Some Cross-Section Theorems on the Tangent Bundle over a Finslerian Manifold (*) (**).

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Summary. – *Let $T(M)$ be the tangent bundle over a Finslerian manifold M of n-dimension endowed with the Cartan connection ∇. One makes $T(M)$ into a $2n$ dimensional affinely connected manifold by assigning a connection ∇^C to $T(M)$. The cross-section $\mathfrak{V}$ of a vector field V defined in M reveals in $T(M)$ an n-dimensional submanifold and its geometry is developed by means of the affine subspace theory and of the affine collineations in the base Finsler manifold.*

Introduction.

K. YANO and OKUBO made an attempt to construct the geometry of Finslerian manifold by regarding it to be that of its tangent bundle itself [7]. In the theory of fibre bundles the notion of cross-section that provides the links between any object defined in base space and its image π^{-1} in its bundle space plays the important role and this paper tries to discuss the geometrical properties of the cross-section of a vector field defined in the base FINSLER manifold.

§ 1 is devoted to the introduction of the tangent bundle $T(M)$ over a Finsler manifold M with the CARTAN connection ∇. Herein endowing $T(M)$ with the vector fields X^V, X^H and X^C derived directly from the theory of connection and also by assigning the connection ∇^C, $T(M)$ is made into an affinely connected manifold. § 2 deals with the infinitesimal affine collineations in FINSLER manifolds, which together with § 1 serves the main discussion presented in § 3. Since the condition of complete integrability of affine collineations defined in Finslerian manifold is in our knowledge hard to find in any extant texts on the theory of LIE derivatives, it would not be less noteworthy to leave it in record (cf. (2.4)).

§ 3 discusses the geometry of the cross-section $\mathfrak{V}$ in $T(M)$ with respect to an arbitrary vector field $V(x)$ defined in M along its differentiable curve C. $\mathfrak{V}$ reveals itself in $T(M)$ a submanifold with the dimension same as that of M and along which we define the vertical, horizontal and complete vector fields denoted respectively by X^V, X^H and X^C. Then we prove that X^V lies in the plane normal to $\mathfrak{V}$ and X^H is

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tangential to $\mathfrak{V}$ if $\mathfrak{V}$ is a horizontal submanifold of $T(M)$, while X^C is so if $\mathfrak{V}$ is horizontal and the LIE derivative of X with respect to the vector field $V(x)$ vanishes. After establishing the subspace theory on $\mathfrak{V}$ we present the condition for the curvature vector field of $\mathfrak{V}$ to be tangential to $\mathfrak{V}$ (cf. Theorem 9).

1. – Tangent bundle $T(M)$ over a Finslerian space.

Let M be an n-dimensional manifold of class C^∞ and T_P be the tangent plane at a point P of M. Then $T(M) = \bigcup_{P \in M} T_p(M)$ is by definition the tangent bundle over the base space M. A point $\tilde{P}$ of $T(M)$ is an ordered pair (P, y_P) of a point P of M and a vector $y_P \in T_p(M)$. π is the projection of $T(M)$ into M defined by $\tilde{P} \to P$. The set $\pi^{-1}(P)$ is the fibre over P. On supposing that M satisfies the second axiom of countability we introduce in $T(M)$ the local coordinate system $x^A = (x^i, y^i)$ (1) in $\pi^{-1}\{U; (x^i)\}$ where U is a local coordinate neighbourhood of P in M and y^i denote the components of vector y at p having the expression $y = y^i \partial_i$ with respect to the natural frame $\partial_i = \partial/\partial x^i$. Then, corresponding to the coordinate transformation $(x^i) \to (x^{i'})$ at $P \in \{U(x)\} \cap \{U'(x')\}$, (x^A) obeys the law of transformation

$$x^{A'} = x^{A'}(x^B) \quad (1) \tag{1.1}$$

such that

$$\begin{aligned} &x^{i'} = x^{i'}(x^i), \ \det. (\partial_i x^{i'}) \neq 0 , \\ &x^{\bar{i}'} = y^{i'} = (\partial_i x^{i'}) y^i = (\partial_i x^{i'}) x^{\bar{i}} . \end{aligned} \tag{1.2}$$

and its Jacobian matrix is given by

$$(\partial_B x^{A'}) = \begin{bmatrix} \partial_i x^{h'} & 0 \\ y^a \partial_a \partial_i x^{h'} & \partial_i x^{h'} \end{bmatrix} . \tag{1.3}$$

