Some Curvature Properties of Complex Surfaces (*).

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Summary. – *In this paper, we are investigating curvature properties of complex two-dimensional Hermitian manifolds, particularly in the compact case. To do this, we start with the remark that the fundamental form of such a manifold is integrable, and we use the analogy with the locally conformal Kähler manifolds, which follows from this remark. Among the obtained results, we have the following: a compact Hermitian surface for which either the Riemannian curvature tensor satisfies the Kähler symmetries or the Hermitian curvature tensor satisfies the Riemannian Bianchi identity is Kähler; a compact Hermitian surface of constant sectional curvature is a flat Kähler surface; a compact Hermitian surface M with nonnegative non-identical zero holomorphic Hermitian bisectional curvature has vanishing plurigenera,* $c_1(M) \geqslant 0$, *and no exceptional curves; a compact Hermitian surface with distinguished metric, and positive integral Riemannian scalar curvature has vanishing plurigenera, etc.*

The present paper started from the remark that 2-dimensional Hermitian metrics have an interesting particular property namely: the corresponding fundamental form is integrable (its differential belongs to the ideal generated by the form itself). In this respect, they are similar to the locally conformal Kähler metrics [13], a fact which suggests us to use the methods of the theory of the locally conformal Kähler manifolds in order to study differential geometric properties of general complex analytic surfaces (i.e., complex analytic manifolds of complex dimension 2).

Our results are concerned with two questions. One of them is the characterization of Kähler metrics by curvature properties. For instance, we prove that: a compact Hermitian surface, whose Riemannian curvature has the Kählerian symmetries is a Kähler surface; a compact Hermitian surface of constant sectional curvature is a flat Kähler surface; a compact Hermitian surface whose unitary (or Hermitian) curvature tensor satisfies the Riemannian Bianchi identity is a Kähler surface, etc.

The second question is that of the influence of the curvature on the structure of the surface. Until now, this subject has been studied rather for Kähler metrics [1, 7, 18], and here we shall be discusing general Hermitian metrics on a surface.

We shall establish that if the Hermitian (unitary, Chern) connection of a compact Hermitian surface M has a nonnegative but not identical zero holomorphic bisectional curvature, then M has a nonnegative first Chern class, vanishing plurigenera and no exceptional curve [10]. In particular, if the above mentioned bisec-

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tional curvature is positive, $c_1(M) > 0$, and M is biholomorphically equivalent either to the complex projective plane or (but not sure) to $CP^1 \times CP^1$. We shall provide also a generalization of a theorem of YAU [18] stating that if M is a compact Hermitian surface whose metric g is of a distinguished (more general than Kähler) type, and if the integral Riemannian scalar curvature of g is positive, then M has vanishing plurigenera, etc.

1. – Introduction.

In a general manner, we shall denote by M a complex surface, and by J its tensor of the complex structure. If a Hermitian metric g is added, we shall speak of a *Hermitian surface*, and we shall denote by Ω its fundamental form

$$\Omega(X, Y) = g(X, JY) . \tag{1.1}$$

Another interesting differential form of (M, J, g) is the *Lee form*

$$\omega = \Lambda\, d\Omega , \tag{1.2}$$

where $\Lambda = i(\Omega)$ is the *interior product* by Ω, i.e. the dual of the operator $L(\alpha) = \Omega \wedge \alpha$.

Now, the following result is well known [11]:

PROPOSITION 1.1. – *The relation*

$$d\Omega = \omega \wedge \Omega \tag{1.3}$$

holds on every Hermitian surface.

PROOF. – In Hermitian geometry we have

$$\Lambda L\alpha - L\Lambda\alpha = (n - \deg \alpha)\alpha , \tag{1.4}$$

for every form α on an n-dimensional manifold. Now, (1.2) and (1.4) imply for $n = 2$

$$L\omega = \Lambda L\, d\Omega + d\Omega ,$$

and, since $\deg (L\, d\Omega) = 5$, this relation is exactly (1.3).

For a Hermitian manifold M with $\dim_c M = n > 2$, a *Lee form* can be defined by

$$\omega = [1/(n-1)]\Lambda\, d\Omega , \tag{1.5}$$

but Proposition 1.1 may not hold.

