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Some concrete operators and their properties

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Abstract: We consider integration and double integration operators, the Hardy operator, and multiplication and composition operators on Lebesgue space $L_p[0,1]$ and Sobolev spaces $W_p^{(n)}[0,1]$ and $W_p^{(n)}([0,1]\times[0,1])$, and we study their properties. In particular, we calculate norm and spectral multiplicity of the Hardy operator and some multiplication operators, investigate its extended eigenvectors, characterize some composition operators in terms of the extended eigenvectors of the Hardy operator, and calculate the numerical radius of the integration operator on the real $L_2[0,1]$ space. The main method for our investigation is the so-called Duhamel products method. Some other questions are also discussed and posed.

Key words: Double integration operator, multiplication operator, composition operator, Sobolev space, Duhamel product, numerical radius

1. Introduction and background

The present paper studies properties of some classical operators, including the Volterra integration operator, double integration operator, multiplication operator, Hardy operator, and composition operator.

Our main method for the proofs of the obtained results is the Duhamel products method, which was essentially used in investigation of various questions of analysis, including differential and integrodifferential equations, the boundary value problems of mathematical physics, operator theory, and Banach algebras in the works of Nagnibida [27], Fage and Nagnibida [11], Dimovski [9], Tkachenko [34], Malamud [25, 26], Domanov and Malamud [10], Bojinov [4], Wigley [36, 37], Karaev [17], and Karaev et al. [19].

Let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on a Banach space X. A subspace $E\subset X$ is called a *cyclic subspace* of an operator $T\in\mathcal{L}(X)$ if

$$span\{T^nE : n=0,1,2,...\}=X,$$

where *span* denotes a closed linear hull; that is:

$$span\{T^nE : n=0,1,2,...\}=clos\{p(T)E : p\in\mathcal{P}\text{ (the set of polynomials)}\}.$$

An element $x\in X$ is said to be a *cyclic vector* for the operator T, if

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$$clos\,\{p\,(T)\,x : p \in \mathcal{P}\} = X.$$

In this case an operator T is said to be a cyclic operator.

The *spectral multiplicity* $\mu\,(T)$ of $T : X \to X$ is the following nonnegative integer or the symbol ∞ :

$$\mu\,(T) := \inf\,\{\dim E : clos\,\{p\,(T)\,E : p \in \mathcal{P}\} = X\}\,.$$

It is clear that T is a cyclic operator if and only if $\mu\,(T) = 1$. The set of all cyclic vectors is denoted by $Cyc\,(T)$.

It is necessary to note that spectral multiplicity is an important invariant of operators. Clearly, the notion of the cyclic vector is important in connection with the general problem of existence of a nontrivial invariant subspace, because an operator $T \in \mathcal{L}\,(X)$ has no nontrivial invariant subspace if and only if every nonzero vector $x \in X$ is a cyclic vector for T. Cyclic vectors are important in weighted polynomial approximation theory.

Let $W_p^{(n)} := W_p^{(n)}\,[0,1]$ denote the Sobolev space, which is a Banach space of continuous functions on the unit segment $[0,1]$ for which $f^{(n)} \in L_p\,[0,1]$. The Duhamel product of the functions $f, g \in W_p^{(n)}\,[0,1]$ is defined by (see Wigley [37]):

$$(f \circledast g)(x) := \frac{d}{dx}\int_0^x f(x-t)g(t)dt = \int_0^x f'(x-t)g(t)dt + f\,(0)\,g\,(x)\,. \tag{1}$$

One can use results from operational calculus to show that $W_p^{(n)}\,[0,1]$ is a commutative and associative algebra with respect to the Duhamel product $\circledast$ with a unit $f\,(x) \equiv 1$. Moreover, it is known (and easy to verify) that $\left(W_p^{(n)}\,[0,1]\,, \circledast\right)$ is a Banach algebra (see Karaev [17]).

Recall that the classical convolution product $*$ is defined in $W_p^{(n)}$ by the formula

$$(f * g)(x) := \int_0^x f(x-t)g(t)dt,\;\; f, g \in W_p^{(n)}. \tag{2}$$

It is classical that $\left(W_p^{(n)}, *\right)$ is a Banach algebra without unit. The function $f \in W_p^{(n)}\,[0,1]$ $(1 \le p < \infty)$ is a $*$-*generator* for the Banach algebra $\left(W_p^{(n)}, *\right)$ if

$$span\,\{f, f * f, f * f * f, ...\} = W_p^{(n)}.$$

The $*$-generators of the Banach algebras $(C\,[0,1]\,, *)$ and $\left(C^{(n)}\,[0,1]\,, *\right)$ are investigated in [12] and [19], respectively.

Consider an operator $A : X \to X$ such that A is bounded and linear. If $B \in \mathcal{L}\,(X)$, it can be happen that there is nonzero operator C such that

$$BA = AC. \tag{3}$$

If we denote by $\mathcal{E}_C$ the set of all B for which there exists an operator C satisfying (3), then it is easy to see that $\mathcal{E}_A$ is an algebra. Furthermore, one can define the map $\Phi_A : \mathcal{E}_A \to \mathcal{L}\,(X)$ by $\Phi_A\,(B) = C$. When $C = \lambda B$, for some complex number $\lambda \in \mathbb{C}$, equation (3) becomes

$$BA = \lambda AB. \tag{4}$$

Furthermore, if A has a dense range and $\ker A = \{0\}$, one can easily see that Φ_A is a uniquely defined algebra homomorphism. It is clear that a pair (B, λ) in $(\mathcal{L}(X) \setminus \{0\}) \times \mathbb{C}$ satisfies (4) if and only if λ is an eigenvalue for Φ_A and B is an eigenvector for Φ_A. Following Biswas et al. [2], an eigenvalue of Φ_A will be referred to as an extended eigenvalue of A. The appropriate eigenvector of Φ_A will be referred to as an extended eigenvector of A.

The notion of extended eigenvalues and extended eigenvectors of operators is closely related, for example, with the theory of invariant subspaces and with the theory of so-called Deddens algebras (see Karaev [18] and Lacruz [20, 22, 21]). More detailed information about "extended spectral theory" can be found in the works of Biswas et al. [2], Domanov and Malamud [10], Karaev [18], Malamud [25, 26], Lacruz et al. [21], Lauric [23, 24], Cassier and Alkanjo [6], Alkanjo [1], Cowen [7], Bourdon and Shapiro [3], Tong and Zhou [35], and Shkarin [30].

Recall that for a bounded linear operator T acting in the Hilbert space H its numerical range and numerical radius are defined by

$$W(T) := \{\langle Tx, x\rangle : x \in H \text{ and } \|x\| = 1\}$$

and

$$w(T) := \sup\{|\langle Tx, x\rangle| : x \in H \text{ and } \|x\| = 1\},$$

respectively. It is well known that $W(T)$ is always a convex set in the complex plane $\mathbb{C}$, $\sigma(T)$ (the spectrum) $\subset \overline{W(T)}$,

$$\frac{1}{dist(\lambda, \sigma(T))} \leq \left\|(T - \lambda I)^{-1}\right\| \leq \frac{1}{dist(\lambda, W(T))}$$

and

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|,$$

and hence the numerical radius defines an equivalent norm in the Banach algebra $\mathcal{L}(H)$ (see, for instance, Halmos [14] and Gustafson and Rao [13]). Thus, one of the interesting and important problems of operator theory is the determination of the numerical range of the concrete operator and calculation of its numerical radius.

