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SOME CONJECTURES ABOUT L^p NORMS OF K-PLANE TRANSFORMS

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ABSTRACT. For $1 \leq k \leq n-1$, the k-plane transform $T_{k,n}$ carries functions f defined on R^n to functions $T_{k,n}f$ defined on the set of affine k-planes in R^n. It is known that $T_{k,n}$ maps L^p into L^q for certain values of p and q. In this article we formulate conjectures for the exact values of the norm of the $T_{k,n}$, and state also a conjecture asserting that the L^q norm of $T_{k,n}f$ changes monotonically when f is replaced by its symmetric decreasing rearrangement.

1 Introduction

For $1 \leq k \leq n-1$, let $G_{k,n}$ denote the Grassmann manifold of all k-dimensional subspaces of $\mathbb{R}^n$, and $M_{k,n}$ the set of all affine k-planes in $\mathbb{R}^n$. As explained, for example, in [M 1995, p.53], there is a unique Borel probability measure $\gamma_{k,n}$ on $G_{k,n}$ which is invariant under the action of the orthogonal group $O(n)$ on $G_{k,n}$. Each $\pi \in M_{k,n}$ has a unique representation of the form $\pi = x + \theta$, with $\theta \in G_{k,n}$, $x \in \mathbb{R}^n \cap \theta^\perp$. Define a measure $\mu_{k,n}$ on $M_{k,n}$ by

$$\mu_{k,n}(A) = \int_{G_{k,n}} \mathcal{L}^{n-k}(\{x \in \theta^\perp : x + \theta \in A\})\, d\gamma_{k,n}(\theta), \quad A \subset M,$$

where $\mathcal{L}^{n-k}$ denotes n-k dimensional Lebesgue measure on $\theta^\perp$, and m-planes in $\mathbb{R}^n$ are identified with $\mathbb{R}^m$ in the natural way. Then $\mu_{k,n}$ is invariant under the action on $M_{k,n}$ by rotations and translations of $\mathbb{R}^n$, and is

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the unique such measure, up to multiplication by positive constants. When the context is clear, we'll write dx for Lebesgue measure of the appropriate dimension, $d\pi$ for $d\mu_{k,n}(\pi)$, and $d\theta$ for $d\gamma_{k,n}(\theta)$. For nonnegative functions g on $M_{k,n}$, we have

$$\int_{M_{k,n}} g(\pi)\,d\pi = \int_{G_{k,n}} d\theta \int_{\theta^\perp} g(\theta + w)\,dw, \tag{1.1}$$

where $\pi = \theta + w$ with $\theta \in G_{k,n}$, $w \in \theta^\perp$.

For functions $f : \mathbb{R}^n \to \mathbb{R}$, the k-plane transform $T_{k,n}f : M_{k,n} \to \mathbb{R}$ is defined by

$$T_{k,n}f(\pi) = \int_\pi f(x)\,dx, \quad \pi \in M_{k,n}.$$

$T_{1,n}$ is called the x-ray transform, and $T_{n-1,n}$ the Radon transform. Oberlin and Stein [OS 1982], for $k = n-1$, Drury [D 1984] for $k \geq \frac{1}{2}(n-1)$, and Christ [C 1984], for $1 \leq k \leq n-1$, found that for each $q \in [1, n+1]$ there is a unique $p \in [1, (n+1)/(k+1)]$ such that $T_{k,n}$ maps $L^p(\mathbb{R}^n)$ boundedly into $L^q(M_{k,n}, \mu)$. The relation between p and q is given by the equivalent formulas

$$p' = \frac{n}{n-k}q', \quad p = \frac{nq}{n-k+kq}, \quad q = \frac{p(n-k)}{n-pk}, \tag{1.2}$$

where $1 \leq q \leq n+1$, $1 \leq p \leq q\frac{n+1}{k+1}$, and the primes denote Hölder conjugation. For $k = 1$, $1 \leq p < \frac{n+1}{k+1}$ the result had been found by Drury [D 1983]. These authors actually found more complete results involving mixed norms, but we'll consider here only the $L^p - L^q$ theorems. For $q > n+1$, that is $p > \frac{n+1}{k+1}$, there is no p such that $T_{k,n}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(M_{k,n})$.

For $1 \leq q \leq n+1$, let $||T_{k,n}||_{p,q}$ denote the corresponding operator norm. This note is devoted to discussion of the following problem:

Can one determine the exact value of $||T_{k,n}||_{p,q}$, or the associated extremal functions?

Here is a conjecture. The numbers a and b denote positive constants.

Conjecture 1 *Let $f_0(x) = (a + b|x|^2)^{-\frac{1}{2}\frac{n-k}{p-1}}$. Then f_0 is extremal for $||T_{k,n}||_{p,q}$.*

The statement means, of course, that

$$||T_{k,n}f_0||_{L^q(M_{k,n})}/||f_0||_{L^p(\mathbb{R}^n)} = ||T_{k,n}||_{p,q}.$$

We'll use analogous language when discussing other operators and the possible functions at which the operators achieve their norms. If Conjecture 1 is true, then functions of the form $cf_0(x - x_0)$ are also extremals. The wording of Conjecture 1 is chosen for expositional simplicity; we'll make no attempt in this article to characterize all extremals, or to consider uniqueness questions of any kind.

Conjecture 1 will follow if Conjectures 2 and 3 below are true. Conjecture 2 proposes a symmetrization inequality for k-plane transforms; Conjecture 3 is an extremal problem for a one-dimensional integral transform. In Conjecture 2, f denotes a nonnegative function on $\mathbb{R}^n$, and $f^\#$ its symmetric decreasing rearrangement. In Conjecture 3, f denotes a nonnegative function on $\mathbb{R}^+$.

Conjecture 2

$$||T_{k,n}f||_q \le ||T_{k,n}f^\#||_q, \quad 1 \le q \le n+1.$$

To state Conjecture 3, define, for $\alpha > 0$, the operator U_α by

$$U_\alpha f(x) = \int_x^\infty f(y)(y-x)^{\alpha-1}\,dy, \quad x \in (0,\infty),$$

The operators U_α are known variously as Riemann-Liouville operators, Weyl operators, or fractional integral operators.