For an arbitrary $n \geqslant 2$, if (1.3) holds and if ω is closed, we have locally $\omega = d\sigma$, and $\tilde{g} = e^{-\sigma} g$ are local Kähler metrics on M. Such a manifold M is called *locally conformal Kähler*, or, if ω is an exact form, *globally conformal Kähler* [13].

If $n > 2$ and (1.3) holds, ω must be closed [11], but this may not be true for $n = 2$, even if M is compact. For example, let us consider the complex torus $T^2_c = C^2/G$, where G is the group generated by $(z^1, z^2) \mapsto (z^1 + a, z^2 + b)$, a and b being two complex numbers of imaginary part 2π. Then, the formula

$$ds^2 = \exp \sin \sqrt{-1}\, \frac{\bar{z}^2 - z^2 + \bar{z}^1 - z^1}{2}\, dz^1 \otimes d\bar{z}^1 + dz^2 \otimes d\bar{z}^2 \tag{1.6}$$

defines a Hermitian metric on T^2_c for which

$$\omega = \frac{\sqrt{-1}}{2} \cos \sqrt{-1}\, \frac{\bar{z}^2 - z^2 + \bar{z}^1 - z^1}{2} (d\bar{z}^2 - dz^2)\,. \tag{1.7}$$

The relation (1.3) holds, but $d\omega \neq 0$.

However, we have [11]

PROPOSITION 1.2. – *The relation*

$$\Lambda\, d\omega = 0 \tag{1.8}$$

holds on every Hermitian surface.

PROOF. – A differentiation of (1.3) yields

$$L\, d\omega = 0\,, \tag{1.9}$$

whence (1.8) follows by applying Λ and using (1.4).

Another important fact is

PROPOSITION 1.3. – *If M is either a compact locally conformal Kähler manifold of an arbitrary dimension or a compact Hermitian surface, then M has a conformally related metric $g' = fg$ $(f > 0)$ whose Lee form ω' is co-closed.*

PROOF. – This is an easy consequence of some rather deep results of GAUDUCHON [2]. In [2], one defines a *Hermitian metric of vanishing eccentricity* as one which satisfies $\delta v = 0$, where $v = \delta\Omega \circ J$, and one states [2, p. 137, footnote] that any Hermitian metric on a compact manifold has a conformally related metric of vanishing eccentricity (1). But it is known that the Lee forms (1.2), (1.5) are exactly

(1) See the proof in: P. GAUDUCHON, *Le théorème de l'excentricité nulle*, C. R. Acad. Sci. Paris, **285** A (1977), pp. 387-390.

equal to $[1/(n-1)]v$ [11, 14]. Then the stated result follows by combining these facts.

REMARK. – In particular, *a compact locally conformal Kähler manifold has a conformally related metric with a harmonic Lee form.* This may be useful in the study of the topology of such a manifold.

In this paper, a Hermitian metric of vanishing eccentricity will be called a *distinguished Hermitian metric.*

On the other hand, it is clear that a Hermitian metric is Kähler iff $\omega = 0$, which justifies to consider $l = |\omega|^2/2 \geqslant 0$ and to call it the *modul of non-Kählerianity* of the respective Hermitian metric.

2. – Connections and curvatures.

Let (M, g) be a Hermitian surface and let us denote by ∇ the Levi-Civita connection of g.

Following [13], let us define the *Weyl connection* of M by

$$\tilde{\nabla}_X Y = \nabla_X Y - \tfrac{1}{2}\omega(X)Y - \tfrac{1}{2}\omega(Y)X + \tfrac{1}{2}g(X, Y)B\,, \tag{2.1}$$

where the *Lee field* B is defined by

$$g(B, X) = \omega(X)\,. \tag{2.2}$$

Then we have

PROPOSITION 2.1. – *The Weyl connection of a Hermitian surface M has no torsion, and it satisfies the relations*

$$\tilde{\nabla}_X J = 0\,, \quad \tilde{\nabla}_X g = \omega(X)g\,, \quad \tilde{\nabla}_X \Omega = \omega(X)\Omega \tag{2.3}$$

PROOF. – The same proof like for locally conformal Kähler manifolds in [13].

