1$.

(1) A sequence $\{u_n\}$ in S is said to be $\tau-$convergent to $u \in S$ (we write $u_n \xrightarrow{\tau} u$) if for any given $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{u_n,u}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{u_n\}$ in S is called a $\tau-$Cauchy sequence if for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{u_n,u_m}(\epsilon) > 1 - \lambda$, whenever $n, m \geq N$.

(3) A Menger PM-space $(S, F, *)$ is said to be τ-complete if each τ-Cauchy sequence in S is τ-convergent to some point in S.

Example 2.5. *If $(E, \|.\|_E)$ be a normed real vector space and define*

$$\hat{F} : E \to D^+,$$

by

$$\hat{F}_x(t) = \begin{cases} \frac{t}{t+\|x\|_E} & t > 0, \\ 0 & t \leq 0. \end{cases} \tag{2.2}$$

Then $(E, \hat{F}, *)$ is a Menger PN-space.

Lemma 2.6. *If $(\mathbb{R}, F, *)$ be a Menger PN-space, then*

$$|x| \leq |y| \Longrightarrow F_x(t) \geq F_y(t), \tag{2.3}$$

for all $x, y \in \mathbb{R}$ and $t \geq 0$.

Proof. Note that, if $|x| = 0$, then (2.3) is obvious. Suppose $|x| > 0$, then

$$F_x(t) = F_{\frac{x}{y}y}(t) = F_y(\frac{t}{|\frac{x}{y}|}) \geq F_y(t).$$

The last inequality holds, because $F_y(.)$ is a nondecreasing function. □

Definition 2.7. Let $(\mathbb{R}, F, *)$ be a Menger PN-space and $(E, \|.\|_E)$ be a normed real vector space, we define a mapping $\tilde{F} : E \to D^+$ by

$$\tilde{F}_x(t) = F_{\|x\|_E}(t). \tag{2.4}$$

Proposition 2.8. *Let $(\mathbb{R}, F, *)$ be a Menger PN-space, then $(E, \tilde{F}, *)$ is also a Menger PN-space.*

Proof. First of all note that

$$\tilde{F}_x(t) = F_{\|x\|_E}(t) \in D^+.$$

Secondly, $\tilde{F}_x(t)$ satisfies all conditions of Definition 2.2. In order to prove this, Note that $\tilde{F}_x(0) = F_{\|x\|_E}(0) = 0$, thus condition (i) is fulfilled. Also

$$\tilde{F}_x(t) = 1 \Longleftrightarrow F_{\|x\|_E}(t) = 1 \Longleftrightarrow \|x\|_E = 0 \Longleftrightarrow x = 0,$$

whenever $t > 0$, so condition (ii) is satisfied. Moreover,

$$\tilde{F}_{\alpha x}(t) = F_{\|\alpha x\|_E} = F_{|\alpha|\|x\|_E}(t) = F_{\|x\|_E}(\frac{t}{|\alpha|}) = \tilde{F}_x(\frac{t}{|\alpha|})$$

and condition (iii) is fulfilled. Finally, by Lemma 2.6

$$\begin{aligned}\tilde{F}_{x+y}(t_1+t_2) &= F_{\|x+y\|_E}(t_1+t_2) \\ &\geq F_{\|x\|_E+\|y\|_E}(t_1+t_2) \\ &\geq F_{\|x\|_E}(t_1) * F_{\|y\|_E}(t_2) \\ &= \tilde{F}_x(t_1) * \tilde{F}_y(t_2).\end{aligned}$$

This proves condition (iv) and ends the proof. □

We now give a lemma which will be used in the next section.

Lemma 2.9. *In a fuzzy metric space $(X, M, *)$, for any $\lambda > 0$ and $k \in \mathbb{N}$, there exists a $\lambda' > 0$ such that* $\underbrace{(1-\lambda') * (1-\lambda') * \cdots * (1-\lambda')}_{k-\text{times}} \geq (1-\lambda)$.

Proof. Note that

$$\sup_{\mu \in [0,1]} (1-\mu) * (1-\mu) * \cdots * (1-\mu) = 1 * 1 * \cdots * 1 = 1.$$

□

3 Main results

In this section, some new results concerning completeness theory.