We now take the direction $\dot{x} = dx/dt$ of any curve $C: x = x(t)$ of class C^r, $r \geqq 2$, as the fibre y of $T(M)$ over P and suppose that there is in $T(M)$ a differentiable function $L(x, \dot{x})$ which is positively homogeneous of degree one in the $\dot{x}$'s. On putting

$$F(x, \dot{x}) = L^2(x, \dot{x})/2$$

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(1) We adopt the following conventions for indices

$$A, B, C, D = 1, 2, \ldots, n , \quad \bar{1}, \bar{2}, \ldots, \bar{n} ,$$

$$a, b, c, h, i, j, k = 1, 2, \ldots, n .$$

and denoting $\partial/\partial\dot{x}^i$ by $\partial_{\bar{i}}$, we obtain n^2 symmetric functions $g_{ij}(x, \dot{x})$ which are homogeneous degree zero in the $\dot{x}$'s. If we assume that the quadratic form $g(A, A)$ defined for $\sum_{j=1}^{n} A^j A^j = 1$ is positive definite, we call the base manifold M a Finslerian space ([2], [3]). Then $T(M)$ is the *tangent bundle over a Finslerian space M.*

Since $T(M)$ is a $2n$-dimensional manifold, we can introduce two complementary n-dimensional distributions $\tilde{V}$ and $\tilde{H}$ at $\pi^{-1} P$, called the *vertical* and *horizontal subspaces.* $\tilde{V}$ and $\tilde{H}$ are respectively spanned by n independent base (∂_i) and

$$\delta_i = \partial_i - \Gamma_i^{\ h} \partial_{\bar{h}} , \tag{1.4}$$

where $\Gamma_i^h = \Gamma_{ji}^{\ h} y^j$ and $\Gamma_{ji}^{\ h}(x, \dot{x})$ are n^3 functions defined in $\pi^{-1}\{U; (x^i)\}$ and undergo the law of tranformation subject to (1.1):

$$\Gamma_{j'i'}^{h'} = (\partial_h x^{h'})\{\partial_{j'} \partial_{i'} x^h + (\partial_{j'} x^j)(\partial_{i'} x^i)\Gamma_{ji}^{\ h}\} .$$

We say that if there are given such n^3 functions in $T(M)$, the manifold is endowed with a non-linear connection ∇ and A. KAWAGUCHI called δ_i the *operator of basic connection.* It was E. CARTAN who provided M with ∇ which is torsion-free and keeps the metric g covariantly constant [7]

$$\Gamma_{ji}^h(x, x) = \begin{Bmatrix} h \\ ji \end{Bmatrix} - g^{ha}(\Gamma_j^b C_{bia} + \Gamma_i^b C_{bja} - \Gamma_a^b C_{bji}) , \tag{1.5}$$

where $\begin{Bmatrix} h \\ ji \end{Bmatrix}$ is the CHRISTOFFEL symbol formed with g_{ji} and C_{hji} is defined by $\frac{1}{2}\partial_{\bar{h}} g_{ji}$ and satisfies

$$C_{kji}\, \dot{x}^k = C_{kji}\, \dot{x}^j = C_{kji}\, \dot{x}^i = 0 .$$

Hence we have for (1.5)

$$\Gamma_{ji}^h \dot{x}^j \dot{x}^i = \Gamma_i^h \dot{x}^i = \begin{Bmatrix} h \\ ji \end{Bmatrix} \dot{x}^j \dot{x}^i . \tag{1.6}$$