The main motivation in this paper is to show the role of the Duhamel products method in various questions for the different kinds of classical operators and thus demonstrate the "universality" of this method.

Here we calculate the norm and spectral multiplicity of the Hardy operator V_n defined on the Lebesgue space $L_p[0,1]$ by the formula

$$(V_n f)(x) := x^n \int_0^x f(t)\,dt, \tag{5}$$

where $n \in \mathbb{N}$, $n \geq 1$, is a fixed number. Clearly,

$$V_n = M_{x^n} V = M_{x^{n+1}} H,$$

where M_{x^m}, $M_{x^m} f(x) := x^m f(x)$ is a multiplication operator; V, $Vf(x) := \int_0^x f(t)\,dt$, is the classical Volterra integration operator; and H, $Hf(x) := \frac{1}{x}\int_0^x f(t)\,dt$, is the classical Hardy operator on the space $L_p[0,1]$. We also investigate the extended eigenvectors of the Hardy operator V_n (Section 2). The characterization of some composition operator is also studied (Section 3). Moreover, we describe $*$-generators of the Banach algebra $\left(W_p^{(n)}, *\right)$ (Section 4). The extended eigenvectors of the integral operator $K_{\ln\frac{x}{y}}$, defined on $W_p^{(n)}[0,1]$ by

$$K_{\ln\frac{x}{y}} f(x) := \int_0^x \ln\frac{x}{y} f(y)\,dy,$$

is also investigated (Section 5). We also estimate the numerical radius of the Volterra integration operator V on the space $L_2[0,1]$ of real-valued functions (Section 6). In Section 7, we give two different proofs for the spectral multiplicity of the multiplication operator on the space $C(\Gamma)$ of continuous functions on the unit circle Γ of the complex plane $\mathbb{C}$. Some open questions are also posed in the Section 8.

2. On the norm, spectral multiplicity, and extended eigenvectors of Hardy operators

For any fixed integer $n \geq 1$, let us consider the Hardy operator V_n defined on the Lebesgue space $L_p := L_p[0,1]$ $(1 \leq p < \infty)$ by the formula

$$V_n f(x) := x^n \int_0^x f(t)\,dt, \ \ f \in L_p. \tag{6}$$

In this section, we calculate the norm and spectral multiplicity of V_n and study extended eigenvectors of this operator. Recall that the composition operator C_φ on L_p is defined by $(C_\varphi f)(x) = (f \circ \varphi)(x)$ for a suitable measurable function $\varphi : [0,1] \to [0,1]$.

Theorem 2.1 *For any fixed $n \geq 1$, let V_n be the Hardy operator defined on L_p by formula (6). Then we have:*

(a) $\|V_n\| \leq \frac{1}{\sqrt[p]{np+1}}$;

(b) V_n is a Volterra operator (that is compact and quasinilpotent) on $L_p[0,1]$; in particular, $V_n \xrightarrow{s} 0$ $(n \to \infty)$.

(c) $\mu(V_n) = 1$.

Proof (a) Indeed, for any $f \in L_p$, we have:

$$\begin{aligned}
\|V_n f\|_p^p &= \left\| x^n \int_0^x f(t)\,dt \right\|_p^p = \int_0^1 \left| x^n \int_0^x f(t)\,dt \right|^p dx \\
&\leq \int_0^1 x^{np} \left(\int_0^x |f(t)|\,dt \right)^p dx \leq \left(\int_0^1 |f(t)|\,dt \right)^p \left(\int_0^1 x^{np} dx \right) \\
&= \frac{1}{np+1} \left(\int_0^1 |f(t)|\,dt \right)^p \\
&\leq \frac{1}{np+1} \left(\left(\int_0^1 |f(t)|^p\,dt \right)^{\frac{1}{p}} \left(\int_0^1 1^q dt \right)^{\frac{1}{q}} \right)^p \\
&= \frac{1}{np+1} \int_0^1 |f(t)|^p\,dt = \frac{1}{np+1} \|f\|_p^p,
\end{aligned}$$

which shows that $\|V_n\| \leq \frac{1}{\sqrt[p]{np+1}}$, as desired.

(b) For any $\lambda \in \mathbb{C} \setminus \{0\}$ the eigenvalue equation of V_n amounts to the differential equation $\lambda g'(x) = x^n g(x)$ with initial condition $g(0) = 0$ where $g(x) = \int\limits_0^x f(t)\,dt$ with f as the eigenvector. The solution of this integrodifferential equation is $g(x) = f(x) \equiv 0$, which implies that V_n has no nonzero eigenvalues. Since V_n is compact, this implies that $\sigma(V_n) = \{0\}$, and this implies that V_n is quasinilpotent. Since V_n is compact, this shows that V_n is a Volterra operator.

(c) We have:

$$\begin{aligned}
V_n^k \mathbf{1} &= V_n^{k-1}(V_n \mathbf{1}) = V_n^{k-1}\left(x^n \int_0^x \mathbf{1} dt\right) = V_n^{k-1} x^{n+1} \\
&= V_n^{k-2}\left(V_n x^{n+1}\right) = V_n^{k-2}\left(x^n \int_0^x t^{n+1} dt\right) = V_n^{k-2} \frac{x^{2(n+1)}}{n+2} \\
&= \frac{1}{n+2}\left(V_n^{k-3}\left(V_n x^{2(n+1)}\right)\right) = \frac{1}{(n+2)(2n+3)} V_n^{k-3} x^{3(n+1)} \\
&= \frac{1}{(n+2)(2n+3)(3n+4)} V_n^{k-4} x^{4(n+1)} = \ldots = \\
&= \frac{1}{(n+2)(2n+3)(3n+4)\ldots((k-1)n+k)} x^{k(n+1)},
\end{aligned}$$

and hence

$$V_n^k \mathbf{1} = \frac{1}{(n+2)(2n+3)(3n+4)\ldots((k-1)n+k)} x^{k(n+1)} \quad (k \geq 1), \tag{7}$$

which implies by Müntz approximation theorem that

$$Span\left\{V_n^k \mathbf{1} : k \geq 0\right\} = L_p;$$

that is, $\mu(V_n) = 1$. The theorem is proved. □

Our next result characterizes in terms of extended eigenvectors a special class of the composition operators C_φ on $L_p[0,1]$ $(1 \leq p < +\infty)$ (for the related result, see [19]). We set $C_\lambda := C_{\lambda x}$; that is, $C_\lambda f(x) = f(\lambda x)$ for all $f \in L_p[0,1]$.

Theorem 2.2 *Let $A : L_p[0,1] \to L_p[0,1]$ be any nonzero bounded linear operator. Then $A = C_\lambda$ for some $\lambda \in (0,1)$ if and only if:*

(a) $A\mathbf{1} = \mathbf{1}$;

(b) there exists an integer $k \geq 1$ such that A is an extended eigenvector of the operator V_n^k corresponding to the extended eigenvalue $\lambda^{(n+1)k} \in (0,1)$, where V_n is the Hardy operator on L_p defined by $V_n f = x^n \int_0^x f(t)\,dt$.

