Open access • Journal Article • DOI:10.2298/FIL1004025B

Some counterexamples related to the theory of HQC mappings — Source link

Vladimir Bozin, Miodrag Mateljević

Published on: 01 Jan 2010 - Filomat (National Library of Serbia)

Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **24:4** (2010), 25–34 DOI: 10.2298/FIL1004025B

SOME COUNTEREXAMPLES RELATED TO THE THEORY OF HQC MAPPINGS

Vladimir Božin and Miodrag Mateljević

Abstract

We give examples that show that several of the theorems in the theory of HQC mappings cannot be strenghtened by weakening or omitting some of the conditions. In particular, we study Kellogg type problems and characterizations of HQC condition.

1 Introduction

This note deals with several examples that clarify necessity of conditions in theorems related to the problems of harmonic quasiconformal maps, or short HQC maps. These maps have been studied extensively in recent years, cf [6],[9],[19],[10] [11],[12] [13], [15], [16], [21], [23], [24], [25],[27], [31][30],[29], [7], [28].

In particular, our attention will be on the extensions of the classical Kellogg theorem, cf [20], as we will show that this result is not possible to extend even in the context of the conformal maps, where various relaxations of smoothness of the boundary condition are examined. Also, we will examine various possible relaxations of the known characterizations of the HQC condition.

Let $\mathbb{U}$ denote the unit disc $\{z : |z| < 1\}$, and $\mathbb{T}$ the unit circle, $\{z : |z| = 1\}$ and we will use notation $z = re^{i\theta}$ and $z = x + iy$. Also, $\mathbb{H}$ will denote the upper half plane $\{z : \mathrm{Re} z > 0\}$. By $\partial_\theta h$ and $\partial_r h$ (or h'_r and h'_θ if it is convenient), h'_x and h'_y we denote partial derivatives with respect θ and r, x and y respectively.

Every harmonic function h in $\mathbb{U}$ can be written in the form $h = f + \bar{g}$, where f and g are holomorphic functions in $\mathbb{U}$. We have

$\partial_\theta h(z) = i(zf'(z) - \overline{zg'(z)})$, $h'_r = e^{i\theta} f' + \overline{e^{i\theta} g'}$, $h'_\theta + irh'_r = 2izf'$ and therefore rh'_r is the harmonic conjugate of h'_θ. Let

$$P_r(t) = \frac{1 - r^2}{2\pi(1 - 2r\cos(t) + r^2)}$$

2000 *Mathematics Subject Classifications.* Primary 30C55, Secondary 31C62.
Key words and Phrases. Complex functions, planar convex harmonic mappings.
Received: May 15, 2010
Communicated by Dragan S. Djordjević

denote the Poisson kernel.

If $\psi \in L^1[0, 2\pi]$ and

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, \psi(t)\, dt,$$

then the function $h = P[\psi]$ so defined is called Poisson integral of ψ. It gives a harmonic extension of given boundary values to the unit disc.

For $f : \mathbb{U} \to \mathbb{C}$, define

$$f_*(\theta) = f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

when this limit exists. For $f : \mathbb{T} \to \mathbb{C}$, define $f_*(\theta) = f^*(e^{i\theta})$.

The Hilbert transform $H(\psi)$ is defined as

$$H(\psi)(\varphi) = \int_0^{\pi} \frac{\psi(\varphi + t) - \psi(\varphi - t)}{\tan t/2} dt,$$

where ψ is considered to be 2π periodic, or alternatively a function from $L^1(\mathbb{T})$.

Hilbert transform has the property that it maps radial limits of a harmonic function on $\mathbb{U}$ to radial limits of its conjugate harmonic function a.e. on $\mathbb{T}$.

Note that, if ψ is 2π-periodic, absolutely continuous on $[0, 2\pi]$ (and therefore $\psi' \in L^1[0, 2\pi]$), then

$$h'_\theta = P[\psi'].$$

Hence, since rh'_r is the harmonic conjugate of h'_θ, we find

$$rh'_r = P[H(\psi')] \tag{1.1}$$

$$(h'_r)^*(e^{i\theta}) = H(\psi')(\theta)\,\text{a.e.} \tag{1.2}$$

A homeomorphism $f \colon D \mapsto G$, where D and G are subdomains of the complex plane $\mathbb{C}$, is said to be K-quasiconformal (K-qc), $K \geq 1$, if f is absolutely continuous on a.e. horizontal and a.e. vertical line and

$$|f_{\bar z}| \leq k|f_z| \quad \text{a.e. on } D \tag{1.3}$$

where $K = \frac{1+k}{1-k}$ i.e. $k = \frac{K-1}{K+1}$.

