SOME CONSENSUS MEASURES AND THEIR APPLICATIONS IN GROUP DECISION MAKING

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In this paper we consider that a group of decision makers rank a set of alternatives by means of weak orders for making a collective decision. Since decision makers could have very different opinions and it should be important to reach a consensuated decision, we have introduced indices of contribution to consensus for each decision maker for prioritizing them in order of their contributions to consensus. These indices are defined by means of a consensus measure which assigns a number between 0 and 1 to each subset of decision makers. For putting in practice this idea, we have introduced a class of consensus measures based on distances on weak orders and we have analyzed some of their properties. We have illustrated the weighted decision procedure with an example.

Keywords: Consensus; Distances; Group decision making.

1. Preliminaries

Consider a set of decision makers or voters $V = \{v_1, \ldots, v_m\}$ $(m \geq 3)$ who show their preferences over a set of alternatives $X = \{x_1, \ldots, x_n\}$ $(n \geq 3)$. With $L(X)$ we denote the set of linear orders on X, and with $W(X)$ the set of weak orders on X. Given $R \in W(X)$, the *inverse* of R is the weak order R^{-1} defined by $x_i \, R^{-1} \, x_j \Leftrightarrow x_j \, R \, x_i$, for all $x_i, x_j \in X$.

A *profile* is a vector $\boldsymbol{R} = (R_1, \ldots, R_m)$ of weak or linear orders, where R_i contains the preferences of the voter v_i, with $i = 1, \ldots, m$. Given a profile $\boldsymbol{R} = (R_1, \ldots, R_m)$, we denote $\boldsymbol{R}^{-1} = (R_1^{-1}, \ldots, R_m^{-1})$. If π is a permutation on $\{1,\ldots,m\}$ and $\emptyset \neq I \subseteq V$, we denote $\boldsymbol{R}_\pi = (R_{\pi(1)},\ldots,R_{\pi(m)})$ and $I_\pi = \{v_{\pi^{-1}(i)} \mid v_i \in I\}$, i.e., $v_j \in I_\pi \Leftrightarrow v_{\pi(j)} \in I$. Given a permutation σ on $\{1,\ldots,n\}$, we denote with $\boldsymbol{R}^\sigma = (R^\sigma_1,\ldots,R^\sigma_m)$ the profile that results of recalling in $\boldsymbol{R}$ the alternatives according to σ, i.e., $x_i\, R_k\, x_j \Leftrightarrow x_{\sigma(i)}\, R^\sigma_k\, x_{\sigma(j)}$ for all $i,j \in \{1,\ldots,n\}$ and $k \in \{1,\ldots,m\}$. The cardinal of I is denoted by $|I|$. With $\mathcal{P}(V)$ we denote the power set of V $(I \in \mathcal{P}(V) \Leftrightarrow I \subseteq V)$. Moreover, $\mathcal{P}_2(V) = \{I \in \mathcal{P}(V) \mid |I| \geq 2\}$.

We now introduce a system for codifying linear and weak orders by means of vectors which represent the relative position of each alternative in the corresponding order. Similar procedures have been considered in the generalization of scoring rules from linear orders to weak orders (see Smith,[1] Black[2] and Cook and Seiford,[3] among others).

Given a profile $(R_1,\ldots,R_m) \in L(X)^m$ of linear orders, consider the mapping $o_i : X \longrightarrow \{1,\ldots,n\}$ which assigns the position of each alternative in R_i. Thus, the vector $(o_i(x_1),\ldots,o_i(x_n)) \in \{1,\ldots,n\}^n$ determines the corresponding linear order.

There does not exist a unique system for codifying weak orders. We propose one based on linearizing the weak order and to assign each alternative the average of the positions of the alternatives within the same equivalence class. As an example, consider 7 alternatives arranged in the weak order: $x_2 \sim x_3 \sim x_5 \succ x_1 \succ x_4 \sim x_7 \succ x_6$. Then, this weak order is codified by the vector $(4,\, 2,\, 2,\, 5.5,\, 2,\, 7,\, 5.5)$. Taking into account this idea, given a profile of weak orders $(R_1,\ldots,R_m) \in W(X)^m$, the mapping $o_i : X \longrightarrow \{1,\, 1.5,\, 2,\, 2.5,\ldots,\, n-0.5,\, n\}$ assigns the relative position of each alternative in R_i.