Also we have from (1.4)

$$(\delta_k \delta_j - \delta_j \delta_k) f(x, \dot{x}) = - K_{kja}^{\ \ \ h} y^a \partial_{\bar{h}} f(x, \dot{x})$$

for any function $f(x, \dot{x})$ defined in $\pi^{-1} U$, where

$$K_{kji}^{\ \ \ h} = \delta_k \Gamma_{ji}^h - \delta_j \Gamma_{ki}^h + \Gamma_{ka}^h \Gamma_{ji}^a - \Gamma_{ja}^h \Gamma_{ki}^a ,$$

and it is called the component of the curvature tesor of the CARTAN connection ∇.

Let $X^i(x, \dot{x})$ be any n functions that obey the law of transformation

$$X^{h'}(x', \dot{x}') = (\partial_h x^{h'}) X^h(x, \dot{x}) . \tag{1.7}$$

We define in $T(M)$ the vertical, horizontal and complete vector fields denoted respectively by X^V, X^H and X^C by [7]

$$X^V = \begin{bmatrix} 0 \\ X^h \end{bmatrix}, \quad X^H = \begin{bmatrix} X^h \\ -\Gamma^h_{ji} y^j X^i \end{bmatrix}, \quad X^C = \begin{bmatrix} X^h \\ y^j \delta_j X^h \end{bmatrix}, \tag{1.8}$$

with respect to the natural frame $\partial_A = (\partial_h, \partial_{\bar{h}})$ in $\pi^{-1}(U)$.

Likewise, we define for n functions $\omega_i(x, x)$ satisfying

$$\omega_{i'}(x', x') = (\partial_{i'} x^i)\, \omega_i(x, x),$$

their vertical, horizontal and complete vector fields by

$$\omega^V = (\omega_i, 0), \quad \omega^H = (-\Gamma^h_{ji} y^j \omega_h, \omega_i), \quad \omega^C = (y^j \delta_j \omega_i, \omega_i).$$

Let P and Q be the two sets of n^{r+s} and n^{t+u} functions having in $\pi^{-1}(U)$ respectively the expression

$$P = P_{i_r i_{r-1} \dots i_1}{}^{h_s h_{s-1} \dots h_1}, \quad Q = Q_{j_t j_{t-1} \dots j_1}{}^{k_u k_{u-1} \dots k_1}$$

and obeying the law of transformation

$$P_{i'_r \dots i'_1}{}^{h'_s \dots h'_1} = (\partial_{i'_r} x^{i_r}) \dots (\partial_{i'_1} x^{i_1}) \cdot (\partial_{h_s} x^{h'_s}) \dots (\partial_{h_1} x^{h'_1}) P_{i_r \dots i_1}{}^{h_s \dots h_1},$$

$$Q_{j'_t \dots j'_1}{}^{k'_u \dots k'_1} = (\partial_{j'_t} x^{j_t}) \dots (\partial_{j'_1} x^{j_1}) \cdot (\partial_{k_u} x^{k'_u}) \dots (\partial_{k_1} x^{k'_1}) Q_{j_t \dots j_1}{}^{k_u \dots k_1},$$

subject to (1.1). Denoting by $P \times Q$ the formal functional product of P and Q, we define the vertical, horizontal and complete tensor fields respectively by

$$\begin{cases} (P \times Q)^V = P^V \otimes Q^V, \\ (P \times Q)^H = P^H \otimes Q^V + P^V \otimes Q^H \\ (P \times Q)^C = P^C \otimes Q^V + P^V \otimes Q^C. \end{cases}$$