2 Kellogg type problems

Kellogg theorem states that Riemann maps between domains with $C^{1,\alpha}$ boundary have to be bi-Lipschitz. This result has finally been generalized to the case of HQC maps in [6], after series of partial results in this direction.

The question then arises can one make further generalizations, for instance, by considering C^1 domains instead. Even in the case of conformal maps this cannot be done, as the following example shows.

Example 2.1. Let $w = A(z) = z \ln \frac{1}{z}$, $U_r^+ = \{z : |z| < r, y > 0\}$.

For r small enough, A maps interval $(0, r)$ onto interval $(0, r \ln \frac{1}{r})$, interval $(-r, 0)$ onto arc $\gamma_1(x) = x \ln \frac{1}{x} = x \ln \frac{1}{|x|} + i\pi x$, semicircle $C_r^+ = \{re^{it} : 0 \le t \le \pi\}$ onto a curve close the semicircle of radius $r \ln \frac{1}{r}$. For r small enough A is univalent in U_r^+. We can check that there is a smooth domain $D \subset U_r^+$ st interval $(-r_0, r_0)$, $r_0 > 0$, is a part of the boundary of D, $A(D)$ is C^1 domain and A is not bi-Lipschitz on D.

We can also consider

$$w = \frac{z}{\ln \frac{1}{z}}, \quad w(0) = 0\,.$$

Note $\ln \frac{1}{z} = -\ln z$, $w'(z) = -(\ln z)^{-1} + (\ln z)^{-2}$ and $w'(z) \to 0$ if $z \to 0$ throughout $\mathbb{H}$.

We can make a more delicate example which has boundary that is even "smoother" than in the case just described. The boundary can be C^2 everywhere except at one point, but at that point, it will be twice differentiable. The map will however not be Lipschitz. This shows that Kellogg type theorems cannot rely on pointwise property of the boundary curve, but on properties of global character.

Example 2.2. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be the function defined as $g(x) = e^{-1/x^2}$ for $x < 0$ and $g(x) = 0$ for $x \ge 0$. Define $D = \{z : \mathrm{Im} z > g(\mathrm{Re} z)\}$ and let $f : D \mapsto \mathbb{C}$ be given as

$$f(z) = z + \sum_{n=3}^{\infty} 2^{-n}(z + 2^{-n})^{1-3^{-n}}$$

(here we take a branch of the argument that is between $-\pi$ and π to define exponentiation). Let $D^* = f(D)$.

Proposition 2.3. *Mapping f defined above is not Lipschitz, yet it is biholomorphic and the boundary of D^* is C^2 except at $f(0)$, and at $f(0)$ the boundary is also twice differentiable.*

Proof. The derivative f' is

$$f'(z) = 1 + \sum_{n=3}^{\infty} (1 - 3^{-n}) 2^{-n} (z + 2^{-n})^{-3^{-n}}\,.$$

Our function f is analytic except at zero, where it has an essential singularity. We can see that the map f is not Lipschitz by considering points $z_n = -2^{-n} + ie^{-2^{2n}}$. Also, we conclude that except at $f(0)$, boundary of D^* is of class C^2, in fact it will be analytic.

Note that argument of the derivative is certainly between $-\pi/4$ and $\pi/4$, and from this it follows that we will have a bijection. To see this, consider $f(z_0) - f(z_1) = \int_{s(z_0,z_1)} f'(z)dz$, where $s(z_0, z_1)$ is the straight line segment joining z_0 and z_1. Suppose that $|\mathrm{Re}(z_0 - z_1)| \ge |\mathrm{Im}(z_0 - z_1)|$, then we will have from the integration formula and condition $-\pi/4 < arg(f') < \pi/4$ that $\mathrm{Re} f(z_0) \ne \mathrm{Re} f(z_1)$

is not zero. When $|\mathrm{Re}(z_0 - z_1)| < |\mathrm{Im}(z_0 - z_1)|$ similarly we will have $\mathrm{Im} f(z_0) \neq \mathrm{Im} f(z_1)$, in any case, $f(z_0) \neq f(z_1)$. Alternatively, this follows from the argument principle.