Furthermore, it is well known that g defines a *unitary connection* (Hermitian connection) $\nabla_{(c)}$, which is characterized by

$$\nabla_{(c)}J = 0\,, \quad \nabla_{(c)}g = 0\,, \quad T_{(c)}(X, JY) = T_{(c)}(JX, Y)\,, \tag{2.4}$$

where $T_{(c)}$ is the torsion of $\nabla_{(c)}$. By straightforward technical computation we can deduce

PROPOSITION 2.2. – *The unitary connection of a Hermitian surface is given by*

$$\begin{aligned}\nabla_{(c)X} Y &= \tilde{\nabla}_X Y + \tfrac{1}{2}\omega(X)Y - \tfrac{1}{2}\omega(JX)JY = \\ &= \nabla_X Y - \tfrac{1}{2}\omega(Y)X - \tfrac{1}{2}\omega(JX)JY + \tfrac{1}{2}g(X, Y)B\,,\end{aligned} \tag{2.5}$$

Now, let us denote by R, $\tilde{R}$, $R_{(c)}$, respectively, the curvatures of ∇, $\tilde{\nabla}$, $\nabla_{(c)}$ with the sign convention of [9], and set

$$R(X_1, X_2, X_3, X_4) = g(R(X_3, X_4)X_2, X_1)\,, \tag{2.6}$$

and something similar for $\tilde{R}$ and $R_{(c)}$.

Then, we get as in [15]

$$\begin{aligned}\tilde{R}(X_1, X_2, X_3, X_4) &= R(X_1, X_2, X_3, X_4) - \tfrac{1}{2}\{L(X_3, X_2)g(X_4, X_1) - \\ &- L(X_4, X_2)g(X_3, X_1) + L(X_4, X_1)g(X_3, X_2) - L(X_3, X_1)g(X_4, X_2)\} - \\ &- \tfrac{1}{2}g(X_1, X_2)\, d\omega(X_3, X_4) - \frac{|\omega|^2}{4}\{g(X_4, X_2)g(X_3, X_1) - g(X_3, X_2)g(X_4, X_1)\}\end{aligned} \tag{2.7}$$

where

$$L(X, Y) = (\nabla_X\omega)(Y) + \tfrac{1}{2}\omega(X)\omega(Y)\,, \tag{2.8}$$

and this is a symmetric tensor iff M is locally conformal Kähler.

Furthermore:

$$\begin{aligned}R_{(c)}(X_1, X_2, X_3, X_4) &= \tilde{R}(X_1, X_2, X_3, X_4) + \\ &\tfrac{1}{2}g(X_1, X_2)\, d\omega(X_3, X_4) - \tfrac{1}{2}\Omega(X_1, X_2)\, d\theta(X_3, X_4)\,,\end{aligned} \tag{2.9}$$

where

$$\theta = \omega \circ J\,. \tag{2.10}$$

And finally:

$$\begin{aligned}R_{(c)}(X_1, X_2, X_3, X_4) &= R(X_1, X_2, X_3, X_4) - \tfrac{1}{2}\{L(X_3, X_2)g(X_4, X_1) - \\ &- L(X_4, X_2)g(X_3, X_1) + L(X_4, X_1)g(X_3, X_2) - L(X_3, X_1)g(X_4, X_2) - \\ &- \frac{|\omega|^2}{4}\{g(X_4, X_2)g(X_3, X_1) - g(X_3, X_2)g(X_4, X_1)\} - \tfrac{1}{2}\Omega(X_1, X_2)\, d\theta(X_3, X_4)\,.\end{aligned} \tag{2.11}$$

As it is well known, various contractions of the curvature tensors are also of interest. Thus are: the *Ricci tensor*

$$\varrho(X, Y) = \sum_{i=1}^{4} R(E_i, Y, E_i, X) \tag{2.12}$$

where E_i $(i = 1, 2, 3, 4)$ is an arbitrary orthonormal local tangent basis; the *Weyl-Ricci form*

$$\tilde{W}(X, Y) = \sum_{i=1}^{4} R(E_i, E_i, X, Y)\,, \tag{2.13}$$

the *scalar curvature*

$$r = \sum_{i,j=1}^{4} R(E_i, E_j, E_i, E_j)\,, \tag{2.14}$$

etc. We shall speak also of the various *sectional curvatures* analogous to those usually considered for Kähler manifolds, while these may not necessarilly be functions on the 2-*section* only.

