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SOME CONSEQUENCES OF THE BEURLING-HELSON THEOREM

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ABSTRACT. Let G be a locally compact abelian group and C a closed subset of G. Denote by $A(C)$ the algebra of restrictions to C of functions on G which are Fourier transforms of elements of $L'(\hat{G})$, where $\hat{G}$ denotes the group dual to G. Denote by $B(G)$ the algebra of functions on G which are Fourier-Stieltjes transforms of regular Borel measures on $\hat{G}$. In 1951 Beurling and Helson proved a classical theorem characterizing the algebra automorphisms of $A(R)$; see [1]. The point of this paper is that a seemingly slight generalization, the Theorem below, contains a great deal of information about various related questions.

THEOREM (Beurling-Helson; see [2], p. 86). *Let C be a compact interval in R. Suppose $\phi : C \to R$ is continuous, and that $f \circ \phi \in A(C)$ whenever $f \in A(R)$. Then ϕ is affine, that is, $\phi(t) = \alpha t + \beta$, for some $\alpha, \beta \in R$.*

The first corollary extends the theorem to R^N.

COROLLARY 1. *Let C be a closed convex set in R^M, and suppose that ϕ is a continuous function from C into R^N, with the property that $f \circ \phi \in A(C)$ whenever $f \in A(R^N)$. Then ϕ is affine.*

PROOF. We may assume that $0 \in C$, and that $\phi(0) = 0$; we must show that ϕ is linear, that is, that $\phi(x) + \phi(y) = \phi(x + y)$ when x, y, and $x + y$ are in C. Let Q be any affine map from R to R^M which carries $[0, 1]$ into C, and let P be any affine map from R^N to R. Fix $F \in A(R^N)$, with $F = 1$ on the compact set $\phi(Q([0, 1]))$.

Then if $f \in A(R)$,

$$f \circ P \circ \phi \circ Q = ((f \circ P)F) \circ \phi \circ Q \tag{1}$$

on $[0,1]$. Since $f \in A(R)$, $f \circ P \in B(R^N)$, so $(f \circ P)F \in A(R^N)$ (recall that $A(R^N)$ is an ideal in $B(R^N)$). Then $((f \circ P)F) \circ \phi \in A(C)$ so $(f \circ P)F) \circ \phi \circ Q$ is the restriction to $[0, 1]$ of a Fourier-Stieltjes transform, and is therefore in $A[0, 1]$. So, by (1), $f \circ P \circ \phi \circ Q$ is in $A[0, 1]$ whenever $f \in A(R)$, so every such composition $P \circ \phi \circ Q$ must be affine.

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This of course implies the desired conclusion; one can see it this way: I claim that

$$\frac{\phi(x) + \phi(y)}{2} = \phi\left(\frac{x + y}{2}\right) \tag{2}$$

for all $x, y \in C$. Suppose (2) fails for some $x, y \in C$. Choose P to separate the points $(\phi(x) + \phi(y))/2$ and $\phi((x + y)/2)$, and let Q map $[0, 1]$ affinely onto the interval $[x, y] \subset C$. Then the composition $P \circ \phi \circ Q$ is not affine on $[0, 1]$. This proves (2).

Now take $y = 0$ in (2), and obtain $\phi(x/2) = \phi(x)/2$, for all $x \in C$, since $\phi(0) = 0$. If now x, y and $x + y$ are in C, then $\phi(x + y) = 2\phi((x + y)/2) = 2((\phi(x) + \phi(y))/2) = \phi(x) + \phi(y)$. This completes the proof.

It should be clear that we can take C to be the closure of any connected open set, in Corollary 1. In fact, let K be any closed ball contained in the interior of C. Then restricting ϕ to K, we have $f \circ \phi \in A(K)$ when $f \in A(R^N)$, and thus ϕ is affine on K, that is $\phi(x) = y_0 + L(x)$ where L is linear. If K' is another ball with $K \cap K' \neq \emptyset$ and $\phi(x) = y_0' + L'(x)$ in K', then the linear function $L - L'$ is equal to the constant value $y_0' - y_0$ in the opening set $K \cap K'$, so $y_0 = y_0'$ and $L = L'$. This shows that ϕ is affine in any component of the interior of C; in particular (using continuity) in all of C, when the interior of C is connected.