The main point of the example is that the boundary will also be twice differentiable at $f(0)$, though clearly from the Kellogg theorem it cannot be C^2.

Note that the series for f' converges for $z = 0$ and that this sum is real valued. Of course, f' is not continuous on ∂D, as it is not bounded near 0. However, $arg(f')$ is continuous on the real line. Moreover, f' is absolutely integrable on ∂D, and $s(z)$, the path length from point $f(0)$ to point $f(z)$ on the boundary of D^*, is commensurable with $|z|$. In fact $arg(f'(z))$ will be $o(z)$ when z is approaching 0. Indeed, argument for each term in the series satisfies this: when $|z| \leq 2^{-n-1}$, the real part of $z + 2^{-n}$ is at least third of $|z|$, and imaginary part is clearly $o(z^2)$; when $|z| > 2^{-n-1}$ the argument of the term is $O(3^{-n})$, which is clearly $o(z)$.

Thus, we conclude that in the natural parameter, ∂D will also be twice differentiable at zero, as required. □

Note that we can take part of ∂D containing zero and make a smooth bounded domain, which by Kellogg theorem can be mapped to the disc with the Riemann map which is bi-Lipschitz, so that we get an example of the same sort but with disc as the domain of definition.

Continuity of the argument of the derivative was a key point in the original proof of the Kellogg theorem. However, in the case of HQC maps, there is no such continuity. Derivatives do not have to extend continuously to the boundary, which can happen even at a dense subset of the boundary.

Example 2.4. Consider $h : \mathbb{H} \to \mathbb{H}$, a harmonic function given by the following expression

$$h(z) = h^{\phi}(z) = \Phi_1(z) + icy + c_1,$$

where $c > 0$, $c_1 \in \mathbb{R}$, $\Phi(z) = \int_i^z \phi(\zeta)d\zeta$, and $\Phi_1 = \mathrm{Re}\,\Phi$.

Note that $h'_x(z) = \mathrm{Re}\,\phi(z)$ and $h'_y(z) = -\mathrm{Im}\,\phi(z) + ic$.

Let $\phi(z) = 2 + e^{-i/z} = 2 + e^{-y/|z|^2}(\cos\frac{x}{|z|^2} - i\sin\frac{x}{|z|^2})$ and $h = h^{\phi}$. Then $h'_x(z) = \mathrm{Re}\,\phi(z) = 2 + e^{-y/|z|^2}\cos\frac{x}{|z|^2}$.

Hence $h'(x) = \mathrm{Re}\,\phi(x) = 2 + \cos\frac{1}{x}$; so h' is not continuous at 0. In polar coordinates, $h'_y(z) = -\mathrm{Im}\,\phi(z) + ic = e^{-\sin\theta/\rho}\sin(\cos\theta/\rho) + ic$; hence $h'_y(z) \to \sin(1/\rho) + ic$ when $\theta \to 0$ for fixed $\rho > 0$.

Let $G \subset \mathbb{H}$ be a smooth domain such that $\partial G \cap \mathbb{R} = [-a, a]$, $a > 0$, ϕ be conformal mapping of $\mathbb{U}$ onto G, $\phi(1) = 0$, $z = \phi(\zeta)$, and $\breve{h} = h \circ \phi$. One can check that $\breve{h}_r$ is not continuous at 1. However $\breve{h}$ is bi-Lipschitz.

We can give a more delicate example. Let $\phi_k(z) = 2 + e^{i/(x_k - z)}$ and x_k, $k \in \mathbb{N}$ be a sequence of real numbers. Define

$$\phi(z) = 2 + \sum_{k=1}^{\infty} 2^{-k}\phi_k(z)\,.$$

For example if x_k is a sequence of all rational numbers, i.e. enumerating $\mathbb{Q}$, then h_y will have no continuous extension to $\mathbb{Q}$, where $h = h^{\phi}$.

These examples can also be translated to the unit disc.

Note that in [6] a continuous function on the closure of the domain was found, that is in some sense analogous for HQC maps to the argument of the derivative in the case of conformal maps. It plays an essential role in the proof of the full version of the generalization of the Kellogg theorem to HQC maps.

3 Characterization of HQC

David Kalaj, in [12], has stated the following characterization for HQC maps.