By the use of the CARTAN connection ∇ we introduce in $T(M)$ the affine connection ∇^C so that $T(M)$ is now made into an affine manifold. ∇^C is supposed to have in $\pi^{-1}U$ the components of the form [6]

$$\begin{aligned} \tilde{\Gamma}^h_{ji} &= \Gamma^h_{ji}, \; \tilde{\Gamma}^h_{\bar{j}i} = \tilde{\Gamma}^h_{j\bar{i}} = \tilde{\Gamma}^h_{\bar{j}\bar{i}} = 0, \\ \tilde{\Gamma}^{\bar{h}}_{ji} &= y^a \delta_a \Gamma^h_{ji}, \; \tilde{\Gamma}^{\bar{h}}_{\bar{j}i} = \tilde{\Gamma}^{\bar{h}}_{j\bar{i}} = \Gamma^h_{ji}, \; \tilde{\Gamma}^{\bar{h}}_{\bar{j}\bar{i}} = 0. \end{aligned} \tag{1.9}$$

Let $X(x, \dot{x})$ and $Y(x, \dot{x})$ be any vector fields in $T(M)$.

$$\nabla^C_{X^C} Y^C = \begin{bmatrix} X^j \nabla_j Y^h \\ \dot{x}^a \partial_a (X^j \nabla_j y^h) \end{bmatrix} - \tag{1.10}$$

$$- \begin{bmatrix} 0 \\ K_{kji}{}^a X^k \dot{x}^j \dot{x}^i \partial_{\bar{a}} Y^h \end{bmatrix} - \begin{bmatrix} 0 \\ (\dot{x}^a \nabla_a X^i)\{\dot{x}^b \partial_b \partial_{\bar{i}} Y^h + \dot{x}^b \partial_{\bar{i}} (\Gamma^a_b \partial_{\bar{a}} Y^h)\} \end{bmatrix}$$

where $\nabla_j Y^h$ is defined by

$$\nabla_j Y^h = \partial_j Y^h + \Gamma^h_{ji} Y^i , \tag{1.11}$$

and as we have by (1.8),

$$(\nabla_X Y)^C = \begin{bmatrix} X^j \nabla_j Y^h \\ \dot{x}^a \partial_a (X^j \nabla_j Y^h) \end{bmatrix}, \tag{1.12}$$

we can state

THEOREM 1. – *$\nabla^C{}_{X^C} Y^C$ coincides with $(\nabla_X Y)^C$ if Y does not depend upon the $\dot{x}$'s.*

The curvature tensor K^C of ∇^C is by definition given by

$$K^C(\tilde{X}, \tilde{Y})\tilde{Z} = \nabla^C_{\tilde{X}} \nabla^C_{\tilde{Y}} \tilde{Z} - \nabla^C_{\tilde{Y}} \nabla^C_{\tilde{X}} \tilde{Z} - \nabla^C_{[\tilde{X}, \tilde{Y}]} \tilde{Z}$$

where $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ are any vector fields in $T(M)$, and for the variable range of indices K^C is found to have in $\pi^{-1}(U)$ the components of the form given by

$$\left\{ \begin{aligned}
&\tilde{K}_{kji}{}^h = K_{kji}{}^h + \Gamma^l_k \partial_{\bar{l}} \Gamma^h_{ji} - \Gamma^l_j \partial_{\bar{l}} \Gamma^h_{ki} , \\
&\tilde{K}_{\bar{k}ji}{}^h = -\tilde{K}_{j\bar{k}i}{}^h = \partial_{\bar{k}} \Gamma^h_{ji} , \\
&\tilde{K}_{kji}{}^{\bar{h}} = \dot{x}^a \partial_a K_{kji}{}^h - \dot{x}^b \dot{x}^a K_{kba}{}^m \partial_{\bar{m}} \Gamma^h_{ji} + \dot{x}^b \dot{x}^a K_{jba}{}^m \partial_{\bar{m}} \Gamma^h_{ki} + \\
&\qquad + \dot{x}^b \Gamma^a_k \partial_{\bar{a}} \partial_b \Gamma^h_{ji} - \dot{x}^b \Gamma^a_j \partial_{\bar{a}} \partial_b \Gamma^h_{ki} , \\
&\tilde{K}_{\bar{k}ji}{}^{\bar{h}} = -\tilde{K}_{j\bar{k}i}{}^{\bar{h}} = K_{kji}{}^h + \dot{x}^a \partial_a \partial_{\bar{k}} \Gamma^h_{ji} - x^a \partial_a \partial_{\bar{j}} \Gamma^h_{ki} - \Gamma^a_j \partial_{\bar{a}} \Gamma^h_{ki} , \\
&\tilde{K}_{kj\bar{i}}{}^{\bar{h}} = K_{kji}{}^h + \Gamma^a_k \partial_{\bar{a}} \Gamma^h_{ji} - \Gamma^a_j \partial_{\bar{a}} \Gamma^h_{ki} , \\
&\tilde{K}_{\bar{k}\bar{j}i}{}^{\bar{h}} = \partial_{\bar{k}} \Gamma^h_{ji} - \partial_{\bar{j}} \Gamma^h_{ki} , \\
&\tilde{K}_{\bar{k}j\bar{i}}{}^{\bar{h}} = -\tilde{K}_{j\bar{k}\bar{i}}{}^{\bar{h}} = \partial_{\bar{k}} \Gamma^h_{ji} , \\
&\tilde{K}_{k\bar{j}\bar{i}}{}^h = \partial_{\bar{j}} \Gamma^h_{ki} ,
\end{aligned} \right. \tag{1.13}$$