Theorem 3.1. *Let $(S, F, *)$ be a Meneger PM-space with a continuous t-norm $*$. Suppose $\{x_n\}$ is a Cauchy sequence which has a convergent subsequence, then $\{x_n\}$ is convergent.*

Proof. Let $\lambda \in (0,1)$ be arbitrary, by Lemma 2.9 there exists a $\lambda' \in (0,1)$ such that $(1-\lambda') * (1-\lambda') \geq 1-\lambda$. For any $\varepsilon > 0$, since $\{x_n\}$ is a Cauchy sequence, there exists $N_0 \in \mathbb{N}$ such that for any $n, m \geq N_0$

$$F_{x_n, x_m}(\frac{\varepsilon}{2}) > 1-\lambda'.$$

Suppose $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n\}$ and converges to $x \in S$. it means that there exists $N_1 \in \mathbb{N}$ such that for any $k \geq N_1$

$$F_{x_{n_k}, x}(\frac{\varepsilon}{2}) > 1-\lambda'.$$

Now, let $N = \text{Max}\{N_0, N_1\}$ then for any $m \geq N$

$$\begin{aligned}F_{x_m,x}(\varepsilon) \geq\ & F_{x_m, x_{n_m}}(\tfrac{\varepsilon}{2}) * F_{x_{n_m}, x}(\tfrac{\varepsilon}{2}) \\ \geq\ & 1-\lambda' * 1-\lambda' \\ >\ & 1-\lambda.\end{aligned}$$

It means that x_m converges to x. □

Theorem 3.2. *Let* $(E, \tilde{F}, *)$ *be a complete Meneger PN-space, where* E *is a real vector space and* $\tilde{F}$ *is defined by (2.4). Then* $(\mathbb{R}, F, *)$ *is complete.*

proof. Suppose $(E, \tilde{F}, *)$ is a complete Meneger PN-space, and $\{\alpha_n\}$ is a Cauchy sequence in $(\mathbb{R}, F, *)$. Due to Theorem 3.1, it is enough to show that there exits a convergent subsequence of $\{\alpha_n\}$.
There is a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that $\alpha_{n_k} \geq 0$ or $\alpha_{n_k} \leq 0$ for all $k \in \mathbb{N}$. Now, let $\{\alpha_{n_k}\}$ be a subsequence such that $\alpha_{n_k} \geq 0$ for all $k \in \mathbb{N}$, (for $\alpha_{n_k} \leq 0$, proof is similar). Set $\alpha_{n_k} = \beta_k$ for simplicity. Choose $e \in E$ such that $\|e\| = 1$ and consider the sequence $\{\beta_k e\}$ in E. We show that $\{\beta_k e\}$ is a Cauchy sequence in $(E, \tilde{F}, *)$. To prove this, first note that $\{\beta_k\}$ is a Cauchy sequence in $(\mathbb{R}, F, *)$, it means that for any $\lambda \in (0, 1)$ and any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for any $n, m \geq N_0$,

$$F_{(\beta_m - \beta_n)}(\varepsilon) > 1 - \lambda.$$

Now, for $n, m \geq N_0$,

$$\begin{aligned} \tilde{F}_{\beta_m e - \beta_n e}(\varepsilon) &= F_{\|(\beta_m - \beta_n)e\|_E}(\varepsilon) \\ &= F_{|(\beta_m - \beta_n)|\|e\|_E}(\varepsilon) \\ &= F_{|(\beta_m - \beta_n)|}(\varepsilon) \\ &= F_{(\beta_m - \beta_n)}(\varepsilon) \\ &> 1 - \lambda. \end{aligned}$$