Furthermore, let us consider local orthonormal bases of the form $\{E_\alpha, JE_\alpha = E_{\alpha^*}\}$ $(\alpha = 1, 2)$, and use them to define forms representing the first Chern class of M. Firstly, if we use the unitary connection to this purpose, we get as in [6] that the form

$$C_{(c)}(X, Y) = \frac{1}{2\pi}\sum_\alpha R_{(c)}(E_\alpha, E_{\alpha^*}, X, Y)\,, \tag{2.15}$$

which we call the *unitary Chern form* represents the first real Chern class of M.

Secondly, if we use the Weyl connection which is also compatible with the complex structure of M, we get by the same method [6] the form:

$$\tilde{C}'(X, Y) = \frac{1}{2\pi}\left\{\sum_\alpha \tilde{R}(E_\alpha, E_{\alpha^*}, X, Y) + \sqrt{-1}\sum_\alpha \tilde{R}(E_\alpha, E_\alpha, X, Y)\right\} \tag{2.16}$$

which represents the complex first Chern class of M. But from (2.9) we have, because of the skew-symmetry of $R_{(c)}$ in its two first arguments,

$$\tilde{R}(E_\alpha, E_\alpha, X, Y) = -\tfrac{1}{2}\, d\omega(X, Y)\,, \tag{2.17}$$

whence we deduce that (2.16) is cohomologous with the real *Weyl-Chern form*

$$\tilde{C}(X, Y) = \frac{1}{2\pi}\sum_\alpha \tilde{R}(E_\alpha, E_{\alpha^*}, X, Y)\,, \tag{2.18}$$

and this also represents the first Chern class of M.

It is worthwhile noting that (2.9) yields

$$C_{(c)}(X, Y) = \tilde{C}(X, Y) + \frac{1}{2\pi}\, d\theta(X, Y)\,. \tag{2.19}$$

Furthermore, there is one additional interesting interpretation of $C_{(c)}(X, Y)$: namely, it represents the Chern class of the contravariant holomorphic canonical bundle of M, while the curvature of the Chern connection of this canonical bundle will be $-2\pi\sqrt{-1}C_{(c)}$. By an easy computation, this implies that the *Ricci scalar* of the contravariant canonical bundle of M [2] is given by

$$k_M = \sum_{\alpha,\beta} R_{(c)}(E_\alpha, E_{\alpha^*}, E_\beta, E_{\beta^*})\,. \tag{2.20}$$

3. – Curvature and the Kähler condition.

Now, we shall proceed with the discussion of the curvature properties of complex surfaces.

Let us begin by transposing an identity which has been established for the locally conformal Kähler manifolds by T. KASHIWADA [8]. It refers to the scalar curvature r of (2.14) and to the invariant

$$r^* = \sum_{i,j=1}^{4} R(E_i, E_j, JE_i, JE_j)\,, \tag{3.1}$$

where E_i is an arbitrary orthonormal local basis.

THEOREM 3.1. – *The following relation holds on any Hermitian surface*

$$\varkappa \overset{\text{def}}{=} r - r^* = 2\delta\omega + |\omega|^2\,. \tag{3.2}$$

PROOF. – Let us start with the obvious relation

$$R_{(c)}(JX_1, JX_2, X_3, X_4) = R_{(c)}(X_1, X_2, X_3, X_4)\,. \tag{3.3}$$

By using (2.11) this yields

$$\begin{aligned} &2R(X_1, X_2, X_3, X_4) - 2R(JX_1, JX_2, X_3, X_4) = L(X_3, X_2)g(X_4, X_1) - \\ &- L(X_4, X_2)g(X_3, X_1) + L(X_4, X_1)g(X_3, X_2) - L(X_3, X_1)g(X_4, X_2) - \\ &- L(X_3, JX_2)\Omega(X_4, X_1) + L(X_4, JX_2)\Omega(X_3, X_1) - L(X_4, JX_1)\Omega(X_3, X_2) + \\ &+ L(X_3, JX_1)\Omega(X_4, X_2) + \frac{|\omega|^2}{2}\{g(X_4, X_2)g(X_3, X_1) - g(X_3, X_2)g(X_4, X_1) - \\ &- \Omega(X_4, X_2)\Omega(X_3, X_1) + \Omega(X_3, X_2)\Omega(X_4, X_1)\}\,. \end{aligned} \tag{3.4}$$