all other being zero.

On computing the BIANCHI identities

$$K^c(\tilde{X}, \tilde{Y})\tilde{Z} + K^c(\tilde{Y}, \tilde{Z})\tilde{X} + K^c(\tilde{Z}, \tilde{X})\tilde{Y} = 0\,,$$

$$\nabla^c_{\tilde{X}} K^c(\tilde{Y}, \tilde{Z})\tilde{W} + \nabla^c_{\tilde{Y}} K^c(\tilde{Z}, \tilde{X})\tilde{W} + \nabla^c_{\tilde{Z}} K^c(\tilde{X}, \tilde{Y})\tilde{W} = 0\,,$$

we have

THEOREM 2. – *The curvature tensor K of the Cartan connection satisfies*

$$K_{kji}{}^h + K_{jik}{}^h + K_{ikj}{}^h = 0 \tag{1.14}$$

$$\nabla_l K_{kji}{}^h + \nabla_k K_{jli}{}^h + \nabla_j K_{lki}{}^h = -K_{lkb}{}^a \dot{x}^b \partial_{\bar{a}} \Gamma^h_{ji} - K_{kjb}{}^a \dot{x}^b \partial_{\bar{a}} \Gamma^h_{li} - K_{jlb}{}^a \dot{x}^b \partial_{\bar{a}} \Gamma^h_{ki}\,.$$

The formulas (1.14) and (1.15) are called *the* BIANCHI *identities of the first and second kinds* respectively [2].

2. – Infinitesimal affine collineations.

Since $T(T(M))$ is spanned by $\tilde{V}$ and $\tilde{H}$ determined respectively by $(\partial_{\bar{i}})$ and (δ_i), the cotangent plane $'T(T(M))$ is spanned by the two planes dual to $\tilde{V}$ and $\tilde{H}$ and their base are respectively given by (δx^i) and (dx^i), where

$$\delta x^i = dx^i + \Gamma^i_j x^j\,. \tag{2.1}$$

Let $C: x = x(t)$, be a curve of class C^r, $r \geqq 2$, and $\dot{x}$ be its direction. If in $T(M)$ the 1-form (2.1) vanishes along C so that we have

$$\frac{\mathrm{d}^2 x^h}{\mathrm{d}t^2} + \Gamma^h_{ji} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^i}{\mathrm{d}t} = 0\,,$$

we call C the geodesic of M. Let $V(x)$ be any vector field defined along C in M and consider the infinitesimal transformation