Take $\beta > \alpha$. Let $p, q \in [1,\infty)$, and assume the parameters are related by the equivalent formulas

$$p' = \frac{\beta}{\beta-\alpha}q', \quad p = \frac{\beta q}{\beta - \alpha + \alpha q}, \quad q = \frac{p(\beta-\alpha)}{\beta - p\alpha}. \tag{1.3}$$

Let $||U_\alpha||_{\alpha,\beta,p,q}$ denote the norm of U_α acting as an operator from

$$L^p(\mathbb{R}^+, x^{\beta-1}\,dx) \to L^q(\mathbb{R}^+, x^{\beta-\alpha-1}\,dx).$$

A theorem of Flett [F 1958] implies that the norm is finite when (1.3) is satisfied.

Conjecture 3 *Let $f_0(x) = (a+bx)^{-\frac{\beta-\alpha}{p-1}}$. Then f_0 is extremal for $||U_\alpha||_{\alpha,\beta,p,q}$.*

In sections 2 and 3 we'll discuss background and partial results for Conjectures 2 and 3, respectively. Among other facts, we'll see that Conjecture 1 is a consequence of Conjecture 2 together with the cases $\alpha = k/2$, $\beta = n/2$ of Conjecture 3. We'll also see, in Theorem 1, that Conjecture 1 is known to be true when $k = 2$ and q is an integer, or when $q = 2$. Once the extremal functions are known, the sharp constants can be explicitly determined in some cases, but we will not present them here.

The search for sharp constants and extremal functions is also of interest when $0 < q < 1$. Section 4 contains a brief discussion.

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2 A symmetrization question for k-plane transforms.

Throughout this section, all functions are understood to be **nonnegative** and measurable Let f be a function on $\mathbb{R}^n$. Its *symmetric decreasing rearrangement* is the unique function $f^\#$ on $\mathbb{R}^n$ such that $f^\#$ is radial, i.e. $f^\#(x) = f_1(|x|)$ for a function f_1 on $\mathbb{R}^+$, f_1 is decreasing (=nonincreasing) on $\mathbb{R}^+$, f_1 is continuous from the right (a normalization), and $f^\#$ has the same distribution as f, i.e

$$\mathcal{L}^n(f^\# > t) = \mathcal{L}^n(f > t), \quad \forall t > 0.$$

From the equidistribution property it follows that f and $f^\#$ have the same L^p norms for every p. Information about symmetric decreasing rearrangements can be found, for example, in [BS 1988], [LL 1997], and [B 1994]. There are numerous results which assert that various functionals change monotonically when functions f are replaced by their symmetric decreasing rearrangements. One of the basic such theorems concerns three functions f_1, f_2, f_3 on $\mathbb{R}^n$.

Riesz-Sobolev Inequality

$$\int_{\mathbb{R}^{2n}} f_1(x) f_2(x-y) f_3(y)\, dy\, dx \le \int_{\mathbb{R}^{2n}} {f_1}^\#(x) {f_2}^\#(x-y) {f_3}^\#(y)\, dy\, dx. \tag{2.1}$$

For $n = 1$ (2.1) was discovered by F.Riesz in 1930. It was extended to $n \ge 1$ by Sobolev in 1939. A proof may be found in [LL 1997].

Let $*$ denote convolution on $\mathbb{R}^n$. Then, for $p \ge 1$ and (nonnegative) functions f and K on $\mathbb{R}^n$,

$$||f * K||_{L^p(\mathbb{R}^n)} = \sup\{\int_{\mathbb{R}^{2n}} g(x) K(x-y) f(y)\, dx\, dy : ||g||_{L^{p'}} = 1\}.$$

From (2.1), it follows that if K is symmetric decreasing, i.e. $K = K^\#$, then

$$||K * f||_{L^p(\mathbb{R}^n)} \le ||K * f^\#||_{L^p(\mathbb{R}^n)}. \tag{2.2}$$

The k−plane transform $T_{k,n}$ resembles a convolution operator whose kernel is Lebesgue measure on $\mathbb{R}^k \subset \mathbb{R}^n$. In this spirit, the proposed inequality $||T_{k,n} f||_q \le ||T_{k,n} f^\#||_q$ of Conjecture 2 can be viewed as an inequality in the same family as (2.2). However, the differences are likely more prominent than the similarities: To compute $T_{k,n} f$ requires both rotation and translation of the kernel, and Lebesgue measure on $\mathbb{R}^k$ does not much resemble a symmetric decreasing function on $\mathbb{R}^n$. Nevertheless, it is possible to prove Conjecture 2 when q is an *integer*. It is a special case of the following result.

Proposition 1 *Let q be an integer in $[1,...,n+1]$ and $f_1,...,f_q$ be functions on $\mathbb{R}^n$. Then*

$$\int_{M_{k,n}} \prod_{i=1}^{q} T_{k,n} f_i \, d\pi \le \int_{M_{k,n}} \prod_{i=1}^{q} T_{k,n}(f_i^{\#}) \, d\pi. \tag{2.3}$$

From (1.1) and Fubini's theorem, it follows that $\int_{M_{k,n}} T_{k,n} f \, d\pi = \int_{\mathbb{R}^n} f \, dx$. Thus, for $q = 1$ (2.3) holds with equality. For $q = n+1$ (2.3) is due to Christ [C 1984, Theorem D]. For $2 \le q \le n$, (2.3) can be proved using the same ideas, or can be deduced from the $q = n+1$ case by appropriate specialization of $f_{q+1},...,f_{n+1}$ and passage to limits. Here is an outline of a proof of Proposition 1 whose elements come from [C], but whose presentation is somewhat different.

The point of departure is an integral identity of Blaschke [Bk 1935]. For points $x_0,...,x_k \in \mathbb{R}^n$, $1 \le k \le n-1$, let $\det(x_0,...,x_k)$ denote the (unsigned) $\mathcal{L}^k$ measure of the convex hull of $\{x_0,...,x_k\}$. Let $F : \mathbb{R}^{n(k+1)} \to \mathbb{R}^+$.