Proof $\Leftarrow$. Let $AV_n^k = \lambda^{(n+1)k} V_n^k A$. Then

$$A\left(V_n^k\right)^m = \lambda^{(n+1)km}\left(V_n^k\right)^m A,$$

or

$$AV_n^{km} = \lambda^{km(n+1)} V_n^{km} A$$

for all $m \geq 0$. Thus, by considering that $A\mathbf{1} = \mathbf{1}$, we have

$$AV_n^{km}\mathbf{1} = \lambda^{km(n+1)} V_n^{km} A\mathbf{1} = \lambda^{km(n+1)} V_n^{km}\mathbf{1}$$

for all $m \geq 0$. Then by using formula (7), we obtain that

$$\begin{aligned} &\frac{1}{(n+2)(2n+3)(3n+4)\dots((km-1)n+km)} Ax^{km(n+1)} \\ &= \frac{1}{(n+2)(2n+3)(3n+4)\dots((km-1)n+km)} \lambda^{km(n+1)} x^{km(n+1)} \end{aligned}$$

or

$$Ax^{km(n+1)} = \lambda^{km(n+1)} x^{km(n+1)} = (\lambda x)^{km(n+1)}$$

for all $m \geq 0$, which by virtue of the Müntz theorem implies that $Af(x) = f(\lambda x) = C_\lambda f(x)$ for all f in L_p. This shows that $A = C_\lambda$.

$\Rightarrow$. It is obvious that $C_\lambda \mathbf{1} = \mathbf{1}$. Let us prove assertion *(b)*. We shall prove actually a more strong assertion that for every $k \geq 1$ C_λ is an extended eigenvector of the operator V_n^k corresponding to the extended eigenvalue $\lambda^{k(n+1)} \in (0,1)$, i.e.

$$C_\lambda V_n^k = \lambda^{k(n+1)} V_n^k C_\lambda. \tag{8}$$

Indeed, by induction we obtain for any $k \geq 1$ and $f \in L_p$

$$\begin{aligned} V_n^k f(x) = &\frac{x^{kn+(k-1)}}{(k-1)!(n+1)^{k-1}} \int_0^x f(t)\,dt - \frac{x^{(k-1)n+(k-2)}}{(k-2)!(n+1)^{k-1}} \int_0^x t^{n+1} f(t)\,dt + \\ &+ \frac{x^{(k-2)n+(k-3)}}{(k-3)!(n+1)^{k-1}} \int_0^x t^{2(n+1)} f(t)\,dt - \\ &- \frac{x^{(k-3)n+(k-4)}}{(k-4)!(n+1)^{k-1}} \int_0^x t^{3(n+1)} f(t)\,dt + \dots \\ &+ (-1)^{k+1} \frac{x^n}{(k-1)!(n+1)^{k-1}} \int_0^x t^{(k-1)(n+1)} f(t)\,dt. \end{aligned}$$

Then we have

$$\begin{aligned} V_n^k C_\lambda f(x) &= V_n^k f(\lambda x) \\ &= \frac{x^{kn+(k-1)}}{(k-1)!(n+1)^{k-1}} \int_0^x f(\lambda t)\,dt - \frac{x^{(k-1)n+(k-2)}}{(k-2)!(n+1)^{k-1}} \int_0^x t^{n+1} f(\lambda t)\,dt \\ &+ \dots + (-1)^{k+1} \frac{x^n}{(k-1)!(n+1)^{k-1}} \int_0^x t^{(k-1)(n+1)} f(\lambda t)\,dt \end{aligned}$$

for all $f \in L_p$. Since $0 \le t \le x$, by denoting $\eta = \lambda t$ we have that $0 \le \eta \le \lambda x$ and $dt = \frac{d\eta}{\lambda}$. By considering these in the last equality, we obtain:

$$\begin{aligned}
V_n^k C_\lambda f(x) &= \frac{x^{kn+(k-1)}}{(k-1)!\,(n+1)^{k-1}} \int_0^{\lambda x} f(\eta)\, \frac{d\eta}{\lambda} - \frac{x^{(k-1)n+(k-2)}}{(k-2)!\,(n+1)^{k-1}} \int_0^{\lambda x} \frac{\eta^{n+1}}{\lambda^{n+1}} f(\eta)\, \frac{d\eta}{\lambda} \\
&+ ... + (-1)^{k+1} \frac{x^n}{(k-1)!\,(n+1)^{k-1}} \int_0^{\lambda x} \frac{\eta^{(k-1)(n+1)}}{\lambda^{(k-1)(n+1)}} f(\eta)\, \frac{d\eta}{\lambda} \\
&= \frac{(\lambda x)^{kn+(k-1)}}{(k-1)!\,(n+1)^{k-1}\,\lambda^{kn+(k-1)+1}} \int_0^{\lambda x} f(\eta)\, d\eta - \\
&- \frac{(\lambda x)^{(k-1)n+(k-2)}}{(k-2)!\,(n+1)^{k-1}\,\lambda^{(k-1)n+(k-2)+n+2}} \int_0^{\lambda x} \eta^{n+1} f(\eta)\, d\eta + ... + \\
&+ (-1)^{k+1} \frac{(\lambda x)^n}{(k-1)!\,(n+1)^{k-1}\,\lambda^{(k-1)(n+1)+n+1}} \int_0^{\lambda x} \eta^{(k-1)(n+1)} f(\eta)\, d\eta \\
&= \frac{1}{\lambda^{k(n+1)}} \left[\frac{(\lambda x)^{kn+(k-1)}}{(k-1)!\,(n+1)^{k-1}} \int_0^{\lambda x} f(\eta)\, d\eta - \right. \\
&- \frac{(\lambda x)^{(k-1)n+(k-2)}}{(k-2)!\,(n+1)^{k-1}} \int_0^{\lambda x} \eta^{n+1} f(\eta)\, d\eta + ... + \\
&\left. + (-1)^{k+1} \frac{(\lambda x)^n}{(k-1)!\,(n+1)^{k-1}} \int_0^{\lambda x} \eta^{(k-1)(n+1)} f(\eta) \right] \\
&= \frac{1}{\lambda^{k(n+1)}} C_\lambda V_n^k f(x),
\end{aligned}$$

and thus $C_\lambda V_n^k f = \lambda^{k(n+1)} V_n^k C_\lambda f$ for all $f \in L_p$, which proves (8). The theorem is proved. □

Note that for any $y \in [0,1]$ the equation $x^n = y$ is solvable in the unit segment $[0,1]$. In particular, for any $\eta \in (0,1)$ there exists $\lambda = \lambda_\eta \in (0,1)$ such that $\eta = \lambda^{n+1}$. Therefore, Theorem 2.2 shows in particular that $(0,1) \subset ext(V_n)$ where $ext(V_n)$ denotes the set of extended eigenvalues of the operator V_n.

The set of all extended eigenvectors of V_n corresponding to the extended eigenvalue λ is denoted by $\{V_n\}'_\lambda$. Then the following is an immediate corollary of the above Theorem 2.2 and Theorem 4.1 of paper [19].

Corollary 2.1 *If $\lambda \in (0,1)$ is an extended eigenvalue of operators $V_n := M_{x^n} V$ and $M_x + V$ on L_p $(1 \le p < \infty)$ (where M_x is a multiplication operator and V is an integration operator), then*

$$C_\lambda \in \{V_n\}'_\lambda \cap \{M_x + V\}'_\lambda .$$

Note that an implication $\Rightarrow$ in Theorem 4.1 of [19] is proved by using a sufficiently deep result of Malamud (see [25, 26]).

Here we give an immediate proof of this implication.