Now, if we take $X_1 = \partial/\partial x^i$, $X_2 = \partial/\partial x^j$, $X_3 = \partial/\partial x^k$, $X_4 = \partial/\partial x^l$, where x^i ($i = 1, 2, 3, 4$) are real local coordinates on M, and then contract with $g^{ki}g^{lj}$, we get after some technical computations the stated relation (3.2).

The invariant $\varkappa$ will be called the *scalar curvature defect* of M.

Theorem 3.1 provides us with the following interesting corollaries:

COROLLARY 3.2. – *A compact Hermitian surface with nonpositive integral scalar curvature defect* $\left(\int_M \varkappa \leqslant 0\right)$ *is a Kähler surface.*

PROOF. – It suffices to integrate (3.2) over M and to compare the signs of the results.

COROLLARY 3.3. – *If the Riemannian curvature of a compact Hermitian surface M satisfies the Kähler identity*

$$R(JX, JY, Z, W) = R(X, Y, Z, W)\,, \tag{3.5}$$

M is a Kähler surface.

PROOF. – This follows from Corollary 3.2 since the hypothesis implies $\varkappa = 0$.

COROLLARY 3.4. – *If for every two holomorphic sections defined by the unit vectors X, Y, the Riemannian sectional and bisectional curvatures of the compact Hermitian surface M are related by*

$$R(X, JX, Y, JY) \geqslant R(X, Y, X, Y) + R(X, JY, X, JY)\,, \tag{3.6}$$

then M is a Kähler surface, and we have the equality sign in (3.6).

PROOF. – If we use in (3.1) a basis of the form $\{E_\alpha, E_{\alpha^*}\}$, we get because of the Bianchi identity for R

$$r^* = 2 \sum_{\alpha,\beta} R(E_\alpha, E_{\alpha^*}, E_\beta, E_{\beta^*})\,, \tag{3.7}$$

whence we can deduce

$$\begin{aligned} \varkappa = {} & \sum_{\alpha,\beta} [R(E_\alpha, E_\beta, E_\alpha, E_\beta) + R(E_\alpha, E_{\beta^*}, E_\alpha, E_{\beta^*}) - R(E_\alpha, E_{\alpha^*}, E_\beta, E_{\beta^*})] + \\ & + \sum_{\alpha,\beta} [R(E_{\alpha^*}, E_{\beta^*}, E_{\alpha^*}, E_{\beta^*}) + R(E_{\alpha^*}, E_\beta, E_{\alpha^*}, E_\beta) - R(E_{\alpha^*}, E_\alpha, E_{\beta^*}, E_\beta)\,. \end{aligned} \tag{3.8}$$

Now, the stated result follows from (3.6) and Corollary 3.2.

It is also worthwhile noting that formula (3.4) and the Bianchi identity for R are leading us to the following relation between the bisectional and the sectional curvature of M:

$$\begin{aligned} R(X, JX, Y, JY) = {} & R(X, Y, X, Y) + R(X, JY, X, JY) - \\ & - \tfrac{1}{2}\{g(X, Y)[2L(X, Y) + L(Y, X) + L(JY, JX)] + \\ & + \Omega(X, Y)[2L(X, JY) - L(Y, JX) + L(JY, X)] - \\ & - [2L(X, X) - L(Y, Y) - L(JY, JY)]\} - \frac{|\omega|^2}{2}\{1 - g^2(X, Y) - \Omega^2(X, Y)\}\,, \end{aligned} \tag{3.9}$$

where X and Y are unit tangent vectors.

COROLLARY 3.5. – *A compact Hermitian surface of constant sectional curvature is a flat Kähler surface.*

PROOF. – Assume that M has the constant sectional curvature c. We cannot have $c > 0$ since M would be covered by S^4, and the latter has no complex structure. On the other hand, it is easy to get $\varkappa = 8c$, whence the stated result follows from Corollary 3.2, in the case $c \leqslant 0$.