Blaschke's identity [Bk 1935]

$$\int_{M_{k,n}} d\pi \int_{\pi^{k+1}} F(y_0,...,y_k)\, dy_0...dy_k$$
$$= c \int_{\mathbb{R}^{n(k+1)}} F(x_0,...,x_k) \det{}^{k-n}(x_0,...,x_k)\, dx_0...dx_k. \tag{2.4}$$

The integral on the left is taken over $k+1$ copies of π. Here, and later, c will denote a constant depending on the parameters which can change from line to line. Blaschke's identity has appeared often in works on integral geometry and "geometric probability". A nice history, and a proof, can be found in [Mi 1971]. See also [S 1976] and [SW 1992]. The identity is not well-known among analysts at large. Its application to L^p problems for k-plane transforms stems from its rediscovery by Drury [D 1984]. The proofs in [Mi 1971], [S 1976], and [D 1984] use induction on k starting with $k = 1$. In Section 5 we shall present a noninductive proof of (2.4).

Returning now to k-plane transforms, let us take in (2.4) $F = \prod_{i=0}^{k} f_i(y_i)$, where f_i are functions on $\mathbb{R}^n$. Write $T = T_{k,n}$. Then (2.4) gives

$$\int_{M_{k,n}} \prod_{i=0}^{k} T f_i(\pi)\, d\pi$$

$$= c \int_{\mathbb{R}^{n(k+1)}} \prod_{i=0}^{k} f_i(x_i) \det{}^{k-n}(x_0,...,x_k)\, dx_0...dx_k. \tag{2.5}$$

Let $m \geq k+1$. For almost all $x_0, ..., x_k \in \mathbb{R}^n$, the x_i are in general position, that is, they span a k-dimensional simplex. For such x_i, let $\pi(x_0, ..., x_k)$ denote the unique k-plane which contains them. Given $f_0, ..., f_m$, set

$$g_i(x_0, ..., x_k) = \int_{\pi(x_0,...,x_k)} f_i(y_i)\, dy_i.$$

Using again (2.4), we obtain

$$\int_{M_{k,n}} \prod_{i=0}^{m} Tf_i(\pi)\, d\pi$$

$$= c \int_{\mathbb{R}^{n(k+1)}} \prod_{i=0}^{k} f_i(x_i) \prod_{i=k+1}^{m} g_i(x_0, ..., x_k) \det{}^{k-n}(x_0, ..., x_k)\, dx_0 ... dx_k. \quad (2.6)$$

Next, let's convert the g_i to integrals over $\mathbb{R}^k$. Select a regular $k+1$ simplex S in $\mathbb{R}^k$ whose vertices $P_0, ..., P_k$ satisfy $|P_i| = 1$. Then $\sum_{i=0}^{k} P_i = 0$. Given $x_0, ..., x_k \in \mathbb{R}^n$, denote by $A = A(x_0, ..., x_k)$ the affine map of $\mathbb{R}^k$ into $\mathbb{R}^n$ which sends P_i to x_i, $i = 0, ..., k$. If the x_i are in general position and we identify $\pi(x_0, ..., x_k)$ with $\mathbb{R}^k$, then

$$|\det A| = c \, \det(x_0, \cdots, x_k), \quad c = \det{}^{-1}(P_0, ..., P_k).$$

Thus, for $i = k+1, ..., m$,

$$g_i(x_0, ..., x_k) = c \, \det(x_0, ..., x_k) \int_{\mathbb{R}^k} f_i(A(x_0, ..., x_k) w_i)\, dw_i,$$

and (2.6) may be written

$$\int_{M_{k,n}} \prod_{i=0}^{m} Tf_i(\pi)\, d\pi = c \int_{\mathbb{R}^{k(m-k)}} H(w_{k+1}, ..., w_m)\, dw_{k+1} ... dw_m, \quad (2.7)$$

where

$$H(w_{k+1}, ..., w_m) =$$
$$\int_{\mathbb{R}^{n(k+1)}} \prod_{i=0}^{k} f_i(x_i) \prod_{i=k+1}^{m} f_i(A(x_0, ..., x_k) w_i) \det{}^{m-n}(x_0, ..., x_k)\, dx_0 ... dx_k. \quad (2.8)$$

From the relation $P_i \cdot P_j = \frac{k+1}{k}\delta_{ij} - \frac{1}{k}$, it follows that, for $w \in \mathbb{R}^k$,

$$A(x_0, ..., x_k) w = \frac{k+1}{k} \sum_{j=0}^{k} (w \cdot P_j) x_j + \frac{1}{k+1} \sum_{j=0}^{k} x_j, \quad (2.9)$$

since the right hand side is affine, and agrees with the left hand side when w is one of the P_i.

Suppose next that l is an integer with $1 \le l \le k$, and that $F : \mathbb{R}^{n(l+1)} \to \mathbb{R}^+$. We claim that

$$\int_{M_{k,n}} d\pi \int_{\pi^{l+1}} F(y_0, ..., y_l)\, dy_0 ... dy_l$$
$$= c \int_{\mathbb{R}^{n(l+1)}} F(x_0, ..., x_l) \det{}^{k-n}(x_0, ..., x_l)\, dx_0 ... dx_l. \qquad (2.10)$$

This can be proved by induction. For $l = k$ (2.10) coincides with (2.4). Suppose that (2.10) has been established for some integer $l \in [2, k]$. Let us show that (2.10) also holds when l is replaced by $l-1$. Given $F(x_0, ..., x_{l-1})$, take $\epsilon > 0$, and apply (2.10) to the function $F(x_0, ..., x_{l-1})\, G(x_0, ..., x_l)$, where G is the characteristic function of the set $E \subset \mathbb{R}^{n(l+1)}$ defined thus: $(x_0, ..., x_l) \in E$ if the distance δ from x_l to $\pi(x_0, ..., x_{l-1})$ is less than 1, and the distance from the center of mass of $x_0, ..., x_{l-1}$ to the orthogonal projection of x_l onto $\pi(x_0, ..., x_{l-1})$ is less than ϵ. Then $\det^{k-n}(x_0, ..., x_l) \approx c \det^{k-n}(x_0, ..., x_{l-1})\, \delta$ when $(x_0, ..., x_l) \in E$. Integrate on the left with respect to y_l, on the right w.r.t. x_l, divide by ϵ^{l-1}, and let $\epsilon \to 0$. The result is (2.10), with l replaced by $l-1$.