It means that $\{\beta_k e\}$ is a Cauchy sequence in $(E, \tilde{F}, *)$, and since $(E, \tilde{F}, *)$ is a complete Meneger PN-space, then $\{\beta_k e\}$ is convergent to some $x \in E$. Now, we prove $\{\beta_k\}$ is convergent in $(\mathbb{R}, F, *)$.
Since $\beta_k e \to x$, then for any $\lambda \in (0, 1)$ and $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for any $k \geq N_1$

$$\tilde{F}_{\beta_k e - x}(\varepsilon) > 1 - \lambda. \tag{3.1}$$

On the other hand,

$$\tilde{F}_{\beta_k e - x}(\varepsilon) = F_{\|\beta_k e - x\|_E}(\varepsilon) \tag{3.2}$$

and since $\|\beta_k e - x\|_E \geq \|\beta_k e\| - \|x\|_E$, Lemma 2.6 shows that

$$F_{\|\beta_k e - x\|_E}(\varepsilon) \leq F_{\beta_k - \|x\|_E}(\varepsilon). \tag{3.3}$$

Considering (3.1), (3.2) and (3.3), we have

$$F_{\beta_k - \|x\|_E}(\varepsilon) > 1 - \lambda,$$

for any $k \geq N_1$. This shows that $\{\beta_k\}$ is convergent to $\|x\|_E$ in $(\mathbb{R}, F, *)$. □

In the next theorem, we consider $\mathbb{R}^k$ with Euclidean norm.

Theorem 3.3. $(\mathbb{R}^k, \tilde{F}, *)$ *is complete Meneger PN-space if and only if* $(\mathbb{R}, F, *)$ *is complete Menger PN-space.*

Proof. Theorem 3.2 shows that if $(\mathbb{R}^n, \tilde{F}, *)$ is a complete Meneger PN-space, then $(\mathbb{R}, F, *)$ is a complete Menger PN-space.

Now, we prove if $(\mathbb{R}, F, *)$ is a complete Meneger PN-space, then $(\mathbb{R}^k, \tilde{F}, *)$ is a complete Menger PN-space. Suppose $(\mathbb{R}, F, *)$ be a complete Meneger PN-space and $\{x_n\}$ is a Cauchy sequence in $(\mathbb{R}^k, \tilde{F}, *)$. Let $x_n = (\alpha_{1n}, \cdots, \alpha_{kn})$, where $\alpha_{in} \in \mathbb{R}$ for $1 \le i \le k$, then with respect to the norm inequality, we have

$$|\alpha_{in} - \alpha_{im}| \le \|x_n - x_m\| \le \sum_{j=1}^{k} |\alpha_{jn} - \alpha_{jm}|, \quad \text{for } 1 \le i \le k. \tag{3.4}$$

Since $\{x_n\}$ is a Cauchy sequence in $(\mathbb{R}^k, \tilde{F}, *)$, then for any $\lambda \in (0, 1)$ and $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for any $m, n \ge N_0$,

$$\tilde{F}_{x_n - x_m}(\varepsilon) = F_{\|x_n - x_m\|}(\varepsilon) > 1 - \lambda. \tag{3.5}$$

Thus for $m, n \ge N_0$ and each $1 \le i \le k$, by Lemma 2.6, and (3.5), we have

$$\begin{aligned} F_{\alpha_{in} - \alpha_{im}}(\varepsilon) = & \; F_{|\alpha_{in} - \alpha_{im}|}(\varepsilon) \\ \ge & \; F_{\|x_n - x_m\|}(\varepsilon) \\ > & \; 1 - \lambda. \end{aligned}$$

It means that $\{\alpha_{in}\}$ is a Cauchy sequence in $(\mathbb{R}, F, *)$ for all $1 \le i \le k$. Thus there exists α_i such that $\alpha_{in} \to \alpha_i$ for each $1 \le i \le k$. Now, we claim that $x_n \to x = (\alpha_1, \cdots, \alpha_k)$ in $(\mathbb{R}^k, \tilde{F}, *)$.
Let $\lambda > 0$ arbitrary, by Lemma 2.9, there exists a $\lambda' > 0$ such that

$$\underbrace{(1 - \lambda') * (1 - \lambda') * \cdots * (1 - \lambda')}_{k-\text{times}} \ge (1 - \lambda).$$