$$\bar{x}^i = x^i + V^i(x)\,\delta u \tag{2.2}$$

$$\dot{\bar{x}}^i = \dot{x}^i + \dot{x}^a \partial_a V^i\,\delta u\,.$$

Then we say that if (2.2) sends the geodesic C to a geodesic, the vector field $V(x)$ generates an *affine collineation* [7], and its necessary and sufficient condition is given by the vanishing of the LIE derivative $\mathfrak{L}_V \Gamma^h_{ji}$ that is,

$$\mathfrak{L}_V \Gamma^h_{ji} = \nabla_j \nabla_i V^h + K_{kji}{}^h V^k + (\dot{x}^b \nabla_b V^a) \partial_{\bar{a}} \Gamma^h_{ji} = 0 \tag{2.3}$$

and the condition of integrability of the equations is found by a straightforward computation to be

$$\nabla_k \mathfrak{L}_V \Gamma^h_{ji} - \nabla_j \mathfrak{L}_V \Gamma^h_{ki} = \mathfrak{L}_V K_{kji}{}^h + (\dot{x}^b \mathfrak{L}_V \Gamma^a_{kb}) \partial_{\bar{a}} \Gamma^h_{ji} - (\dot{x}^b \mathfrak{L}_V \Gamma^a_{jb}) \partial_{\bar{a}} \Gamma^h_{ki} = 0 . \tag{2.4}$$

Hence we have by taking account of (2.3)

THEOREM 3. – *The condition of integrability of* $\mathfrak{L}_V \Gamma^h_{ji} = 0$ *is given by the vanishing of* $\mathfrak{L}_V K_{kji}{}^h$.

3. – Cross-section of $V(x)$.

Let $V(x)$ be a vector field of M defined along a curve $C: x = x(t)$, of class C^r, $r \geqq 2$, that lies in $\{U; (x^i)\}$. The coordinates x^A of $\pi^{-1}(P \in C)$ in $\pi^{-1}(U)$ corresponding to the vector $V(x)$ issuing from P is then given by

$$x^A = (x^i, V^i(x)) , \tag{3.1}$$

which we call the *cross-section* of $V(x)$ [1], [4] and it reveals itself in $T(M)$ an n-dimensional submanifold. Hereafter we express it by $\mathfrak{V}$. Along $\mathfrak{V}$ we have from (3.1)

$$\dot{x}^A = (\dot{x}^i, \dot{x}^a \partial_a V^i) ,$$

and with those $\dot{x}^i$ involved in x^A the n^3 functions $\Gamma^h_{ji}(x, \dot{x})$ of $T(M)$ are well defined throughout $\tilde{V}$. Subject to these $\Gamma^h_{ji}(x, \dot{x})$ if the vector field $V(x)$ satisfies

$$\nabla_j V^h = \partial_j V^h + \Gamma^h_{ji}(x, \dot{x}) V^i = 0 , \tag{3.2}$$

we say that $\mathfrak{V}$ is the *horizontal submanifold* of $T(M)$.

The base B of the tangent plane of $\mathfrak{V}$ has in $\pi^{-1}(U)$ the expression

$$B_i{}^A = \partial_i x^A = \begin{bmatrix} \delta_i{}^h \\ \partial_i V^h \end{bmatrix} , \tag{3.3}$$

and serves to map any vector field defined in $\mathfrak{V}$ into a vector in $T(M)$ tangent to $\mathfrak{V}$. In order to obtain the WHITNEY sum of frames at each point of $\mathfrak{V}$, we choose n *affine normals* $C_{\bar{i}}^A$ defined by

$$C_{\bar{i}}^A = \partial x^A / \partial V^i = \begin{bmatrix} 0 \\ \delta^h_i \end{bmatrix} . \tag{3.4}$$

Then the coframe $(B^h{}_B, C^{\bar{h}}{}_B)$ dual to the frame $(B_i{}^A, C_{\bar{i}}{}^A)$ has the expression

$$B^h{}_B = (\delta_i^h, 0)\,, \qquad C^{\bar{h}}{}_B = (-\partial_i V^h, \delta_i^h)\,. \tag{3.5}$$