Given $f_0, ..., f_l$ with $1 \le l \le k$, (2.10) implies that

$$\int_{M_{k,n}} \prod_{i=0}^{l} Tf_i(\pi)\, d\pi = c \int_{\mathbb{R}^{n(l+1)}} \prod_{i=0}^{l} f_i(x_i) \det{}^{k-n}(x_0, ..., x_l)\, dx_0 ... dx_l. \qquad (2.11)$$

Proposition 1 follows from (2.11) (for $2 \le q \le k+1$) and (2.7) - (2.9) (for $k+2 \le q \le n+1$), together with the following generalization of the Riesz-Sobolev inequality.

Symmetrization Theorem *Let $f_0, ..., f_M$ be functions on $\mathbb{R}^n$, K be a non-negative decreasing function on $\mathbb{R}^+$, , $1 \le N \le n$, and a_{ij} be real constants. Then*

$$\int_{\mathbb{R}^{n(N+1)}} \prod_{i=0}^{M} f_i(\sum_{j=0}^{N} a_{ij} x_j), K(\det(x_0, ..., x_N))\, dx_0 ... dx_N$$
$$\le \int_{\mathbb{R}^{n(N+1)}} \prod_{i=0}^{M} f_i^{\#}(\sum_{j=0}^{N} a_{ij} x_j)\, K(\det(x_0, ..., x_N))\, dx_0 ... dx_N. \quad (2.12)$$

For $K \equiv 1$, this theorem is due to to Brascamp, Lieb, and Luttinger [BLL 1974]. Their theorem does not require an upper bound on N. Christ [C, 1984] showed that the BLL proof could be adapted to prove results for multiple integrals which contain certain types of functions involving det. When K is decreasing, the BLL-Christ arguments work for the integral (2.12). The proof can be reduced ultimately to proving that if $A_0, ..., A_N$ are subsets of $\mathbb{R}$ and B is a convex centrally symmetric set in $\mathbb{R}^{N+1}$, then

$$\mathcal{L}^{N+1}((A_0 \times ... \times A_N) \cap B) \leq \mathcal{L}^{N+1}(({A_0}^{\#} \times ... \times {A_N}^{\#}) \cap B), \tag{2.13}$$

where $A^{\#}$ denotes the interval $(-a, a)$ such that $\mathcal{L}^1(A) = \mathcal{L}^1(A^{\#})$. (2.13) can be proved by application of the Brunn-Minkowski inequality, as in the proof of Theorem 1.2 of [BLL 1974].

As pointed out to us by Lieb, when K has the form $K(t) = t^{-\nu}$ with ν a nonnegative integer, the inequality in (2.12) can be easily deduced from the "$K \equiv 1$" case proved in [BLL 1974]. To accomplish this, one removes the det terms in (2.12) by means of the formula

$$\det{}^{-1}(x_0, ..., x_N) = c \int_{\mathbb{R}^N} \exp(-|\sum_{i=1}^{N} t_i(x_i - x_0)|^2)\, dt_1 ... dt_N.$$

Inequality (2.13) appears also in a paper by Pfiefer [P 1990, Theorem 3] about "random simplex" inequalities. From (2.13), Pfiefer deduced the special case of (2.12) when K is an arbitrary decreasing function and each $f_i(\sum_{j=0}^{N} a_{ij}x_j) = \chi_E(x_i)$ for some compact $E \subset \mathbb{R}^n$. Pfiefer was apparently unaware of [BLL 1974] and [C 1984]; he cites a paper of T.W. Anderson [A 1955] as a source for his work in this direction.

Let us return to Proposition 1, and contemplate the hypothesis $q \leq n + 1$. If $m \geq n + 1$ then $K(t) = t^{m-n}$ is no longer decreasing. Thus, if in Proposition 1 we take $q \geq n + 2$, we can no longer be sure that the integral in Proposition 1 will increase under symmetrization. And indeed, it sometimes does not. Take $n = 2$, $k = 1$, and let f be the characteristic function of the interior of an ellipse. The integrals in Proposition 1 are equal when $q = 3$. When $q > 3$, the integral with the symmetrized functions is strictly smaller than the original integral, unless the ellipse is a circle.

Consider now the case $q = 2$. From (2.11), we have

$$\int_{M_{k,n}} Tf\, Tg\, d\pi = c \int_{\mathbb{R}^{2n}} f(x)g(y)|x - y|^{k-n}\, dx\, dy. \tag{2.14}$$

For real $k \in (0, n)$, not necessarily an integer, it was shown by Hardy and Littlewood when $n = 1$, and by Sobolev when $n \geq 1$, that if $p = \frac{2n}{n+k}$, then the right hand side is bounded by $C(k, n)||f||_p||g||_p$. Lieb [L 1983] found the best constant and the extremal functions. Among them are $f(x) = g(x) = (a + b|x|^2)^{-\frac{n+k}{2}}$. We'll refer to this as the sharp HLS inequality. Another proof appears in [CL 1990] and [LL 1997].

Specializing to integers k, the sharp HLS inequality and (2.14) give

Proposition 2 *Conjecture 1 is true when* $q = 2$.