In addition, since $\alpha_{in} \to \alpha_i$ for each $1 \le i \le k$, thus there exits N_i such that for any $n \ge N_i$

$$F_{\alpha_{in} - \alpha_i}(\frac{\varepsilon}{k}) > 1 - \lambda'$$

for each $1 \le i \le k$. Let $N = \text{Max}\{N_i | \ 1 \le i \le k\}$, then for any $n \ge N$

$$\begin{aligned} \tilde{F}_{x_n - x}(\varepsilon) = & \; F_{\|x_n - x\|}(\varepsilon) \\ \ge & \; F_{\sum_{i=1}^{k} |\alpha_{in} - \alpha_i|}(\varepsilon) \\ \ge & \; F_{\alpha_{1n} - \alpha_1}(\tfrac{\varepsilon}{k}) * \cdots * F_{\alpha_{kn} - \alpha_k}(\tfrac{\varepsilon}{k}) \\ \ge & \; 1 - \lambda' * \cdots * 1 - \lambda' \\ > & \; 1 - \lambda. \end{aligned}$$

Thus $\{x_n\}$ converges to x and this ends the proof. □

Theorem 3.4. *Let $(E, \|.\|_E)$ be a normed real vector space. Then $x_n \overset{\|.\|_E}{\to} x$ if and only if $x_n \overset{\hat{F}}{\to} x$, where $(E, \hat{F}, *)$ is a Menger PN-space and $\hat{F}$ is defined by (2.2).*

proof. Suppose $x_n \overset{\hat{F}}{\to} x$, then for $\lambda = 1/2$ and $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for any $n \ge N$,

$$\hat{F}_{x_n - x}(\varepsilon) = \frac{\varepsilon}{\varepsilon + \|x_n - x\|_E} > 1 - 1/2,$$

or for any $n \geq N$,

$$\|x_n - x\|_E < \varepsilon$$

and then $x_n \overset{\|.\|_E}{\to} x$.

Conversely, suppose $x_n \overset{\|.\|_E}{\to} x$, then for any $\varepsilon > 0$ and $\lambda \in (0,1)$ there is $N_0 \in \mathbb{N}$ such that $\|x_n - x\|_E < \frac{\varepsilon\lambda}{1-\lambda}$, then for any $n \geq N_0$

$$\hat{F}_{x_n - x}(\varepsilon) = \frac{\varepsilon}{\varepsilon + \|x_n - x\|_E} > \frac{\varepsilon}{\varepsilon + \frac{\varepsilon\lambda}{1-\lambda}} = 1 - \lambda.$$

Then $x_n \overset{\hat{F}}{\to} x$.

If we have continuity assumption of $F_x(t)$ at $t = 0$, then we will have the following theorem.

Theorem 3.5. *Let $(\mathbb{R}, F, *)$ be a Menger PN-space, where $F_x(.)$ is continuous at zero and $*$ is a continuous t-norm. Then $(\mathbb{R}, F, *)$ is complete Menger PN-space.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $(\mathbb{R}, F, *)$. Now, we claim that $\{x_n\}$ is a bounded sequence in $(\mathbb{R}, |.|)$. Otherwise, there exists a subsequence $\{x_{n_k}\}$ which $|x_{n_k}|$ tends to ∞. For a given $\varepsilon > 0$, there exists a N such that for any $n, m \geq N$

$$F_{x_n - x_m}(\varepsilon) > \frac{1}{2}.$$

Now, for every $n_k > N$

$$1/2 < F_{x_{n_k} - x_N}(\varepsilon) = F_1\left(\frac{\varepsilon}{|x_{n_k} - x_N|}\right) \to F_1(0) = 0,$$

which is a contradiction. Then $\{x_n\}$ is a bounded sequence in $(\mathbb{R}, |.|)$ and has a convergent subsequence say x_{n_k} in $(\mathbb{R}, |.|)$. Now, suppose $x_{n_k} \overset{|.|}{\to} x$, then

$$F_{x_{n_k} - x}(\varepsilon) = F_1\left(\frac{\varepsilon}{x_{n_k} - x}\right) \to F_1(\infty) = 1,$$

which it means $x_{n_k} \to x$ to $(\mathbb{R}, F, *)$. Applying Theorem 3.1 completes the proof. □

Corollary 3.6. *$(\mathbb{R}^k, \tilde{F}, *)$ is complete Meneger PN-space when $F_x(.)$ is continuous at zero in the definition of $\tilde{F}$.*

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