Let $X^h(x, \dot{x})$ be any n functions in $T(M)$ defined along $\mathfrak{V}$ satisfying the law of transformation given in (1.7), and by the use of which we construct X^V. If we decompose X^V into the directions tangent and normal to $\mathfrak{V}$, we have

$$B^h{}_A(X^V)^A = 0, \qquad C^{\bar{h}}{}_A(X^V)^A = X^h$$

in virtue of (3.5). Hence we have

$$X^V = C_{\bar{i}}{}^A X^i, \tag{3.6}$$

and thus

THEOREM 4. – *Any vertical vector field defined along $\mathfrak{V}$ is ontained in the normale plane of $\mathfrak{V}$.*

Along $\mathfrak{V}$ we define X^H and X^C by using the above stated n functions $X^h(x, \dot{x})$ and n^3 functions $\Gamma^h_{ji}(x, \dot{x})$ by

$$X^H = \begin{bmatrix} X^h \\ -\Gamma^h_{ji}(x, \dot{x}) V^j X^i \end{bmatrix}, \qquad X^C = \begin{bmatrix} X^h \\ V^j \partial_j X^h \end{bmatrix}, \tag{3.7}$$

respectively. Then we have

$$B^h{}_A(X^H)^A = X^h, \qquad C^{\bar{h}}{}_A(X^H)^A = -(\nabla_i V^h) X^i\,,$$

from which we get

$$(X^H)^A = B_i{}^A X^i - C_{\bar{i}}{}^A(\nabla_j V^i) X^j\,. \tag{3.8}$$

and hence we have by taking account of (3.2)

THEOREM 5. – *Any horizontal vector field defined along $\mathfrak{V}$ is tangential to $\mathfrak{V}$ if $\mathfrak{V}$ is an horizontal submanifold of $T(M)$.*

As for X^C we have

$$B^h{}_A(X^C)^A = X^h, \qquad C^{\bar{h}}{}_A(X^C)^A = \mathfrak{L}_V X^h - (\dot{x}^b \nabla_b V^a)\,\partial_{\bar{a}} X^h$$

from which we get

$$(X^C)^A = B_i{}^A X^i + C_{\bar{i}}{}^A\{\mathfrak{L}_V X^i - (\dot{x}^b \nabla_b V^a)\partial_{\bar{a}} X^i\}\,. \tag{3.9}$$

Thus we have

THEOREM 6. – *Any complete vector field X^C defined along $\mathfrak{V}$ is tangential to $\mathfrak{V}$ if the* LIE *derivative $\mathfrak{L}_V X$ vanishes and $\mathfrak{V}$ is a horizontal submanifold of $T(M)$.*

We have introduced in $T(M)$ the affine connection ∇^C whose components in $\pi^{-1}(U)$ were given in (1.9). Then we have for $\mathfrak{B}$ the van der WAERDEN-BORTOLOTTIE equations

$$\nabla^C_{B\bar{X}} B\bar{Y} = B\,'\nabla_{\bar{X}}\bar{Y} + H^i(\bar{X},\bar{Y})C_{\bar{i}} \tag{3.10}$$

where $\bar{X}$ and $\bar{Y}$ are arbitrary vector fields of $\mathfrak{B}$ and $'\nabla$ is the connection of $\mathfrak{B}$, induced from ∇^C which is symmetric and is called the *induced connection* of $\mathfrak{B}$. H^i are n symmetric tensor fields of type (0.2) and are called the *second fundamental tensor fields* of $\mathfrak{B}$. We say that $\mathfrak{B}$ is a *totally geodesic submanifold* of $T(M)$ if H^i vanish identically. In terms of the local coordinates $'\nabla$ has the components

$$'\Gamma^h_{ij}(x, x) = (\partial_j B_i{}^A + (\Gamma^A_{CB})^C B_j{}^C B_i{}^B)\, B^h{}_A$$

where $(\Gamma^A_{CB})^C$ denote the components of ∇^C with respect to (x^A) of $\pi^{-1}(U)$. Then taking account of (1.9), (3.3) and (3.5) we find

$$'\Gamma^h_{ji} = \Gamma^h_{ji} \tag{3.11}$$

and (3.10) is reducible to

$$'\nabla_j B_i{}^A = \{\mathfrak{L}_V \Gamma_{ji}{}^h - (\dot{x}^b \nabla_b V^a)\partial_{\bar{a}} \Gamma^h_{ji}\} C_{\bar{h}}{}^A\,. \tag{3.12}$$