Another interesting result can be deduced from a theorem of A. GRAY [5]:

THEOREM 3.6. – *Assume that the holomorphic sectional curvature of a complete Hermitian surface satisfies the condition*

$$R(X, JX, X, JX) \geqslant \frac{|\omega|^2}{4} + \delta \qquad (|X| = 1) \tag{3.10}$$

for some $\delta > 0$. Then M is compact and simply connected.

PROOF. – The proof proceeds like in the case of the locally conformal Kähler manifolds [15; Prop. 3.1]. First, we get from (2.1):

$$(\nabla_X J)(Y) = -\tfrac{1}{2}g(X, Y)A - \tfrac{1}{2}\Omega(X, Y)B + \tfrac{1}{2}\theta(Y)X - \tfrac{1}{2}\omega(Y)JX\,, \tag{3.11}$$

where $A = -JB$. Then, we derive from here

$$|(\nabla_X J)X|^2|X|^{-4} = \tfrac{1}{4}\{|\omega|^2 - (\omega^2(X) + \theta^2(X))|X|^{-2}\}\,, \tag{3.12}$$

and the stated result follows from Gray's theorem [5] mentioned above.

REMARKS. – *a*) The example of the Hopf surface $S^1 \times S^3$ proves that Theorem 3.6 does not hold for $\delta = 0$ [15].

b) Formula (3.12) yields the known result [4] that a Hermitian nearly Kähler surface is Kähler, and, also, that a nearly Kähler locally conformal Kähler manifold is Kähler. Indeed, (3.12) and $(\nabla_X J)(X) = 0$ give $\omega^2(X) + \theta^2(X) = |\omega|^2|X|^2$, which, in turn, yields $|\omega| = 0$ if we sum it over $X \in \{E_i\}$, an orthonormal basis.

4. – Unitary curvature and the Kähler condition.

We shall be considering now the curvatures of the Weyl connection and of the unitary connection, and we begin by noticing the following simple result

PROPOSITION 4.1. – *A Hermitian surface is locally conformal Kähler iff it satisfies either one of the conditions*: i) $\tilde{R}(X_1, X_2, X_3, X_4) = -\tilde{R}(X_2, X_1, X_3, X_4)$; ii) *the Weyl connection is equiaffine (volume preserving).*

Proof. – Formula (2.7) proves that i) holds iff $d\omega = 0$. Furthermore, formulas (2.7) and (2.13) yield

$$\tilde{W} = -2\, d\omega\,, \tag{4.1}$$

and, since $\tilde{W} = 0$ is a classical condition for $\tilde{\nabla}$ to be volume preserving [12], the stated proposition is proven.

A more interesting result is given by

Theorem 4.2. – *If a compact connected Hermitian surface M satisfies*

$$R_{(c)}(X_1, X_2, X_3, X_4) = R_{(c)}(X_3, X_4, X_1, X_2)\,, \tag{4.2}$$

M is globally conformal Kähler. If, moreover, $R_{(c)}$ satisfies the Riemannian Bianchi identity, then M is Kähler.

Proof. – Let us define the tensor

$$\mathcal{B}(X_1, X_2, X_3, X_4) = R_{(c)}(X_1, X_2, X_3, X_4) + R_{(c)}(X_1, X_3, X_4, X_2) + R_{(c)}(X_1, X_4, X_2, X_3)\,. \tag{4.3}$$

Since $\tilde{R}$ satisfies the Riemannian Bianchi identity, we obtain from (2.9)

$$\begin{aligned} 2\mathcal{B}(X_1, X_2, X_3, X_4) = {} & g(X_1, X_2)\, d\omega(X_3, X_4) + g(X_1, X_3)\, d\omega(X_4, X_2) + \\ & g(X_1, X_4)\, d\omega(X_2, X_3) - \Omega(X_1, X_2)\, d\theta(X_3, X_4) - \Omega(X_1, X_3)\, d\theta(X_4, X_2) - \\ & - \Omega(X_1, X_4)\, d\theta(X_2, X_3)\,. \end{aligned} \tag{4.4}$$