3 A class of integral operators

Suppose that f is a nonnegative radial function on $\mathbb{R}^n$. Then $f(x) = f_1(|x|)$, where f_1 is a nonnegative function on $\mathbb{R}^+$. Let $1 \le k \le n-1$. For $\pi \in M_{k,n}$, write $\pi = x + \theta$, with $\theta \in G_{k,n}$, $x \in \theta^\perp$. Then $T_{k,n}f(\pi)$ depends only on $|x|$. Write $r = |x|$ and $T_{k,n}f(\pi) = f_2(r)$. Then

$$f_2(r) = c_1 \int_0^\infty f_1((r^2+s^2)^{1/2}) s^{k-1}\, ds, \quad c_1 = |S^{k-1}|.$$

Put $g(u) = f_1(u^{1/2})$, $G(r) = f_2(r^{1/2})$. Then

$$G = \frac{1}{2} c_1 U_{\frac{k}{2}}\, g, \tag{3.1}$$

$$\int_{\mathbb{R}^n} f^p\, dx = \frac{1}{2} c_2 \int_{\mathbb{R}^+} g^p(x) x^{\frac{n}{2}-1}\, dx, \quad c_2 = |S^{n-1}|, \tag{3.2}$$

$$\int_{M_{k,n}} (T_{k,n}f)^q\, d\mu_{k,n} = \frac{1}{2} c_3 \int_{\mathbb{R}^+} G^q(x) x^{\frac{n-k}{2}-1}\, dx, \quad c_3 = |S^{n-k-1}|. \tag{3.3}$$

We remind the reader that the operators U_α were introduced in section 1. Formula (3.3) is proved with the aid of (1.1).

Recall that the parameters p, q, n, k are related by (1.2), and p, q, α, β by (1.3). From (3.1)-(3.3), one sees that for a pair n, k, Conjecture 1 for radial functions is equivalent to the truth of Conjecture 3 when $\alpha = \frac{k}{2}$ and $\beta = \frac{n}{2}$. If Conjectures 2 and 3 are both true for some pair n, k, then so is Conjecture 1 for that pair.

The determination of extremals for $||U_\alpha||_{\alpha,\beta,p,q}$ seems to be interesting for other values of the parameters α, β as well. Since it is a problem about simple one-variable integral transforms, one figures that its solution ought to be known. But this seems to be so only in two special cases, to be discussed below.

For $\alpha > 0$, let V_α denote the operator

$$V_\alpha f(x) = \int_0^x f(y)(x-y)^{\alpha-1}\, dy, \quad x \in \mathbb{R}^+.$$

Let $1 \le p \le q < \infty$, and $w_1, w_2 \in \mathbb{R}$. Suppose that the 5 parameters α, p, q, w_1, w_2 satisfy the relation

$$\frac{w_1}{p} - \frac{w_2}{q} = \alpha. \tag{3.4}$$

Flett [F 1958, Theorem 2] proved that if the parameters also satisfy certain inequalities, then V_α is a bounded mapping of

$$L^p(\mathbb{R}^+, x^{w_1-1}\,dx) \to L^q(\mathbb{R}^+, x^{w_2-1}\,dx).$$

If $F = U_\alpha f$ and we set $g(x) = f(x^{-1})x^{-\alpha-1}$, $G(x) = F(x^{-1})x^{\alpha-1}$, then $G = V_\alpha g$. This enables one to deduce boundedness results for U_α from those for V_α, and vice-versa. When $\alpha < \beta$ and (1.3) is satisfied, then, after transformation, Flett's hypotheses are satisfied. Thus, with the hypotheses of Conjecture 3, we have $||U_\alpha||_{\alpha,\beta,p,q} < \infty$. Observe that Flett's theorem contains 4 free parameters, whereas Conjecture 3 contains only 3. Of course, Flett's theorem is equivalent to a 4-parameter boundedness theorem for the U-operators, but for the extremals to possibly have the form $(a+bx)^{-\lambda}$ the parameters, at least in some cases, need to satisfy a second equation.

Let now $\alpha = 1$. Write $F = U_1 f$. From (1.3), it follows that

$$\beta = \frac{p(q-1)}{q-p}, \quad \frac{p-1}{\beta-p} = \frac{q}{p} - 1, \quad \frac{\beta-1}{p-1} = \frac{q}{p-q}. \tag{3.5}$$

Define $G(x) = F(x^{-\frac{q-p}{p}})$ and $g = G'$. Then $G = V_1 g$ and

$$||f||_{L^p(\mathbb{R}^+, x^{\beta-1}\,dx)} = c||g||_{L^p(\mathbb{R}^+, dx)},$$

$$||F||_{L^q(\mathbb{R}^+, x^{\beta-2}\,dx)} = c||G||_{L^q(\mathbb{R}^+, x^{\frac{q}{p}-q-1}\,dx)}.$$

The problem of finding extremals for

$$V_1 : L^p(\mathbb{R}^+, dx) \to L^q(\mathbb{R}^+, x^{\frac{q}{p}-q-1}\,dx)$$

was raised by Hardy and Littlewood [HL 1930], who proposed a candidate, but remarked that a proof that it did extremize seemed to require more knowledge of calculus of variations than they possessed. A proof was supplied soon after by Bliss [Bl 1930], who indeed did use ideas from calculus of variations. According to Bliss's theorem, the extremizers have the form $g(x) = (a + bx^{\frac{q-p}{p}})^{-\frac{q}{q-p}}$, so that $G(x)$ has the form $(a + bx^{-\frac{1}{p}(q-p)})^{-\frac{p}{(q-p)}}$. (The skeptical reader may check that $G' = g$.) It follows that extremals for $||U_1||_{p,q,\beta,\beta-1}$ have the form $F(x) = (a+bx)^{-\frac{p}{q-p}}$, $f(x) = -F'(x) = (a+bx)^{-\frac{q}{q-p}}$. From the third equation in (3.5), we obtain

Proposition 3 *Conjecture 3 is true when $\alpha = 1$.*

Thus, Conjecture 1 is true for radial functions when $k = 2$.

Bliss's theorem with $\alpha = 1$ plays a role in the proof by Talenti [T 1976] that extremals for the Sobolev inequality $||f||_{L^{np/(n-p)}(\mathbb{R}^n)} \le C||\nabla f||_{L^p(\mathbb{R}^n)}$ $1 < p < n$ are furnished by $f(x) = (a + b|x|^{p'})^{1-\frac{n}{p}}$.