Proposition 1 *Let $\lambda \in (0,1)$, $p \ge 1$ be any integer and C_λ is a composition operator on L^p $(1 \le p < \infty)$ associated to λ defined by $C_\lambda f(x) = f(\lambda x)$, $f \in L_p[0,1]$. Then*

$$C_\lambda (\alpha V^p + \beta M_x^p) = \lambda^p (\alpha V^p + \beta M_x^p) C_\lambda \tag{9}$$

for every $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha| + |\beta| \neq 0$.

Proof Indeed, for every polynomial $q(x) = \sum_{j=0}^{\deg q} q_j x^j$ we have:

$$
\begin{aligned}
& C_\lambda \left[\alpha V^p + \beta M_x^p\right] q(x) \\
& = \alpha C_\lambda V^p q(x) + \beta C_\lambda M_x^p q(x) \\
& = \alpha C_\lambda \int_0^x \frac{(x-t)^{p-1}}{(p-1)!} q(t)\, dt + \beta C_\lambda \left(x^p q(x)\right) \\
& = \alpha \int_0^{\lambda x} \frac{(\lambda x - t)^{p-1}}{(p-1)!} \sum_{j=0}^{\deg q} q_j t^j dt + \beta (\lambda x)^p q(\lambda x) \\
& = \frac{\alpha}{(p-1)!} \int_0^{\lambda x} \sum_{i=0}^{p-1} C_{p-1}^i (\lambda x)^{p-1-i} (-t)^i \sum_{j=0}^{\deg q} q_j t^j dt \\
& + \lambda^p \left(\beta x^p (C_\lambda q)(x)\right) \\
& = \frac{\alpha}{(p-1)!} \int_0^{\lambda x} \sum_{i=0}^{p-1} (-1)^i C_{p-1}^i (\lambda x)^{p-1-i} \sum_{j=0}^{\deg q} q_j t^{i+j} dt \\
& + \lambda^p \left(\beta M_x^p C_\lambda q(x)\right) \\
& = \frac{\alpha}{(p-1)!} \sum_{i=0}^{p-1} (-1)^i C_{p-1}^i (\lambda x)^{p-1-i} \sum_{j=0}^{\deg q} q_j \int_0^{\lambda x} t^{i+j} dt \\
& + \lambda^p \left(\beta M_x^p C_\lambda q(x)\right) \\
& = \frac{\alpha}{(p-1)!} \sum_{i=0}^{p-1} (-1)^i C_{p-1}^i (\lambda x)^{p-1-i} \sum_{j=0}^{\deg q} \frac{q_j (\lambda x)^{i+j+1}}{i+j+1} \\
& + \lambda^p \left(\beta M_x^p C_\lambda q(x)\right) \\
& = \frac{\alpha}{(p-1)!} \sum_{i=0}^{p-1} \sum_{j=0}^{\deg q} (-1)^i C_{p-1}^i \frac{q_j}{i+j+1} \lambda^p \lambda^{-(i+1)} \lambda^{i+j+1} x^{p-1-i} x^{i+j+1} \\
& + \lambda^p \left(\beta M_x^p C_\lambda q(x)\right) \\
& = \frac{\alpha}{(p-1)!} \sum_{i=0}^{p-1} \sum_{j=0}^{\deg q} (-1)^i C_{p-1}^i \frac{q_j}{i+j+1} \lambda^p \lambda^j x^{p+j} + \lambda^p \left(\beta M_x^p C_\lambda q(x)\right) \\
& = \frac{\alpha}{(p-1)!} \sum_{i=0}^{p-1} \sum_{j=0}^{\deg q} (-1)^i C_{p-1}^i (\lambda x)^{p-1-i} q_j \frac{\lambda^{i+j+1} x^{i+j+1}}{i+j+1} + \lambda^p \left(\beta M_x^p C_\lambda q(x)\right) \\
& = \frac{\alpha}{(p-1)!} \sum_{i=0}^{p-1} (-1)^i C_{p-1}^i (\lambda x)^{p-1-i} \sum_{j=0}^{\deg q} q_j \frac{(\lambda x)^{i+j+1}}{i+j+1} + \lambda^p \left(\beta M_x^p C_\lambda q(x)\right) \\
& = \frac{\alpha}{(p-1)!} \sum_{i=0}^{p-1} (-1)^i C_{p-1}^i (\lambda x)^{p-1-i} \lambda^{i+1} \sum_{j=0}^{\deg q} q_j \lambda^j \int_0^x t^{i+j} dt + \lambda^p \left(\beta M_x^p C_\lambda q(x)\right)
\end{aligned}
$$

$$= \lambda^p \frac{\alpha}{(p-1)!} \int_0^x \sum_{i=0}^{p-1} C_{p-1}^i x^{p-1-i} (-t)^i \lambda^{-(i+1)} \lambda^{i+1} \sum_{j=0}^{\deg q} q_j (\lambda t)^j \, dt + \lambda^p (\beta M_x^p C_\lambda q(x))$$

$$= \lambda^p \alpha \int_0^x \frac{(x-t)^{p-1}}{(p-1)!} q(\lambda t) \, dt + \lambda^p (\beta M_x^p C_\lambda q(x))$$

$$= \lambda^p \alpha \int_0^x \frac{(x-t)^{p-1}}{(p-1)!} (C_\lambda q)(t) \, dt + \lambda^p (\beta M_x^p C_\lambda q)(x)$$

$$= \lambda^p (\alpha V^p + \beta M_x^p) C_\lambda q,$$

which obviously gives (9). The proposition is proved. □

3. Extended eigenvectors of the Volterra double integration operator

Let $W_p^{(n)}([0,1] \times [0,1])$ denote the Sobolev space of functions in two variables defined in the unit square $[0,1] \times [0,1]$, where $1 \leq p < \infty$ and $n \geq 2$. In the space $W_p^{(n)}([0,1] \times [0,1])$, we consider the Volterra integration operator in two variables,

$$(Wf)(x,y) := \int_0^x \int_0^y f(t,\tau) \, d\tau dt.$$

Denote by E_{xy} the subspace of the space $W_p^{(n)}([0,1] \times [0,1])$ consisting of functions that depend on the product xy. It is easy to see that

$$E_{xy} = span \left\{ (xy)^k : k = 0, 1, 2, ... \right\}$$

and $E_{xy} \in Lat(W)$; that is, $WE_{xy} \subset E_{xy}$. Set $W_{xy} := W|E_{xy}$, i.e.

$$(W_{xy} f)(xy) = \int_0^x \int_0^y f(t\tau) \, d\tau dt.$$

The study of this operator is important at least in view of its following relation with the integral operator

$$(\mathcal{K}_{\log x} f)(x) := \int_0^x \log \frac{x}{y} f(y) \, dy$$

on $W_p^{(n)}[0,1]$ with the kernel function $\mathcal{K}(x,y) = \log \frac{x}{y}$:

$$W_{xy} f(xy) = \int_0^{xy} \log \frac{xy}{\nu} f(v) \, d\nu$$

(see [17]).