Theorem 3.1 ([12]). *Let $f : \mathbb{T} \to \gamma$ be an orientation preserving homeomorphism of the unit circle onto the Jordan convex curve $\gamma = \partial\Omega \in C^{1,\mu}$.*

Then $h = P[f]$ is a quasiconformal mapping if and only if

$$0 < \operatorname{ess\,inf} |f'(\varphi)|, \tag{3.1}$$

$$\operatorname{ess\,sup} |f'(\varphi)| < \infty \tag{3.2}$$

and

$$\operatorname*{ess\,sup}_{\varphi} |H(f')(\varphi)| < \infty, \tag{3.3}$$

where

$$H(f')(\varphi) = \int_0^{\pi} \frac{f'(\varphi + t) - f'(\varphi - t)}{\tan t/2} dt,$$

denotes the Hilbert transformations of f'.

In the proof the assumption of absolute continuity was tacitly used. The following example shows that we cannot get rid of this condition.

Example 3.2. Let $c : [0, 1] \mapsto [0, 1]$ denote the Cantor staircase function. Let $\psi : [0, 2\pi] \mapsto [0, 2\pi]$ be defined as $\psi(2\pi t) = t\pi + c(t)\pi$, $f(t) = e^{i\psi(t)}$ and $h = P[f]$. Then f satisfies (3.1), (3.2) and (3.3), h maps $\mathbb{U}$ bijectively to $\mathbb{U}$ but is not quasiconformal.

Proof. By the fundamental theorem of Rado, Kneser and Choquet, h maps $\mathbb{U}$ bijectively to $\mathbb{U}$. Clearly, it is not going to be Lipschitz, and therefore this map cannot be quasiconformal since all HQC maps from disc to disc are bi-Lipschitz.

The only point which is left to be clarified is that Hilbert transform of $f'(t)$ will be bounded. Note that $f'(t)$ a.e. equals $i\pi f(t)$, since derivative of c is a.e. zero. Analysing the expression for the Hilbert transform, we see that essentially it suffices to prove that $\int_0^1 c(t)dt/t$ is finite. From this it will follow that Hilbert transform of $e^{\pi i(c(t)+t)}$ is bounded. Indeed, we just need to examine the integral on a δ neighbourhood of a point with small but fixed δ, and this is bounded if the integral

$$\int_{t_0-\delta}^{t_0+\delta} \frac{c(t) - c(t_0) + t - t_0}{t - t_0} dt$$

is bounded.

The part coming from the Cantor function when point is on some interval where function is constant is what we need to examine. Note that near the points 0 and 1 we need to extend the domain of c by setting $c(t+n) = n + c(t)$, $n \in \mathbb{Z}$. We look at integrals to and from t_0 separately. The pattern of $I_0 = \int_0^1 c(t)dt/t$ everywhere repeats, rescaled on x and y axes by factors 3^{-n} and 2^{-n}. Rescaling the pattern on x axis gives the same value of the integral. Rescaling it on y axis q times gives value of integral $q \cdot I_0$, and hence pasting patterns that add together to a certain jump on y axis gives integral that is no more than that for a single pattern of the same jump on the y axis. We need to add contributions from intervals where function c is constant that do not fit into any of the patterns. The size of the intervals has to increase as we go away from t_0, and if the value of difference $c(t) - c(t_0)$ is 2^{-n}, then $t - t_0 > 3^{-n}$. Contribution from this interval is at most $\int_{3^{-n}}^{\delta} 2^{-n} dt/t$ which is less than $2^{-n} n \cdot ln(3)$, and since we are adding contributions from n that will be different (since interval sizes not fitting into a whole pattern increse as we go away from t_0), this will be bounded by $O(\sum_{n=1}^{\infty} 2^{-n} n \cdot ln(3))$, which is finite. So we are left to prove that contribution from patterns is also finite, i.e. to prove that $\int_0^1 c(t)dt/t$ converges.