One can infer the existence of ϕ in Corollary 1 beginning with a homomorphism from $A(R^N)$ to $A(C)$. We take this approach in the next corollary, because it is more natural. We wish to give a partial description of the form of a homomorphism from an ideal in $A(R^N)$ into $A(R^M)$. Let J be a closed ideal in $A(R^N)$, and let U be the set of points $x \in R^N$ such that $f(x) \neq 0$ for some $f \in J$. U is an open subset of R^N, and is identified with the maximal ideal space of J; to each complex homomorphism h on J corresponds a point $x \in U$ with $h(f) = f(x)$, for all $f \in J$. If Φ is a homomorphism from J into $A(R^M)$, then it follows from the Gelfand theory (see [3], page 213) that there is an open set V in R^M and a continuous function ϕ from V to U, and Φ has the form $\Phi f(t) = f(\phi(t))$ when $t \in V$, and $\Phi f(t) = 0$ when $t \notin V$.

Corollary 2. *ϕ is affine in each component of V.*

Proof. This is a corollary of Corollary 1 and two other facts: $A(R^N)$ contains every $(N + 1)$-times differentiable function with compact support, and J contains every function in $A(R^N)$ whose support is contained in U. Let C be a compact convex subset of V, and fix F in J with $F = 1$ on $\phi(C)$. For example, F can be taken to be an $(N+1)$-times differentiable function supported in U. If $f \in A(R^N)$, then $fF \in J$, so $(fF) \circ \phi \in A(R^M)$, hence also $(fF) \circ \phi$, restricted to C, is in $A(C)$; but on C, $(fF) \circ \phi = f \circ \phi$. Thus $f \to f \circ \phi$ takes $A(R^N)$ into $A(C)$, so by Corollary 1, ϕ must be affine on C. Since ϕ must be affine on any compact convex subset of V, it must be affine on components of V.

The last corollary has to do with the space $L^1(R^N) \cap L^2(R^N)$. This is a Banach algebra with convolution as multiplication and with norm $\|f\|_{1,2} = \|f\|_1 + \|f\|_2$. Let $A_2(R^N)$ denote the collection of Fourier transforms of functions in $L^1(R^N) \cap L^2(R^N)$; with the induced norm, $A_2(R^N)$ is a Banach algebra. $A_2(R^N)$ is a (dense) ideal in $A(R^N)$, and the maximal ideal space of $A_2(R^N)$ is R^N, in the usual way. The Plancherel theorem tells us that $A_2(R^N) = A(R^N) \cap L^2(R^N)$. If C is a closed subset of R^N, let $A_2(C)$ be the collection of restrictions to C of functions in $A_2(R^N)$. Note that if C is compact, then $A_2(C) = A(C)$. To see this, fix $F \in A(R^N)$ with compact support and with $F = 1$ on C. If $f \in A(C)$, then f is the restriction to C of some $g \in A(R^N)$, so also f is the restriction to C of gF. But gF is in $A(R^N)$ on the one hand, and in $L^2(R^N)$ on the other hand, since gF is continuous with compact support. This observation, combined with Corollary 1, gives us the following.

Corollary 3. *Let Φ be an algebra homomorphism from $A_2(R^N)$ into $A(R^M)$. Then Φ has the form $\Phi f = f \circ \phi$, where ϕ is affine from R^M to R^N.*

Given the conclusion, we see that ϕ must be one-to-one if it is to induce a homomorphism from $A_2(R^N)$ to $A(R^M)$, for if ϕ should have nontrivial "kernel," then the functions $f \circ \phi$ would be constant on entire straight lines. Of course, functions $A(R^M)$ vanish at infinity. Thus the hypothesis implicitly assumes that $M \leqq N$.

Proof. By the Gelfand theory, there is an open subset V of R^M and a continuous function ϕ from V into R^N, and $\Phi f = f \circ \phi$. Let C be a compact convex subset of V. Then ϕ restricted to C induces a homomorphism from $A_2(\phi(C))$ to $A(C)$. But $\phi(C)$ is compact, so $A_2(\phi(C)) = A(\phi(C))$. We obtain then the composite homomorphism $A(R^N) \to A(\phi(C)) \to A(C)$, the first part being the canonical restriction homomorphism. By Corollary 1, now, ϕ is affine on C. Therefore ϕ is affine on components, and then the fact that functions in $A(R^M)$ are continuous forces $V = R^M$.

Here are a couple of related questions to which I would like to know answers:

1. Is Corollary 2 still true if R^N is replaced by any locally compact abelian group?

2. Let A_0 denote the factor algebra obtained from $A(R)$ by identifying two functions in $A(R)$ if they agree in some neighborhood of 0. What are the automorphisms of A_0?

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