Hence we have by taking account of (2.3) and (3.2)

THEOREM 7. – *The cross-section $\mathfrak{B}$ of a vector field $V(x)$ is a totally geodesic submanifold, if $V(x)$ generates an affine collineation in M and $\mathfrak{B}$ is a horizontal submanifolds in $T(M)$.*

The local expression of the WEINGARTEN equations of $\mathfrak{B}$ is given by

$$'\nabla_j C_{\bar{i}}^A = \partial_j C_{\bar{i}}{}^A + (\Gamma^A_{CB})^C B_j{}^C C_{\bar{i}}{}^B - {}'\Gamma^h_{ji}\, C_{\bar{h}}{}^A$$

and by taking account of (1.9), (3.3) and (3.11) we find

$$'\nabla_j C_{\bar{i}}{}^A = 0 \tag{3.13}$$

Thus

THEOREM 8. – *The affine normal vector fields $C_{\bar{i}}$ are transported parallel along $\mathfrak{B}$.*

Since we have (3.10) the structure equations of $\mathfrak{B}$ will only be given by

$$\nabla^C_{B\bar{X}}\nabla^C_{B\bar{Y}} B\bar{Z} - \nabla^C_{B\bar{Y}}\nabla^C_{B\bar{X}} B\bar{Z} - \nabla^C_{[B\bar{X}, B\bar{Y}]} B\bar{Z} = K^C(B\bar{X}, B\bar{Y})B\bar{Z}\,,$$

and it has the local expression

$$ {}'\nabla_k {}'\nabla_j B_i{}^A - {}'\nabla_j {}'\nabla_k B_i{}^A = (K)^C_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B - K_{kji}{}^h B_h{}^A \tag{3.14} $$

in virtue of (3.10) and (3.11). If we take account of the components $(K)^C{}_{DCB}{}^A$ of K^C given in (1.13) and also of (3.3), the right hand side of (3.14) is found to have the form

$$ \begin{aligned} &[\mathfrak{L}_V K_{kji}{}^h + (\dot{x}^b \mathfrak{L}_V \Gamma^a_{kb})\partial_{\bar{a}}\Gamma^h_{ji} - (\dot{x}^b \mathfrak{L}_V \Gamma^a_{jb})\partial_{\bar{a}}\Gamma^h_{ki} - \\ &\quad - \dot{x}^d\{\partial_{\bar{b}} K_{kji}{}^h + (\partial_{\bar{b}}\Gamma^a_{kc})\dot{x}^c \partial_{\bar{a}}\Gamma^h_{ji} - (\partial_{\bar{b}}\Gamma^a_{jc})\dot{x}^c \partial_{\bar{a}}\Gamma^h_{ki}\}\nabla_d V^b - \\ &\qquad - \dot{x}^a\{(\partial_{\bar{b}}\Gamma^h_{ji})\nabla_k\nabla_a V^b - (\partial_b \Gamma^h_{ki})\nabla_j\nabla_a V^b\}]C_{\bar{h}}{}^A. \end{aligned} \tag{3.15} $$

Hence we have by taking account of (2.3), (3.2), (3.14), (3.15) and also of Theorem 4:

THEOREM 9. – *Let $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ be any vector fields in $\mathfrak{B}$, then the curvature vector $K^C(B\bar{X}, B\bar{Y})B\bar{Z}$ is tangential to $\mathfrak{B}$, if $\mathfrak{B}$ is a horizontal submanifold in $T(M)$ and $V(x)$ defines an affine collineation in M.*

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