Next, let us consider Conjecture 3 when $q = 2$. Suppose also that $\alpha = k/2$ and $\beta = n/2$, where k and n are integers with $1 \le k \le n-1$. Given $g : \mathbb{R}^+ \to \mathbb{R}^+$, write $G = U_{\frac{k}{2}} g$, and define f on $\mathbb{R}^n$ by $f(x) = g(|x|^2)$. Then (3.2) and (3.3) hold. From Proposition 2, it follows that when $p = 2n/(n+k)$ extremals for $||U_{\frac{k}{2}}||_{p,2,\frac{n}{2},\frac{n-k}{2}}$ are furnished by $g(x) = (a+bx)^{-\frac{1}{2}\frac{n-k}{p-1}}$. Thus, Conjecture 3 is true when $q = 2$, $\alpha = \frac{k}{2}$, $\beta = \frac{n}{2}$.

In fact, more is true.

Proposition 4 *Conjecture 3 is true when $q = 2$, $\beta = \frac{n}{2}$ with n a positive integer, and α is any real number with $0 < \alpha < \frac{n}{2}$.*

Proof. Let $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, and let $0 < \alpha < \beta$ be real numbers. Write $G = U_\alpha g$. Then

$$\int_0^\infty G^2(x) x^{\beta-\alpha-1}\, dx = \int_{\mathbb{R}^+\times\mathbb{R}^+} g(y)g(z)K_1(y,z)\, dy\, dz, \tag{3.6}$$

where

$$K_1(y,z) = \int_0^\infty (y-x)_+{}^{\alpha-1}(z-x)_+{}^{\alpha-1} x^{\beta-\alpha-1}\, dx. \tag{3.7}$$

For an integer $n \ge 1$, define again f on $\mathbb{R}^n$ by $f(x) = g(|x|^2)$. Then, by passing to polar coordinates and changing variables, we obtain

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} f(x)f(y)|x-y|^{2\alpha-n}\, dx\, dy = \int_{\mathbb{R}^+\times\mathbb{R}^+} g(y)g(z)K_2(y,z)\, dy\, dz, \tag{3.8}$$

where, denoting integration over S^{n-1} with respect to surface measure by du and dv,

$$K_2(y,z) = \frac{1}{4} y^{\frac{n}{2}-1} z^{\frac{n}{2}-1} \int_{S^{n-1}\times S^{n-1}} |y^{1/2}u - z^{1/2}v|^{2\alpha-n}\, du\, dv. \tag{3.9}$$

Apart from a constant multiple, the kernels K_1 and K_2 are identical when $\beta = \frac{n}{2}$. One proof makes use of the identities

$$\int_{S^{n-1}} |e_1 - ru|^{2\alpha-n}\, du = c\, F\left(\frac{n}{2}-\alpha,\, 1-\alpha;\, \frac{n}{2};\, r^2\right)$$

$$= c\int_0^1 t^{\beta-\alpha-1}(1-t)^{\alpha-1}(1-tr^2)^{\alpha-1}\, dt, \quad 0 < r < 1, \tag{3.10}$$

where $e_1 = (1, 0, ..., 0) \in \mathbb{R}^n$ and F denotes a hypergeometric function. To prove that each of the two integrals equals the middle term, the reader may consult, for example, Chapter 9 of [Le 1965], especially (9.6.16) (quadratic transformation of hypergeometric functions) and (9.12) (symmetry of hypergeometric functions in their first two parameters.)

By (3.6) and (3.8), when $\beta = \frac{n}{2}$, we have

$$\int_0^\infty G^2(x) x^{\frac{n}{2}-\alpha-1}\, dx = c \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x) f(y) |x-y|^{2\alpha-n}\, dx\, dy. \tag{3.11}$$

The conclusion of Conjecture 3 now follows from (3.2), (3.11), and Lieb's sharp HLS inequality with $p = \frac{2n}{n+2\alpha}$. $\square$

The proofs of the sharp HLS inequality in [L 1983], [CL 1990], and [LL 1996] rely on the fact that the right hand side in (3.11) can be represented as a fractional integral expression on S^n as well as on $\mathbb{R}^n$. This introduces a new set of symmmetries, which can be played off against the symmetries of $\mathbb{R}^n$, to show in the end that extremals on $\mathbb{R}^n$ must have the form $(a + b|x|^2)^{-\frac{2\alpha+n}{2}}$. Thus, an attempt to extend Proposition 4 to the case when β is an arbitrary real number larger than α could lead to contemplation of spheres of fractional dimension, and their possible symmetries.

Here is a summing up of the current state of our knowledge of Conjecture 1:

Theorem 1 *Conjecture 1 is true when* $k = 2$ *and* q *is an integer between* 1 *and* $n + 1$, *inclusive, or when* $q = 2$.

The first statement follows from Propositions 1 and 3;, the second is a restatement of Proposition 2.

The only other situations we know of in which extremals or best constants are known for the U and V operators occur in the limiting case $q = p$. If $p = q = 1$ then Fubini's theorem implies that $||U_\alpha f||_{L^1(\mathbb{R}^+, x^{\beta-\alpha-1}\,dx)} = c||f||_{L^1(\mathbb{R}^+, x^{\beta-1}\,dx)}$ for all nonnegative f and $\beta > 0$. For $p = q \in (1, \infty)$ and $\alpha > 0$ there are inequalities of Hardy, Littlewood and Pólya [HLP 1934, Theorem 329]:

$$\int_0^\infty (U_\alpha f)^p\, dx < \Big(\Gamma\Big(\frac{1}{p}\Big) \frac{\Gamma(\alpha)}{\Gamma(\frac{1}{p} + \alpha)}\Big)^p \int_0^\infty (x f(x))^p\, dx.$$

$$\int_0^\infty x^{-\alpha p} (V_\alpha f(x))^p\, dx < \Big(\Gamma\Big(1 - \frac{1}{p}\Big) \frac{\Gamma(\alpha)}{\Gamma(1 - \frac{1}{p} + \alpha)}\Big)^p \int_0^\infty f^p\, dx.$$

The constants are best possible, but equality is not achieved unless $f \equiv 0$. For $\alpha = 1$, the second inequality becomes Hardy's inequality:

$$\int_0^\infty (x^{-1} V_1(x))^p\, dx < \Big(\frac{p}{p-1}\Big)^p \int_0^\infty f^p\, dx.$$

These and a large number of kindred inequalities are catalogued in the book [MPF].