We now introduce a simple procedure for constructing a distance on $W(X)$ from a distance on $\mathbb{R}^n$.

Definition 1.1. Given a distance $d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0,\infty)$, the *distance on* $W(X)$ *induced by* d is the mapping $\bar{d} : W(X) \times W(X) \longrightarrow [0,\infty)$ defined by $\bar{d}(R_1,R_2) = d\big((o_1(x_1),\ldots,o_1(x_n)),(o_2(x_1),\ldots,o_2(x_n))\big)$, for all $R_1,R_2 \in W(X)$.

Example 1.1. Typical examples of distances in $\mathbb{R}^n$ are the following:

(1) The discrete distance d',

$$d'\big((a_1,\ldots,a_n),(b_1,\ldots,b_n)\big) = \begin{cases} 1, & \text{if } (a_1,\ldots,a_n) \neq (b_1,\ldots,b_n), \\ 0, & \text{if } (a_1,\ldots,a_n) = (b_1,\ldots,b_n). \end{cases}$$

(2) For every $p \geq 1$, the Minkowski distance d_p,

$$d_p\big((a_1,\ldots,a_n),(b_1,\ldots,b_n)\big) = \left(\sum_{i=1}^{n} |a_i - b_i|^p\right)^{\frac{1}{p}}.$$

For $p = 1$ and $p = 2$ we have the Manhattan and Euclidean distances, respectively.

(3) The Chebyshev distance d_∞,

$$d_\infty\big((a_1,\ldots,a_n),(b_1,\ldots,b_n)\big) = \max\big\{|a_1 - b_1|,\ldots,|a_n - b_n|\big\}.$$

(4) The cosine distance d_c,

$$d_c\big((a_1,\ldots,a_n),(b_1,\ldots,b_n)\big) = 1 - \frac{\displaystyle\sum_{i=1}^{n} a_i\, b_i}{\sqrt{\displaystyle\sum_{i=1}^{n} a_i^2}\sqrt{\displaystyle\sum_{i=1}^{n} b_i^2}}.$$

Definition 1.2. A distance $d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0,\infty)$ is *neutral* if for every permutation σ on $\{1,\ldots,n\}$, it holds

$$d\big((a_{\sigma(1)},\ldots,a_{\sigma(n)}),(b_{\sigma(1)},\ldots,b_{\sigma(n)})\big) = d\big((a_1,\ldots,a_n),(b_1,\ldots,b_n)\big),$$

for all $(a_1,\ldots,a_n),(b_1,\ldots,b_n) \in \mathbb{R}^n$.

Remark 1.1. The distances d', d_p, d_∞ and d_c are neutral for every $p \geq 1$.

2. Consensus measures

Consensus measures have been analyzed by Bosch[4] in the context of linear orders. We now extend this concept to the framework of weak orders.

Definition 2.1. A *consensus measure* on $W(X)^m$ is a mapping

$$\mathcal{M} : W(X)^m \times \mathcal{P}_2(V) \longrightarrow [0,1]$$

that satisfies the following conditions:

(1) *Weak unanimity.* For every $\boldsymbol{R} \in W(X)^m$, $\mathcal{M}(\boldsymbol{R}, V) = 1$ if and only if $R_1 = \cdots = R_m$.
(2) *Anonymity.* For all permutation π on $\{1,\ldots,m\}$, $\boldsymbol{R} \in W(X)^m$ and $I \in \mathcal{P}_2(V)$, $\mathcal{M}(\boldsymbol{R}_\pi, I_\pi) = \mathcal{M}(\boldsymbol{R}, I)$.
(3) *Neutrality.* For all permutation σ on $\{1,\ldots,n\}$, $\boldsymbol{R} \in W(X)^m$ and $I \in \mathcal{P}_2(V)$, $\mathcal{M}(\boldsymbol{R}^\sigma, I) = \mathcal{M}(\boldsymbol{R}, I)$.

We now introduce other properties that consensus measures can satisfy.

Definition 2.2. Let $\mathcal{M} : W(X)^m \times \mathcal{P}_2(V) \longrightarrow [0,1]$ be a consensus measure.