To estimate $\int_0^1 c(t)dt/t$ we sum contribution of intervals of length 3^{-n}, now considering the full pattern of the Cantor function. There will be 2^{n-1} such intervals. The corresponding values of c will be of the form $k2^{-(n-1)} + 2^{-n}$, $0 \le k < 2^{n-1}$, and for binary representation $k = \sum a_j 2^j$, $a_j \in \{0, 1\}$, we have the corresponding interval $3^{-n} + T(k) < t < 2 \cdot 3^{-n} + T(k)$, where
$T(k) = \sum 2a_j 3^{j-n+1}$. For $k = 0$ the contribution of the corresponding interval is less than 2^{-n}. For $1 < k < 2^{n-1}$ we get the contribution to the integral of corresponding interval to be less than $3^{-n}(k2^{-(n-1)} + 2^{-n})/T$. Now for $2^j \le k < 2^{j+1}$, this is less than $3^{-n} 2^{j+1-n}/3^{j-n}$. There are 2^j such values of k, and so the contribution from all those intervals is less than $2^{2j+1-n}/3^j = 2^{1-n}(4/3)^j$. Summing over all j where $j < n$ gives $O(2^{-n}(4/3)^n) = O((2/3)^n)$. Hence the corresponding contribution to the integral on all intervals of size 3^{-n} is $O((2/3)^n)$, which, summing over all n, gives a finite bound. □

4 The convexity condition and further results

In [27], we extend Theorem 3.1 and prove:

Theorem 4.1 (the characterization theorem)**.** *Suppose that $C^{1,\alpha}$ domain D is convex and denote by γ positively oriented boundary of D. Let $h_0 : \mathbb{T} \to \gamma$ be an orientation preserving homeomorphism and $h = P[h_0]$.*

The following conditions are then equivalent

a) h is K-qc mapping

b) h is bi-Lipschitz in the Euclidean metric

c) the boundary function h_* is bi-Lipschitz in the Euclidean metric and Cauchy transform $C[h'_*]$ of its derivative is in L^∞.

d) the boundary function h_* is absolutely continuous, $\operatorname{ess\,inf}|h'_*| > 0$ *and Cauchy transform $C[h'_*]$ of its derivative is in L^∞.*

e) the boundary function h_* is bi-Lipschitz in the Euclidean metric and Hilbert transform $H[h'_*]$ of its derivative is in L^∞.

f) the boundary function h_* is absolutely continuous, $\operatorname{ess\,sup}|h'_*| < +\infty$, $\operatorname{ess\,inf}|h'_*| > 0$ *and Hilbert transform $H[h'_*]$ of its derivative is in L^∞.*

Note that here, in our notation, $(h_0)_* = h_*$ and $h_0 = h^*$.

Our last example in this note shows that in the mentioned characterization of quasiconformality one cannot drop the convexity hypothesis.

Example 4.2 (three-cornered hat domain)**.** Let $h(z) = z + \overline{z}^2/2$,

$\gamma(t) = h(e^{it})$, $z_k = e^{\pi/3+2k\pi/3}$ and $A = \{z_0, z_1, z_2\}$.

Suppose also that Γ is smooth Jordan closed curve, $G = Int(\Gamma)$ such that $G \subset \mathbb{U}$, $A \subset Ext(\Gamma)$ and Γ has a joint arc J_0 with $\mathbb{T}$.

Let ϕ be conformal mapping of $\mathbb{U}$ onto G, $z = \phi(\zeta)$, and $\check{h} = h \circ \phi$. Then

$d\check{h} = Pd\zeta + Qd\overline{\zeta}$ and

$H(\check{h}'_*) = \zeta P^* + \overline{\zeta Q^*}$ a.e. on $\mathbb{T}$, where $P = \phi'$ and $Q = \phi\phi'$.

Since $h_*(t) = h(e^{it}) = e^{it} - e^{-2it}/2$, we find

$h'_*(t) = ie^{-2it}(e^{3it} - 1)$ and $h'_*(t) = 0$ iff $t = t_k = 2k\pi/3$.

It is easily to check that $\check{h}$ satisfies the all the hypothesis in the characterization except for the convexity condition, but $\check{h}$ is not a qc on $\mathbb{U}$ because $J_{\check{h}}$ is zero on J_0.

We can chose Γ to be C^∞ curve, then $\check{h}_*$ is C^∞; in particular $H(\check{h}'_*)$ is in L^∞.

The domain $h(\mathbb{U})$ is known as a three-cornered hat domain.

Suppose that there is a bi-Lipschitz homeomorphism h from $\mathbb{U}$ onto a domain G (in this setting we call G a bi-Lipschitz domain).

Then

1) partial derivatives of h are bounded and in particular $0 < s_1 \le |h'_*| \le s_2$ a.e. and

2) the boundary of G satisfies chord arc condition.

Also, if in addition h is harmonic then, by (1.1),

3) the Hilbert transform $H(h'_*) \in L^\infty$.