4 Symmetrization for L^q, $0 < q < 1$.

For $0 < q < 1$, it is possible that the inequality in Conjecture 2 might reverse:

Conjecture 4 *For* $f : \mathbb{R}^n \to \mathbb{R}^+$, $\quad ||T_{k,n}f||_q \geq ||T_{k,n}f^\#||_q, \quad 0 < q < 1.$

Let E be a compact subset of $\mathbb{R}^n$, and $1 \leq k \leq n-1$. For $\theta \in G_{k,n}$, denote by $\mathcal{L}^k(E|\theta)$ the Lebesgue k-measure of the orthogonal projection of E on θ. Recall that $d\theta$ denotes the invariant probability measure on $G_{k,n}$. Mattila [M 1990], [M 1995] conjectured an isoperimetric property for the mean value of$\mathcal{L}^k(E|\theta)$:

Mattila's Projection Conjecture

$$\int_{G_{k,n}} \mathcal{L}^k(E|\theta)\, d\theta \geq \int_{G_{k,n}} \mathcal{L}^k(E^\#|\theta)\, d\theta, \quad 1 \leq k \leq n-1 \tag{4.1}$$

Mattila proved (4.1) when $k = n-2$. In [M 1995, p.132], Mattila writes that for $k = 1$ (4.1) has been proved in unpublished work of M. Chlebík. For other values of k the problem is open.

Let $g = T_{n-k,n}(\chi_E)$. Using (1.1), one can show that

$$\int_{G_{k,n}} \mathcal{L}^k(E|\theta)\, d\theta = C_{k,n}\, \mu_{n-k,n}\, (g > 0) = C_{k,n} \lim_{q\to 0} \int_{M_{n-k,n}} g^q\, d\mu.$$

Thus, if Conjecture 4 is true, then so is Mattila's conjecture.

When E is convex, (4.1) is true for all k. See, for example, [G]. For $k = n-1$ and E convex, "Cauchy's area formula" tells us that the integrals in (4.1) are constant multiples of the $n-1$ dimensional Hausdorff measures of ∂E and $\partial(E^\#)$, respectively. Thus, (4.1) reduces in the convex case to the classical isoperimetric inequality between volume and surface measure in $\mathbb{R}^n$. For other values of k the proof of (4.1) for convex E makes use of A.D. Alexandrov's theory of mixed volumes.

5 A proof of Blaschke's identity

We restate the identity (2.4) to be proved, and relabel it (5.1).

$$\int_{M_{k,n}} d\pi \int_{\pi^{k+1}} F(y_0, ..., y_k)\, dy_0 ... dy_k$$

$$= c \int_{\mathbb{R}^{n(k+1)}} \det^{k-n}(x_0, ..., x_k)\, F(x_0, ..., x_k)\, dx_0 ... dx_k. \tag{5.1}$$

To prove (5.1), we will first prove a similar identity when the left hand integration is over $G_{k,n}$: For $F : \mathbb{R}^k \to \mathbb{R}^+$ holds

$$\int_{G_{k,n}} d\theta \int_{\theta^k} F(y_1, ..., y_k)\, dy_1 ... dy_k$$

$$= c \int_{\mathbb{R}^{nk}} \det^{k-n}(0, x_1 ..., x_k)\, F(x_1, ..., x_k)\, dx_1 ... dx_k. \tag{5.2}$$

The first step in the proof of (5.2) is to lift the integral over $G_{k,n}$ to an integral over $\mathbb{R}^{n,k}$. For $x_i \in \mathbb{R}^n$, $i = 1, ..., k$, let $\theta(x_1, ..., x_k)$ denote the k-dimensional subspace in $\mathbb{R}^n$ spanned by the x_i. Then, for each k-tuple of $f_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\int_{\mathbb{R}^n} f_i(|x|)\, dx < \infty$ there is a constant c such that for each $g : G_{k,n} \longrightarrow \mathbb{R}^+$ holds

$$\int_{G_{k,n}} g(\theta)\, d\theta = c \int_{\mathbb{R}^{n,k}} g(\theta(x_1, ..., x_k)) \prod_{j=1}^{k} f_j(|x_i|)\, dx_1 ... dx_k.$$

This is true because neither side changes when g is changed to $g \circ R$ for $R \in O(n)$. Hence, each side is an $O(n)$-invariant integral on $G_{k,n}$, and such invariant integrals are unique up to multiplicative constant. [M 1995, p.49].

In particular,

$$\int_{G_{k,n}} g(\theta)\, d\theta = c \int_{\mathbb{R}^k} g(\theta(x_1, ..., x_k)) e^{-\sum_{j=1}^{k} |x_j|^2}\, dx_1 ... dx_k. \tag{5.3}$$

Given $x_1, ..., x_k \in \mathbb{R}^n$, $y_1, ..., y_k \in \mathbb{R}^n$, and $w_1, ..., w_k \in \mathbb{R}^k$, let X, Y, and W denote the matrices whose i'th row vectors are x_i, y_i and w_i respectively. Then X and Y are $k \times n$, and W is $k \times k$. Write also

$$|X|^2 = \sum_{j=1}^{k} |x_j|^2, \quad dX = dx_1 ... dx_k, \quad \det X = \det(0, x_1, ..., x_k),$$

and use analogous notation for the y_i and w_i. Then (5.3) can be restated as

$$\int_{G_{k,n}} g(\theta)\, d\theta = c \int_{\mathbb{R}^{n,k}} e^{-|X|^2} g(\theta(X))\, dX. \tag{5.4}$$

By our definitions, det W is the k-volume of the simplex in $\mathbb{R}^k$ spanned by $w_1, ..., w_k$ and the origin. Using $\det(0, e_1, ..., e_k) = 1/k!$, it follows that our det W is $1/k!$ times the determinant of the matrix W. We can factor X as $X = SR^*$, where S is a $k \times k$ symmetric matrix and $R \in O(k,n)$. Then it is easy to see that $\det X = \det S = 1/k!$ times the determinant of the matrix S. Of course, the same considerations apply to Y. A good source for the linear algebra we use here is [EG 1992, §3.2].