In the space $W_p^{(n)}\left([0,1]\times[0,1]\right)$ we define the Duhamel product as follows:

$$(f \circledast g)(x,y) := \frac{\partial^2}{\partial x \partial y}\int\limits_0^x\int\limits_0^y f(x-t,y-\tau)g(t,\tau)d\tau dt$$

$$= \int\limits_0^x\int\limits_0^y \frac{\partial^2}{\partial x \partial y} f(x-t,y-\tau)g(t,\tau)d\tau dt +$$

$$+\int\limits_0^x \frac{\partial}{\partial x} f(x-t,0)g(t,y)dt +$$

$$+\int\limits_0^y \frac{\partial}{\partial y} f(0,y-\tau)g(x,\tau)d\tau + f\left(0,0\right)g\left(x,y\right).$$

This formula implies that if $f,g\in E_{xy}$, then

$$(f \circledast g)(xy) = \int\limits_0^x\int\limits_0^y \frac{\partial^2}{\partial x \partial y} f((x-t)\left(y-\tau\right))g(t\tau)d\tau dt + f\left(0\right)g\left(xy\right)$$

$$= \int\limits_0^x\int\limits_0^y \left[f'\left(\left(x-t\right)\left(y-\tau\right)\right) + \left(x-t\right)\left(y-\tau\right)f''\left(\left(x-t\right)\left(y-\tau\right)\right)\right]g(t\tau)d\tau dt +$$

$$+ f\left(0\right)g\left(xy\right). \tag{10}$$

Formula (10) easily implies that $\mathbf{1}\circledast g\left(xy\right) = g\left(xy\right)\circledast \mathbf{1} = g\left(xy\right)$ for all $g\in E_{xy}$,

$$W_{xy}^k g\left(xy\right) = \frac{\left(xy\right)^k}{\left(k!\right)^2}\circledast g\left(xy\right) \quad \left(k\geq 1\right), \tag{11}$$

and the “Duhamel operator” $\mathcal{D}_f$ defined by

$$\left(\mathcal{D}_f g\right)\left(xy\right) := \left(f\circledast g\right)\left(xy\right) \tag{12}$$

is bounded in E_{xy}. More generally, it can be also proved by the arguments of paper [17] that $\left(E_{xy},\circledast\right)$ is the Banach algebra.

Here we shall study the extended eigenvalues and extended eigenvectors of the double integration operator W_{xy} on the space E_{xy}. Namely, we describe in terms of the Duhamel operator and composition operator the extended eigenvectors of the operator W_{xy}.

Recall that if $\theta:[0,1]\times[0,1]\to[0,1]\times[0,1]$ is a measurable complex-valued function, the composition operator C_θ is defined by the formula

$$C_\theta f\left(x,y\right) := \left(f\circ\theta\right)\left(x,y\right) = f\left(\theta\left(x,y\right)\right).$$

The composition operator $C_{\lambda(x,y)}$, where $\lambda\in\left(0,1\right)$, will be denoted simply as C_λ.

The main result of this section is the following.

Theorem 3.1 *Let* $\lambda \in \mathbb{C}$, $A \in \mathcal{L}(E_{xy}) \setminus \{0\}$, *and let* W_{xy} *be the Volterra double integration operator on* $E_{xy} \subset W_p^{(n)}([0,1] \times [0,1])$. *If* $\lambda \in (0,1)$ *then* $AW_{xy} = \lambda W_{xy} A$ *if and only if an operator* A *has the form* $A = \mathcal{D}_{A1} C_\lambda$, *i.e.*

$$(Af)(xy) = \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y (A1)((x-u)(y-v)) C_\lambda f(uv)\, dv du,$$

where $\mathcal{D}_{A1}$ *is the Duhamel operator in* E_{xy} *defined by formula* (18) *and* $(C_\lambda f)(xy) := f(\lambda(xy))$ *is a composition operator in* E_{xy}.

Proof We shall use some arguments from [18]. Indeed, suppose that $AW_{xy} = \lambda W_{xy} A$. Then it is clear that

$$AW_{xy}^n = \lambda^n W_{xy}^n A$$

for each $n \geq 0$; that is,

$$AW_{xy}^n f(xy) = \lambda^n W_{xy}^n Af(xy)$$

for each $f \in E_{xy}$. In particular,

$$AW_{xy}^n 1 = \lambda^n W_{xy}^n A1 \tag{13}$$

for each $n \geq 0$. Since

$$W_{xy}^n f(xy) = \frac{(xy)^n}{(n!)^2} \circledast f(xy) \tag{14}$$

for all $f \in E_{xy}$ and $n \geq 0$ (see formula (11)), we obtain from (11) and (14) that

$$A\left(\frac{(xy)^n}{(n!)^2} \circledast 1\right) = \lambda^n \left(\frac{(xy)^n}{(n!)^2} \circledast A1\right),$$

i.e.

$$A(xy)^n = (\lambda(xy))^n \circledast A1,\ n \geq 0. \tag{15}$$

It follows from (15) that

$$Ap(xy) = p(\lambda(xy)) \circledast A1$$

for every polynomial p in xy. Since polynomials in xy are dense in the space E_{xy} and $(E_{xy}, \circledast)$ is a Banach algebra, we have from the last equality that

$$\begin{aligned} Af(xy) &= A1 \circledast f(\lambda(xy)) = A1 \circledast C_\lambda f(xy) \\ &= \mathcal{D}_{A1} C_\lambda f(xy) \\ &= \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y (A1)((x-u)(y-v)) C_\lambda f(uv)\, dv du \end{aligned}$$

for all $f \in E_{xy}$.

We now show that every operator of the form $A = \mathcal{D}_{A1} C_\lambda$ satisfies the equation

$$AW_{xy} = \lambda W_{xy} A.$$

Indeed, we have for each $f \in E_{xy}$ that:

$$\begin{aligned}
(AW_{xy}f)(xy) &= (\mathcal{D}_{A1}C_{\lambda}W_{xy}f)(xy) = \mathcal{D}_{A1}(W_{xy}f)(\lambda(xy)) \\
&= A1 \circledast (W_{xy}f)(\lambda(xy)) = A1 \circledast ((\lambda(xy)) \circledast f(\lambda(xy))) \\
&= \lambda(xy) \circledast (A1 \circledast f(\lambda(xy))) = \lambda(xy) \circledast \mathcal{D}_{A1}C_{\lambda}f(xy) \\
&= \lambda[xy \circledast \mathcal{D}_{A1}C_{\lambda}f(xy)] = \lambda W_{xy}\mathcal{D}_{A1}C_{\lambda}f(xy) \\
&= \lambda W_{xy}Af(xy).
\end{aligned}$$

Theorem 3.1 is proved. □

Corollary 3.1 *The composition operator C_{θ} with $\theta(x,y) = \theta(xy)$ satisfies the equation $C_{\theta}W_{xy} = \lambda W_{xy}C_{\theta}$, where $\lambda \in \mathbb{D}\setminus\{0\}$, if and only if $\theta(xy) = \lambda(xy)$.*

Proof By using assertion of Theorem 3.1 and the obvious fact that $C_{\theta}1 = 1$, we have that $C_{\theta}W_{xy} = \lambda W_{xy}C_{\theta}$ if and only if

$$C_{\theta}f(xy) = \frac{\partial^2}{\partial x \partial y}\int_0^x\int_0^y f(\lambda uv)\,dudv = f(\lambda xy) = C_{\lambda}f(xy)$$

for all $f \in E_{xy}$, which proves that $C_{\theta} = C_{\lambda}$, i.e. $\theta(xy) = \lambda(xy)$. The corollary is proved. □

4. $*$-generators of the Banach algebra $\left(W_p^{(n)}[0,1], *\right)$

Recall that the Sobolev space $W_p^{(n)}[0,1]$ $(1 \le p < \infty)$ is a Banach algebra with respect to the classical convolution product $*$ and the Duhamel product $\circledast$. The following lemma is also known, which gives a $\circledast$-invertibility criterion for the elements of the Banach algebra $\left(W_p^{(n)}[0,1], \circledast\right)$ (see, for instance, Karaev [17]).