(1) $\mathcal{M}$ satisfies *strong unanimity* if for all $\boldsymbol{R} \in W(X)^m$ and $I \in \mathcal{P}_2(V)$, $\mathcal{M}(\boldsymbol{R}, I) = 1$ if and only if $R_i = R_j$ for all $v_i, v_j \in I$.
(2) $\mathcal{M}$ satisfies *maximum dissension* if for all $\boldsymbol{R} \in W(X)^m$ and $v_i, v_j \in V$ such that $i \neq j$, $\mathcal{M}(\boldsymbol{R}, \{v_i, v_j\}) = 0$ if and only if $R_i, R_j \in L(X)$ and $R_j = R_i^{-1}$.
(3) $\mathcal{M}$ is *reciprocal* if for all $\boldsymbol{R} \in W(X)^m$ and $I \in \mathcal{P}_2(V)$, $\mathcal{M}(\boldsymbol{R}^{-1}, I) = \mathcal{M}(\boldsymbol{R}, I)$.

Obviously, strong unanimity implies weak unanimity.

Definition 2.3. Given a distance $\bar{d} : W(X) \times W(X) \longrightarrow [0, \infty)$, the mapping $\mathcal{M}_{\bar{d}} : W(X)^m \times \mathcal{P}_2(V) \longrightarrow [0,1]$ is defined by

$$\mathcal{M}_{\bar{d}}(\boldsymbol{R}, I) = 1 - \frac{\displaystyle\sum_{\substack{v_i, v_j \in I \\ i<j}} \bar{d}(R_i, R_j)}{\displaystyle\binom{|I|}{2} \cdot \Delta_n},$$

where $\Delta_n = \max\{\bar{d}(R_i, R_j) \mid R_i, R_j \in W(X)\}$.

Proposition 2.1. *For every distance* $\bar{d} : W(X) \times W(X) \longrightarrow [0, \infty)$*,* $\mathcal{M}_{\bar{d}}$ *satisfies strong unanimity and anonymity.*

If $\mathcal{M}_{\bar{d}}$ is neutral, then we say that $\mathcal{M}_{\bar{d}}$ is the *consensus measure associated with* $\bar{d}$.

Proposition 2.2. *If* $d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0, \infty)$ *is a neutral distance, then* $\mathcal{M}_{\bar{d}}$ *is a consensus measure.*

Corollary 2.1. *If* $\bar{d}$ *is the distance on* $W(X)$ *induced by* d'*,* d_p*, with* $p \geq 1$*,* d_∞ *or* d_c*, then* $\mathcal{M}_{\bar{d}}$ *is a reciprocal consensus measure.*

Proposition 2.3. *If* $\bar{d}$ *is the distance induced by* d_p*, with* $p > 1$*, or* d_c*, then* $\mathcal{M}_{\bar{d}}$ *satisfies the maximum dissension property.*

Remark 2.1. If $\bar{d}$ is the distance induced by d', d_∞ or d_1, then $\mathcal{M}_{\bar{d}}$ does not satisfy the maximum dissension property.

3. Application to group decision making

Definition 3.1. Given a consensus measure $\mathcal{M}$, $\boldsymbol{c} : W(X)^m \longrightarrow [-1,1]^m$ is defined by $\boldsymbol{c}(\boldsymbol{R}) = \big(c_1(\boldsymbol{R}), \ldots, c_m(\boldsymbol{R})\big)$ with

$$c_i(\boldsymbol{R}) = \frac{\displaystyle\sum_{I \in S(i)} \Big(\mathcal{M}\big(\boldsymbol{R}, I \cup \{v_i\}\big) - \mathcal{M}(\boldsymbol{R}, I) \Big)}{|S(i)|}, \qquad i = 1, \ldots, m,$$

where $S(i) = \{I \in \mathcal{P}_2(V) \mid v_i \notin I\}$.

Proposition 3.1. *Let* $\bar{d} : W(X) \times W(X) \longrightarrow [0, \infty)$ *be a distance. For the consensus measure* $\mathcal{M}_{\bar{d}}$*, it holds* $c_1(\boldsymbol{R}) + \cdots + c_m(\boldsymbol{R}) = 0$*, for every profile* $\boldsymbol{R} \in W(X)^m$.