For example, $C^{1,\alpha}$ domains are bi-Lipschitz.

One may ask whether there is a nice geometric characterization of bi-Lipschitz domains and a version of the characterization theorem for such domains.

In [27], we have also outlined the proof of the following theorem.

Theorem 4.3. *Suppose that*

a1) D is $C^{1,\alpha}$ domain and

a2) h is a harmonic orientation preserving map of the unit disc onto D and homeomorphism of $\overline{\mathbb{U}}$ onto $\overline{D}$.

The following conditions are then equivalent

A1) h is K-qc mapping

A2) the boundary function h_* is absolutely continuous, $\operatorname{ess\,sup}|h'_*| < +\infty$, $Hh'_* \in L^\infty$ *and* $\operatorname{ess\,inf}|(Hh'_*, ih'_*)| > 0$.

If we only suppose the hypothesis $a2$), then $A2$) implies $A1$).

In an unpublished manuscript [14], cf Theorem 1.2 (stated below as Theorem A), D. Kalaj has announced (as the main result) an extension of Rado-Choquet-Kneser theorem and of a recent result of Alessandrini and Nesi [2] for the mappings with Lipschitz boundary data and essentially positive Jacobian at the boundary, without the convexity of the image domain assumption:

Theorem A. Suppose that
$b1$) γ is a $C^{1,\mu}$ smooth Jordan curve and D the domain bounded by γ,
$b2$) $h_0 : \mathbb{T} \to \partial D$ is an orientation preserving weak homeomorphic Lipschitz mapping of the unit circle onto ∂D, $h = P[h_0]$ and
$b3$) $\operatorname{essinf}\{J_h(e^{it}) : t \in [0, 2\pi]\} > 0$.
Then
B) the mapping h is a diffeomorphism of $\mathbb{U}$ onto D and h_0 is bi-Lipschitz continuous.

A characterization of quasiconformal harmonic mappings between smooth Jordan domains without hypothesis of the convexity of the image domains has also been given in [14]:

Theorem B. Suppose that
$c1$) γ is a C^2 smooth Jordan curve, D domain bounded by γ and
$c2$) $h_0 : \mathbb{T} \to \partial D$ is an orientation preserving homeomorphism of the unit circle onto the Jordan curve γ and $h = P[h_0]$.
The following conditions are equivalent:
$C1$) h is a qc mapping
$C2$) the boundary function h_* is absolutely continuous, $\operatorname{ess\,inf}|h'_*| > 0$, $\operatorname{ess\,sup}|h'_*| < +\infty$, $Hh'_* \in L^\infty$ and $\operatorname{ess\,inf}|\lambda_{h_*}| > 0$

Here for $h = f + \bar{g}$, a representation of a harmonic function h as sum of analitic and antianalytic function, $\lambda_h = |f'| - |g'|$.

Applications of Theorem 4.3 and Theorem A would lead to the following result: Under hypotheses $b1$) and $c2$), $A1$) is equivalent to $A2$).

References

[1] L. Ahlfors: *Lectures on quasiconformal mappings*, Van Nostrand Princeton, N.J., 1966.

[2] G. Alessandrini and V. Nesi: *Invertible harmonic mappings, beyond Kneser*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (5) VIII (2009), 451-468.

[3] S. Axler, P. Bourdon and W. Ramey: *Harmonic function theory*, Springer-Verlag, New York 1992.

[4] C. A. Berenstein and R. Gay: *Complex Variables. An Introduction.* New York, Springer-Verlag, 1991.

[5] A. Beurling and L. Ahlfors: *The boundary correspondence under quasiconformal mappings*, Acta Math. 96 (1956), 125-142.

[6] V. Božin and M. Mateljević: *Quasiconformal harmonic mapping between Jordan domains*, to appear

[7] X. Chen and A. Fang: *A Schwarz-Pick inequality for harmonic quasiconformal mappings and its applications*, J. Math. Anal. Appl. (2010), doi:10.1016/j.jmaa.2010.02.031

[8] G. M. Goluzin: *Geometric function theory* , Nauka Moskva 1966 (Russian).

[9] D. Kalaj: *On harmonic diffeomorphisms of the unit disc onto a convex domain.* Complex Variables, Theory Appl. 48, No.2, 175-187 (2003).