Lemma 4.1 *Let $f \in W_p^{(n)}[0,1]$. Then f is $\circledast$-invertible if and only if $f(0) \neq 0$.*

Actually, this lemma shows that $f \in W_p^{(n)}[0,1]$ generates $\left(W_p^{(n)}[0,1], \circledast\right)$ if and only if $f(0) \neq 0$.

For any function $k \in W_p^{(n)}[0,1]$, let us define the usual convolution operator $\mathcal{K}_k$ on $W_p^{(n)}[0,1]$ by the formula

$$(\mathcal{K}_k f)(x) := \int_0^x k(x-t)\,f(t)\,dt. \tag{16}$$

Our following result gives some equivalent characterization of $*$-generators of the radical Banach algebra $\left(W_p^{(n)}[0,1], *\right)$.

Theorem 4.1 *Let $f \in W_p^{(n)}[0,1]$ and $f(0) \neq 0$. Then f is a $*$-generator of the algebra $\left(W_p^{(n)}[0,1], *\right)$ if and only if*

$$span\left\{1, F, \mathcal{K}_f F, \mathcal{K}_f^2 F, ...\right\} = W_p^{(n)}[0,1],$$

where $F(x) := \int_0^x f(t)\,dt$ *and* $\mathcal{K}_f$ *is an operator on* $W_p^{(n)}$ *defined by* (16)*.*

Proof Since $F'(x) = f(x)$, we have $\mathcal{K}_f = \mathcal{D}_F$; that is, $\mathcal{K}_f g = \mathcal{D}_F g$ for all $g \in W_p^{(n)}[0,1]$, where $\mathcal{D}_F$ is the Duhamel operator defined $W_p^{(n)}[0,1]$ by $\mathcal{D}_F g = F \circledast g$. In particular,

$$\begin{aligned}
\mathcal{K}_f f = \mathcal{D}_F f &= \frac{d}{dx}\int_0^x f(x-t)\,F(t)\,dt \\
&= \int_0^x f'(x-t)\,F(t)\,dt + f(0)\,F(x) \\
&= \mathcal{D}_f F,
\end{aligned}$$

where $\mathcal{D}_f$ is an invertible operator in $W_p^{(n)}[0,1]$ (see Lemma 4.1). Thus,

$$f = \mathcal{D}_f 1 \tag{17_1}$$

and

$$f * f = \mathcal{D}_f F. \tag{17_2}$$

Further, we have:

$$\begin{aligned}
f * f * f = \mathcal{K}_f^2 f = \mathcal{K}_f(\mathcal{K}_f f) &= \mathcal{K}_f(\mathcal{D}_f F) \\
&= \mathcal{K}_f(\mathcal{K}_{f'} + f(0)\,I)\,F \\
&= (\mathcal{K}_f\mathcal{K}_{f'} + f(0)\,\mathcal{K}_f)\,F \\
&= (\mathcal{K}_{f'} + f(0)\,I)(\mathcal{K}_f F) \\
&= \mathcal{D}_f(\mathcal{K}_f F),
\end{aligned}$$

and thus

$$f * f * f = \mathcal{D}_f(\mathcal{K}_f F); \tag{17_3}$$

$$\begin{aligned}
f * f * f * f = \mathcal{K}_f^3 f = \mathcal{K}_f\left(\mathcal{K}_f^2 f\right) &= \mathcal{K}_f\mathcal{D}_f(\mathcal{K}_f F) \\
&= \mathcal{D}_f\mathcal{K}_f(\mathcal{K}_f F) \\
&= \mathcal{D}_f\left(\mathcal{K}_f^2 F\right),
\end{aligned}$$

and thus

$$f * f * f * f = \mathcal{D}_f\left(\mathcal{K}_f^2 F\right). \tag{17_4}$$

By induction we deduce that

$$\mathcal{K}_f^m f = \mathcal{D}_f\left(\mathcal{K}_f^{m-1} F\right) \quad (\forall m \geq 1). \tag{17_{m+1}}$$

Now, from formulas (17_{m+1}), $m \geq 0$, we have:

$$\begin{aligned}
span\{f, f*f, f*f*f, ...\} &= span\left\{\mathcal{D}_f 1, \mathcal{D}_f F, \mathcal{D}_f(\mathcal{K}_f F), \mathcal{D}_f\left(\mathcal{K}_f^2 F\right), ...\right\} \\
&= clos\mathcal{D}_f span\left\{1, F, \mathcal{K}_f F, \mathcal{K}_f^2 F, ...\right\}.
\end{aligned}$$

From this, by considering that the condition $f(0) \neq 0$ means invertibility of the corresponding Duhamel operator $\mathcal{D}_f$ (see Lemma 4.1), we deduce that

$$span\{f, f*f, f*f*f, ...\} = W_p^{(n)}[0,1]$$

if and only if

$$span\left\{1, F, \mathcal{K}_f F, \mathcal{K}_f^2 F, ...\right\} = W_p^{(n)}[0,1],$$

which proves Theorem 4.1. □

5. On the Volterra integral equations in the Sobolev space $W_p^{(n)}[0,1]$

In this section, we consider the Volterra integral equation

$$\int_0^x k(x-t) f(t)\, dt = g(x), \tag{18}$$

where $k \in W_p^{(n)}[0,1]$ is the kernel function $k(x,y)$ depending on $x-y$ and $g \in W_p^{(n)}[0,1]$ is a given nonzero function. It is well known from the theory of integral equations that every equation (18) has a solution in the Sobolev space $W_p^{(n)}[0,1]$. Let us denote

$$\mathcal{G}_g := \left\{u \in W_p^{(n)}[0,1] : u \text{ is a solution of } (18)\right\}.$$

It is clear that $g \notin \mathcal{G}_g$, because $\sigma_p(\mathcal{K}_k) = \emptyset$ and $\sigma(\mathcal{K}_k) = \{0\}$ (that is, $\mathcal{K}_k$ is quasinilpotent in $W_p^{(n)}[0,1]$).

Let us denote $(\mathcal{G}_g)_1 := \left\{u \in \mathcal{G}_g : \|u\|_{W_p^{(n)}} = 1\right\}$. Thus, the following problem naturally arises.

Problem 1 *To calculate* $dist\left(g, (\mathcal{G}_g)_1\right)$.

In this short section, we prove the following theorem, which estimates $dist\left(g, (\mathcal{G}_g)_1\right)$ in terms of the kernel function k. The proof is based on the Duhamel product method.

Theorem 5.1 *Let* $g \in W_p^{(n)}[0,1] \setminus \{0\}$ *be any function, and consider equation* (18) *in the Sobolev space* $W_p^{(n)}[0,1]$. *Then*

$$\inf_{g \in W_p^{(n)} \setminus \{0\}} dist\left(g, (\mathcal{G}_g)_1\right) \geq \frac{1}{\left\|\left(-1 + \int_0^x k(t)\, dt\right)^{-1\circledast}\right\|_{W_p^{(n)}}},$$

where $f^{-1\circledast}$ *denotes the* $\circledast$*-inverse of the function* $f \in W_p^{(n)}[0,1]$, *and* $\circledast$ *is the usual Duhamel product in* $W_p^{(n)}[0,1]$.