We can use the vector $\boldsymbol{c}(\boldsymbol{R})$ for prioritizing the decision makers in order of their contribution to consensus, as suggested by Cook, Kress and Seiford.[5] In this way, we introduce a new index

$$c'_i(\boldsymbol{R}) = c_i(\boldsymbol{R}) - \min\{c_1(\boldsymbol{R}), \ldots, c_m(\boldsymbol{R})\},$$

for every $i \in \{1, \ldots, m\}$. We now define a weight for each voter:

$$w_i(\boldsymbol{R}) = \begin{cases} \dfrac{c'_i(\boldsymbol{R})}{c'_1(\boldsymbol{R}) + \cdots + c'_m(\boldsymbol{R})}, & \text{if } c'_1(\boldsymbol{R}) + \cdots + c'_m(\boldsymbol{R}) \neq 0 \\ \dfrac{1}{m}, & \text{if } c'_1(\boldsymbol{R}) + \cdots + c'_m(\boldsymbol{R}) = 0, \end{cases}$$

for $i = 1, \ldots, m$. Notice that $w_i(\boldsymbol{R}) \in [0,1]$ for every $i \in \{1, \ldots, m\}$, and $w_1(\boldsymbol{R}) + \cdots + w_m(\boldsymbol{R}) = 1$.

Now, for each decision maker $v_i \in V$ we multiply the position of each alternative $o_i(x_j)$ by his/her weight $w_i(\boldsymbol{R})$ for assigning collective positions to the alternatives:

$$O(x_j) = \sum_{i=1}^{m} w_i(\boldsymbol{R}) \cdot o_i(x_j), \quad j = 1, \ldots, n. \tag{1}$$

Thus, we can order the alternatives through the weak order $\succeq$ on X defined by $x_j \succeq x_k \Leftrightarrow O(x_j) \geq O(x_k)$.

3.1. *An illustrative example*

In order to illustrate the above decision making procedure, we now consider the set of voters $V = \{v_1, v_2, v_3, v_4, v_5\}$ that rank order the alternativas of $X = \{x_1, \ldots, x_7\}$ by means of the following weak orders:

R_1	R_2	R_3	R_4	R_5
$x_2\ x_3\ x_5$	x_5	x_3	$x_1\ x_2$	$x_1\ x_4$
x_1	$x_1\ x_3$	x_4	x_5	x_7
$x_4\ x_7$	x_2	x_1	x_3	x_2
x_6	$x_4\ x_6\ x_7$	$x_2\ x_5$	x_4	$x_3\ x_5\ x_6$
		$x_6\ x_7$	$x_6\ x_7$	

Taking into account the consensus measure $\mathcal{M}_{\bar{d}}$ for d_p with $p = 2$, d_∞ and d_c, we obtain the following coefficients:

	$c_1(\boldsymbol{R})$	$c_2(\boldsymbol{R})$	$c_3(\boldsymbol{R})$	$c_4(\boldsymbol{R})$	$c_5(\boldsymbol{R})$
d_2	0.05938261	0.04309471	−0.01037855	0.06547564	−0.15757441
d_∞	0.08080808	0.02020202	−0.06060606	0.10101010	−0.14141414
d_c	0.05114639	0.03468305	0.01296238	0.07872037	−0.17751219

These coefficients induce the following weights:

	$w_1(\boldsymbol{R})$	$w_2(\boldsymbol{R})$	$w_3(\boldsymbol{R})$	$w_4(\boldsymbol{R})$	$w_5(\boldsymbol{R})$
d_2	0.27537088	0.25469759	0.18682711	0.28310441	0
d_∞	0.31428571	0.22857143	0.11428571	0.34285714	0
d_c	0.25762578	0.23907680	0.21460449	0.28869292	0

We now weight the position of each alternative $o_i(x_j)$ by the individual weight $w_i(\boldsymbol{R})$. Then, taking into account the collective positions $O(x_j)$ defined in (1), we have for each distance the following orders on X:

$$\begin{array}{ll} d_2 & x_5 \succ x_3 \succ x_1 \succ x_2 \succ x_4 \succ x_7 \succ x_6 \\ d_\infty & x_5 \succ x_2 \succ x_1 \sim x_3 \succ x_4 \succ x_7 \succ x_6 \\ d_c & x_3 \succ x_5 \succ x_1 \succ x_2 \succ x_4 \succ x_7 \succ x_6 \end{array}$$

Because of each distance has a different sensitiveness towards heterogeneity, the election of the distance can be crucial for determining the outcome. This is the reason why the outcomes in the previous example have been different.

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