[10] D. Kalaj: *Quasiconformal harmonic functions between convex domains*, Publ. Inst. Math., Nouv. Ser. 76(90), 3-20 (2004).

[11] D. Kalaj: *Univalent harmonic mappings between Jordan domains.* Publ. Inst. Math., Nouv. Ser. 69(83), 108-112 (2001).

[12] D. Kalaj: *Quasiconformal harmonic mapping between Jordan domains* Math. Z. Volume 260, Number 2, 237-252, 2008.

[13] D. Kalaj: *Lipschitz spaces and harmonic mappings*, Ann. Acad. Sci. Fenn. Math. 34(2), (2009), 475-485.

[14] D. Kalaj, *Invertible Harmonic mappings beyond Kneser and harmonic quasiconformal mappings*, arXiv:1003.2740v1 [math.CV], 13 Mar 2010

[15] D. Kalaj and M. Mateljević: *Inner estimate and quasiconformal harmonic maps between smooth domains*, Journal d'Analise Math. 100. 117-132, (2006).

[16] D. Kalaj and M. Mateljević: *Quasiconformal and harmonic mappings between Jordan smooth domains*, Novi Sad J. Math. Vol. 38, No. 3, 2008, 147-156

[17] D. Kalaj and M. Mateljević: *On quasiconformal harmonic surfaces with rectifiable boundary*, to appear in Complex Analysis and Operator Theory

[18] D. Kalaj and M. Mateljević: *On certain nonlinear elliptic PDE and quasiconfomal mapps between Euclidean surfaces*, to appear in Potential Analysis

[19] D. Kalaj and M. Pavlović: *Boundary correspondence under harmonic quasiconformal homeomorfisms of a half-plane* Ann. Acad. Sci. Fenn., Math. 30, No.1, (2005) 159-165.

[20] O. Kellogg: *On the derivatives of harmonic functions on the boundary,* Trans. Amer. Math. Soc. 33 (1931), 689-692.

[21] M. Knežević and M. Mateljević: *On the quasi-isometries of harmonic quasiconformal mappings*, *On the quasi-isometries of harmonic quasi-conformal mappings* J. Math. Anal. Appl., 2007, 334(1), 404-413

[22] F. D. Lesley and S. E. Warschawski: *Boundary behavior of the Riemann mapping function of asymptotically conformal curves.* [J] Math. Z. 179, 299-323 (1982).

[23] O. Martio: *On harmonic quasiconformal mappings*, Ann. Acad. Sci. Fenn., Ser. A I 425 (1968), 3-10.

[24] M. Mateljević: *Estimates for the modulus of the derivatives of harmonic univalent mappings*, Proceedings of International Conference on Complex Analysis and Related Topics (IX^{th} Romanian-Finnish Seminar, 2001), Rev Roum Math Pures Appliq (Romanian Journal of Pure and Applied mathematics) 47 (2002) 5-6, 709 -711.

[25] M. Mateljević: *Distortion of harmonic functions and harmonic quasiconformal quasi-isometry*, Revue Roum. Math. Pures Appl. Vol.**51**,(2006)), 5-6, 711-722

[26] M. Mateljević: *On quasiconformal harmonic mappings*, unpublished manuscript, 2000, 2006.

[27] M. Mateljević, V. Božin and M. Knežević: *Quasiconformality of harmonic mappings between Jordan domains*, Filomat 24:3 (2010), 111-124

[28] M. Mateljević and M. Vuorinen: *On uniform continuity of harmonic maps*, to appear in Journal of Inequalities and Applications, Volume 2010, Article ID 178732, 19 pages doi:10.1155/2010/1787; arXiv: 0709.4546v1

[29] D. Partyka and K. Sakan: *Quasiconformality of harmonic extensions*, J. Comput. Appl. Math. 105 (1999), no. 1-2, 425–436.

[30] D. Partyka and K. Sakan: *On bi-Lipschitz type inequalities for quasiconformal harmonic mappings,* Ann. Acad. Sci. Fenn. Math. Vol 32 (2007), 579-594.

[31] M. Pavlović: *Boundary correspondence under harmonic quasiconformal homeomorfisms of the unit disc*, Ann. Acad. Sci. Fenn., Vol 27, 2002, 365-372.

[32] W. Rudin: *Real and complex analysis*, McGraw-Hill, 1966.

Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Belgrade, Seriba

E-mail: `miodrag@matf.bg.ac.rs`