Consider now the mapping from $\mathbb{R}^{k^2} \times \mathbb{R}^{n,k} \longrightarrow \mathbb{R}^{n,k}$ defined by $Y = WX$. Then

$$\det Y = k! \det W \det X. \tag{5.5}$$

For fixed X of rank k the map $W \longrightarrow Y$ carries $\mathbb{R}^{k^2}$ 1-1 onto $\theta^k(X)$. For fixed nonsingular W, the map $X \longrightarrow Y$ carries $\mathbb{R}^{n,k}$ 1-1 onto itself. By considerations left to the reader, the volume distortion factors of these maps are given by

$$dY = (k! \det X)^k \, dW, \quad dY = (k! \det W)^n \, dX. \tag{5.6}$$

Here now is the proof of (5.2). For brevity, we'll omit the changing multiplicative constant that belongs in front of each factor. The first equality is from (5.4). The second, fourth, and sixth equalities are obtained by the respective variable changes $Y = WX$ with X fixed, $X = W^{-1}Z$ with W fixed, and $W = SV$ with $Z = SR^*$ fixed, where S is $k \times k$ symmetric and $R \in O(k,n)$.

$$\begin{aligned}
&\int_{G_{k,n}} d\theta \int_{\theta^k} F(Y)\, dY = \int_{\mathbb{R}^{nk}} e^{-|X|^2}\, dX \int_{\theta^k(X)} F(Y)\, dY \\
&= \int_{\mathbb{R}^{nk}} e^{-|X|^2}\, dX \int_{\mathbb{R}^{k^2}} F(WX)\, (\det{}^k X)\, dW \\
&= \int_{\mathbb{R}^{k^2}} dW \int_{\mathbb{R}^{nk}} F(WX)\, e^{-|X|^2}\, (\det{}^k X)\, dX \\
&= \int_{\mathbb{R}^{k^2}} dW \int_{\mathbb{R}^{nk}} F(Z)\, e^{-|W^{-1}Z|^2} (\det{}^k Z)(\det{}^{-k-n} W)\, dZ \\
&= \int_{\mathbb{R}^{nk}} F(Z)\, (\det{}^k Z)\, dZ \int_{\mathbb{R}^{k^2}} e^{-|W^{-1}Z|^2} (\det{}^{-k-n} W)\, dW \\
&= \int_{\mathbb{R}^{nk}} F(Z) (\det{}^k Z)\, dZ \int_{\mathbb{R}^{k^2}} e^{-|V^{-1}|^2} (\det{}^{-k-n} Z)(\det{}^{-k-n} V)\, (\det{}^k Z)\, dV \\
&= \int_{\mathbb{R}^{nk}} F(Z)\, (\det{}^{k-n} Z)\, dZ.
\end{aligned}$$

Proof of (5.1). We shall transform the right hand side of (5.1) into the left hand side. The third equality comes from (5.2), and the last equality from (1.1). In the fifth equality, for fixed θ, $x_0 = z_0 + w$ is the orthogonal decomposition of x_0 into $z_0 \in \theta$, $w \in \theta^\perp$. Again, the multiplicative constants in front of each term will be omitted.

$$
\begin{aligned}
&\int_{\mathbb{R}^{n(k+1)}} \det{}^{k-n}(x_0, x_1, ..., x_k) F(x_0, x_1, ..., x_k)\, dx_0\, dx_1 ... dx_k \\
&= \int_{\mathbb{R}^n} dx_0 \int_{\mathbb{R}^{nk}} \det{}^{k-n}(0, x_1 - x_0, ..., x_k - x_0) F(x_0, x_1, ..., x_k)\, dx_1 ... dx_k \\
&= \int_{\mathbb{R}^n} dx_0 \int_{\mathbb{R}^{nk}} \det{}^{k-n}(0, z_1, ..., z_k) F(x_0, x_0 + z_1, ..., x_0 + z_k)\, dz_1 ... dz_k \\
&= \int_{\mathbb{R}^n} dx_0 \int_{G_{k,n}} d\theta \int_{\theta^k} F(x_0, x_0 + z_1, ..., x_0 + z_k)\, dz_1 ... dz_k \\
&= \int_{G_{k,n}} d\theta \int_{\mathbb{R}^n} dx_0 \int_{\theta^k} F(x_0, x_0 + z_1, ..., x_0 + z_k)\, dz_1 ... dz_k \\
&= \int_{G_{k,n}} d\theta \int_{\theta^\perp} dw \int_{\theta} dz_0 \int_{\theta^k} F(z_0 + w, z_0 + w + z_1, ..., z_0 + w + z_k)\, dz_1 ... dz_k \\
&= \int_{G_{k,n}} d\theta \int_{\theta^\perp} dw \int_{\theta} dz_0 \int_{\theta^k} F(z_0 + w, z_1 + w, ..., z_k + w)\, dz_1 ... dz_k \\
&= \int_{G_{k,n}} d\theta \int_{\theta^\perp} dw \int_{\theta^{k+1}} F(z_0 + w, z_1 + w, ..., z_k + w)\, dz_0 ... dz_k \\
&= \int_{G_{k,n}} d\theta \int_{\theta^\perp} dw \int_{(\theta + w)^{k+1}} F(y_0, ..., y_k)\, dy_0 ... dy_k \\
&= \int_{M_{k,n}} d\pi \int_{\pi^{k+1}} F(y_0, ..., y_k)\, dy_0 ... dy_k.
\end{aligned}
$$

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