Proof We set $F(x) := -1 + \int_0^x k(t)\, dt$. Then the equation

$$k * f = g$$

can be rewritten as

$$\frac{d}{dx}\int_0^x F(x-t)\,f(t)\,dt + f(x) = g(x),$$

or $F \circledast f = g - f$. Then, by considering that $F(0) = -1 \neq 0$, by Lemma 4.1 in Section 4, there exists a function $\Phi \in W_p^{(n)}[0,1]$ such that $\Phi \circledast F = \mathbf{1}$. Hence,

$$\Phi \circledast F \circledast f = \Phi \circledast (g - f);$$

that is, $f = \Phi \circledast (g - f)$. Therefore, for any $f \in (\mathcal{G}_g)_1$, we have

$$1 = \|f\|_{W_p^{(n)}} = \|\Phi \circledast (g-f)\|_{W_p^{(n)}} \leq \|\Phi\|_{W_p^{(n)}} \|g-f\|_{W_p^{(n)}} \tag{19}$$

(because $W_p^{(n)}[0,1]$ is a Banach algebra with unit $\mathbf{1}$ with respect to the Duhamel product $\circledast$ (see [17])). It follows from inequality (19) that

$$\|g-f\| \geq \frac{1}{\|\Phi\|_{W_p^{(n)}}} \tag{20}$$

for each $f \in (\mathcal{G}_g)_1$. Since $\Phi = F^{-1\circledast}$, the last inequality (20) implies that

$$\|g-f\| \geq \frac{1}{\|F^{-1\circledast}\|_{W_p^{(n)}}} = \frac{1}{\left\|\left(-1+\int_0^x k(t)\,dt\right)^{-1\circledast}\right\|_{W_p^{(n)}}}$$

for all $f \in (\mathcal{G}_g)_1$, and hence

$$dist\left(g, (\mathcal{G}_g)_1\right) \geq \frac{1}{\left\|\left(-1+\int_0^x k(t)\,dt\right)^{-1\circledast}\right\|_{W_p^{(n)}}}. \tag{21}$$

Since $g \in W_p^{(n)} \setminus \{0\}$ is an arbitrary function, inequality (21) means that

$$\inf_{g \in W_p^{(n)} \setminus \{0\}} dist\left(g, (\mathcal{G}_g)_1\right) \geq \frac{1}{\left\|\left(-1+\int_0^x k(t)\,dt\right)^{-1\circledast}\right\|_{W_p^{(n)}}},$$

which proves the theorem. $\square$

6. On the numerical radius of operator V

Here we will consider the Lebesgue space $L_2 = L_2[0,1]$ of real-valued functions in the unit segment $[0,1]$. Let V, $Vf(x) = \int_0^x f(t)\,dt$, be the Volterra integration operator in this space. Recall that the description of the numerical range of operator V acting in the complex Hilbert space $L_2[0,1] isknown$, and it is also known that $\|V\| = \frac{2}{\pi}$ (see, for example, Halmos [14]). In this short section we calculate the numerical radius $w(V)$ of the operator V acting in the real space $L_2[0,1]$.

Theorem 6.1 *Let V be the Volterra integration operator on the real space $L_2[0,1]$. Then $w(V) = \frac{1}{2}$.*

Proof Let $f \in L_2[0,1]$, $\|f\|_2 = 1$, be an arbitrary real valued function. Then we have:

$$\begin{aligned} |\langle Vf, f\rangle| &= \left|\left\langle \int_0^x f(t)\,dt, f \right\rangle\right| = \left|\int_0^1 \left(\int_0^x f(t)\,dt\right) f(x)\,dx\right| \\ &= \left|\int_0^1 \left(\int_0^x f(t)\,dt\right) d\left(\int_0^x f(t)\,dt\right)\right| = \left.\frac{\left(\int_0^x f(t)\,dt\right)^2}{2}\right|_0^1 \\ &= \frac{1}{2}\left(\int_0^1 f(t)\,dt\right)^2 \\ &\le \frac{1}{2}\left[\left(\int_0^1 f^2(t)\,dt\right)^{1/2}\left(\int_0^1 1^2 dt\right)^{1/2}\right]^2 \\ &= \frac{1}{2}\|f\|_2^2 = \frac{1}{2}. \end{aligned}$$

Thus, $|\langle Vf, f\rangle| \le \frac{1}{2}$ for all $f \in L_2[0,1]$ with $\|f\|_2 = 1$, which shows that $w(V) \le \frac{1}{2}$.

On the other hand, since $\mathbf{1} \in L_2[0,1]$ and $\|\mathbf{1}\|_2 = 1$, we have:

$$\langle V\mathbf{1}, \mathbf{1}\rangle = \left\langle \int_0^x 1dt, \mathbf{1} \right\rangle = \langle x, \mathbf{1}\rangle = \int_0^1 xdx = \frac{1}{2},$$

and therefore, since $\frac{1}{2} = |\langle V\mathbf{1}, \mathbf{1}\rangle| \le w(V) \le \frac{1}{2}$, we obtain that $w(V) = \frac{1}{2}$, which proves the theorem. □

7. On the spectral multiplicity of the shift operator

Let Γ be a unit circle of the complex plane $\mathbb{C}$, and $C(\Gamma)$ be the space of continuous functions on Γ. Let T be the shift operator defined on $C(\Gamma)$ by

$$Tf\left(e^{it}\right) = e^{it} f\left(e^{it}\right),$$

or

$$Tf(\xi) = \xi f(\xi),\ f \in C(\Gamma),$$

where $\xi := e^{it}$, $t \in [0, 2\pi)$.

It is well known that $\mu(T) = 2$, and the proof of the assertion that $\mu(T) \le 2$ is the main step in the proof of this equality (because it is relatively easy to show that the operator T has no cyclic vector). In this section, we give two different proofs of the following proposition.

Proposition 2 *Let T be the shift operator defined on $C(\Gamma)$ by $Tf(\xi) = \xi f(\xi)$. Then $\mu(T) \le 2$.*

Before giving the proofs of this proposition, let us state some auxiliary results on T-invariant subspaces in $C(\Gamma)$ (see Hasumi and Srinivasan [15]).

Let m be, just as earlier, the Lebesgue measure on Γ. The weak* closure of analytic polynomials in $L^\infty(dm)$ is denoted by $H^\infty(dm)$. We denote by $Z(X)$ the space of functions in $C(\Gamma)$ that vanish on a subset X of Γ. Let E be a closed subspace of $C(\Gamma)$. The following two results are due to Hasumi and Srinivasan (see [15, Theorems 1 and 2).

Lemma 7.1 *$TE = E$ if and only if $E = Z(X)$ for some closed subset X of Γ.*

Lemma 7.2 *If $TE \subseteq E$ but $TE \neq E$, then $E = \theta H^{\infty}(dm) \cap Z(X)$, where $\theta \in L^{\infty}(dm)$ with $|\theta| = 1$ $m-$a.e., and X is a closed set in Γ with $m(X) = 0$.*

Proof [First Proof of Proposition 2]We put

$$\Gamma_{+} := \Gamma \cap \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$$

and

$$\Gamma_{-} := \Gamma \cap \{z \in \mathbb{C} : \operatorname{Im} z < 0\}.$$

Let S be a closed subset of $\Gamma \backslash \Gamma_{+}$ such that $m(S) > 0$. Then there exists $f \in C(\Gamma)$ such that $f(\Gamma_{+}) = \{1\}$ and $f(S) = \{0\}$. Now, let K be a closed subset of $\Gamma \backslash \Gamma_{-}$ such that $m(K) > 0$. Then there exists $g \in C(\Gamma)$ such that $g(\Gamma_{-}) = \{1\}$ and $g(K) = \{0\}$. It is easy to see that f and g are linearly independent. Let us show that

$$clos\{p(\xi) f + q(\xi) g : p \text{ and } q \text{ runs over all polynomials}\} = C(\Gamma)$$

(i.e. $span\left\{e^{int}f, e^{imt}g : n, m = 0, 1, 2, ...\right\} = C(\Gamma)$). Assume on contrary that

$$E_{f,g} := clos\{p(\xi) f + q(\xi) g : p \text{ and } q \text{ runs over all polynomials}\} \neq C(\Gamma).$$

Since $E_{f,g} \neq \{0\}$, it is a nontrivial T-invariant subspace. Clearly, $\xi E_{f,g} \subset E_{f,g}$. Since $null(f) \cap null(g) = \emptyset$, it follows from Lemma 7.1 that $\xi E_{f,g} \neq E_{f,g}$ (that is, $E_{f,g}$ is a simply invariant subspace for T). Then, by Lemma 7.2, there exists a measurable function $\theta \in L^{\infty}(dm)$ and a closed subset X in Γ such that $|\theta| = 1$ $m-$a.e., $m(X) = 0$ and $E_{f,g} = \theta H^{\infty}(dm) \cap Z(X)$. Thus, we have that $null(f) \cap null(g) \neq \emptyset$, which is a contradiction. The proof is completed. □

Recall that the rational multiplicity of spectrum $\mu_R(A)$ of the operator $A \in \mathcal{L}(Y)$, where Y be a Banach space, is defined as follows:

$$\mu_R(A) := \min\{\dim E : span(R_{\lambda}(A) E : \lambda \in \mathbb{C} \backslash \sigma(A)) = Y\},$$

where $R_{\lambda}(A) = (\lambda I - A)^{-1}$ is the resolvent of A and $\sigma(A)$ stands for the spectrum of A. The subspace $E \subset Y$ with the property

$$span\{R_{\lambda}(A) E : \lambda \in \mathbb{C} \backslash \sigma(A)\} = Y$$

is called the rational cyclic subspace for A.

The following result belongs to Herrero [16].

Lemma 7.3 *Let $A \in \mathcal{L}(Y)$ and C be a subspace of Y. There exists a subspace $C' \supset C$, $\dim C' \leq \dim C + 1$ such that*

$$span\{A^n C' : n \geq 0\} = span\{R_{\lambda}(A) C : \lambda \in \mathbb{C} \backslash \sigma(A)\}.$$

In particular,

$$\mu(A) \leq \mu_R(A) + 1. \tag{22}$$

Proof [Second Proof of Proposition 2]Since $\sigma(T) = \Gamma,$ we have the following:

(a) If $|\lambda| < 1,$ then

$$R_\lambda(T) f = (\lambda I - T)^{-1} f = \left(\lambda - e^{it}\right)^{-1} f = -e^{it}\left(1 - \lambda e^{-it}\right)^{-1} f$$
$$= \left(-e^{it}\sum_{n=0}^{\infty} e^{-int}\lambda^n\right) f$$

for all $f \in C(\Gamma)$.

(b) If $|\lambda| > 1,$ then

$$R_\lambda(T) f = \left(\lambda - e^{it}\right)^{-1} f = \frac{1}{\lambda}\left(1 - \frac{1}{\lambda}e^{it}\right)^{-1} f$$
$$= \left(\frac{1}{\lambda}\sum_{n=0}^{\infty} e^{int}\lambda^{-n}\right) f$$

for all $f \in C(\Gamma)$.

Now by putting $f = \mathbf{1}$ and considering that $\mathbb{C} = \mathbb{D} \cdot \mathbb{C},$ from these formulas we obtain that

$$span\{R_\lambda(T)\mathbf{1} : \lambda \in \mathbb{C}\backslash\Gamma\} = C(\Gamma),$$

which shows that $\mu_R(T) = 1,$ and hence by virtue of inequality (22) (see Lemma 7.3), we deduce that $\mu(T) \leq 2,$ which completes the proof. □

8. Open problems

Let $-\infty \leq a < b \leq \infty$ and let ν and w be locally integrable nonnegative weight functions on (a, b). Let us consider the Volterra integral operators

$$Kf(x) := w(x)\int_a^x k(x,y) f(y)\nu(y)\,dy, \; x \in (a,b), \tag{23}$$

in Lebesgue spaces. Of course, besides other independent interests, these operators also play an important role in applications to spectral theory, integral and differential equations, and embeddings of Sobolev spaces (see, for instance paper [32] by Stepanov and Ushakova, and references therein).

A generalization of the Volterra integral operator (23) is the Hardy–Steklov type operator:

$$\mathcal{K}f(x) := w(x)\int_{a(x)}^{b(x)} k(x,y) f(y)\nu(y)\,dy, \tag{24}$$

with border functions $a(x)$ and $b(x)$ satisfying the following conditions:

(i) $a(x)$ and $b(x)$ are differentiable and strictly increasing on $(0,\infty)$;

(ii) $a(0) = b(0) = 0,$ $a(x) < b(x)$ for $0 < x < \infty,$ $a(\infty) = b(\infty) = \infty,$ and a continuous kernel $k(x,y) > 0$ on $\mathcal{R} := \{(x,y) : x > 0, \; a(x) < y < b(x)\}$.

In the limiting cases $a(x) = 0$ or $b(x) = \infty$, operator (24) is reduced to the Hardy type operators with only one variable boundary $a(x)$ or $b(x)$ (for more information about Hardy–Steklov operators see Stepanov and Ushakova [33]).

Let φ be a fixed nonnegative measurable function on $(0, \infty)$ that is not equivalent to 0. The multidimensional Hardy operator $H_{n,\varphi}$ is defined by

$$(H_{n,\varphi} f)(x) := \varphi(|x|) \int_{B_{|x|}} f(y)\, dy, \ x \in \mathbb{R}^n, \tag{25}$$

for all functions $f \in L_1^{\ell_\alpha}(\mathbb{R}^n)$, where $n \in \mathbb{N}$ and B_r is the open ball in $\mathbb{R}^n$ centered at the origin of radius $r > 0$. For basic facts about operator (25), see, for instance, Burenkov and Oinarov [5], Persson and Samko [28], and Samko [29], and references therein.

Note that there are many different necessary and sufficient conditions for $L_p - L_q$, $0 < p, q < \infty$, boundedness of operators (23), (24), and (25) (see, for instance, [5, 28, 29, 31, 32, 33] and their references). In particular, there exist sufficient conditions ensuring $L_p - L_p$ boundedness of operators $(23)-(25)$. Thus, for such bounded operators $(23)-(25)$ on L_p $(0 < p < \infty)$, it would be very interesting and important to solve the following problems:

Problem 2 *To calculate spectral multiplicities of operators* $(23)-(25)$.

Problem 3 *To investigate extended eigenvalues and extended eigenvectors of operators* $(23)-(25)$.

Problem 4 *For* $p = 2$, *to investigate the numerical range and numerical radius of operators* $(23)-(25)$.

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