Now let us consider local complex coordinates $\{z^\lambda, \bar{z}^\lambda\}$ $(\lambda = 1, 2)$, and put

$$\omega = \omega' + \omega''\,, \qquad \omega' = \omega_\lambda\, dz^\lambda\,, \qquad \omega'' = \bar{\omega}_\lambda\, d\bar{z}^\lambda\,, \tag{4.5}$$

whence

$$\theta = \sqrt{-1}(\omega' - \omega'')\,. \tag{4.6}$$

It is by (4.3) that the local components of $\mathcal{B}$ are zero unless they have two indices λ, μ, and two $\bar{\lambda}$, $\bar{\mu}$, and that, actually, the only «independent» component is

$$\mathcal{B}_{\lambda\bar{\mu}\sigma\bar{\tau}} = g_{\lambda\bar{\mu}}\, \partial_\sigma \bar{\omega}_\tau - g_{\lambda\bar{\tau}}\, \partial_\sigma \bar{\omega}_\mu\,. \tag{4.7}$$

Furthermore, by a classical computation [9], we can get

$$\begin{aligned} 2\{R_{(c)}(X_1, X_2, X_3, X_4) - R_{(c)}(X_3, X_4, X_1, X_2)\} = {} & \mathcal{B}(X_1, X_2, X_3, X_4) - \\ & - \mathcal{B}(X_2, X_3, X_4, X_1) - \mathcal{B}(X_3, X_4, X_1, X_2) + \mathcal{B}(X_4, X_1, X_2, X_3)\,, \end{aligned} \tag{4.8}$$

which yields

$$R_{(c)\lambda\bar{\mu}\sigma\bar{\tau}} - R_{(c)\sigma\bar{\tau}\lambda\bar{\mu}} = \tfrac{1}{2}g_{\lambda\bar{\mu}}(\partial_\sigma\bar{\omega}_\tau + \partial_{\bar{\tau}}\omega_\sigma) - \tfrac{1}{2}g_{\sigma\bar{\tau}}(\partial_\lambda\bar{\omega}_\mu + \partial_{\bar{\mu}}\omega_\lambda) - \tfrac{1}{2}g_{\lambda\bar{\tau}}(\partial_\sigma\bar{\omega}_\mu - \partial_{\bar{\mu}}\omega_\sigma) + \tfrac{1}{2}g_{\sigma\bar{\mu}}(\partial_\lambda\bar{\omega}_\tau - \partial_{\bar{\tau}}\omega_\lambda)\,. \tag{4.9}$$

Now, (4.2) implies the vanishing of (4.9). Then, by contracting in (4.9) with $g^{\bar{\tau}\lambda}$ we get

$$d'\omega'' + d''\omega' = \psi\Omega \tag{4.10}$$

for some function ψ, and where d' is the (1, 0)-type part of d, and d'' the (0,1)-type part of d. Moreover, if we apply to (4.10) the operator Λ, and use the formulas (1.4) and (1.8), we obtain $\psi = 0$ i.e.,

$$d'\omega'' + d''\omega' = 0\,. \tag{4.11}$$

On the other hand, let us contract in (4.9) $= 0$ with $g^{\bar{\mu}\lambda}$. This will provide us with a relation of the form

$$d'\omega'' - d''\omega' = \varphi\Omega\,, \tag{4.12}$$

for some scalar function φ. Together with (4.11), this implies

$$d'\omega'' = \alpha\Omega\,, \quad d''\omega' = \beta\Omega\,, \tag{4.13}$$

where $\alpha = \bar{\beta}$ is some scalar function on M, and, because of (1.9), we deduce from here

$$d'\omega' \wedge d'\omega'' = 0\,. \tag{4.14}$$

Now, if we have $d'\omega' \neq 0$, we obviously have

$$\int_M (d'\omega') \wedge (d''\omega'') > 0\,, \tag{4.15}$$

which, because of (4.14) and (4.11), yields

$$0 < \int_M (d'\omega') \wedge (d\omega'') = \int_M d[(d'\omega') \wedge \omega''] = 0\,,$$

i.e., a contradiction.

Therefore, $d'\omega' = \overline{d''\omega''} = 0$, which, together with (4.11) implies $d\omega = 0$, and, consequently, M must be locally conformal Kähler. But then, as we proved in [16; Prop. 3.3], M is, in fact, globally conformal Kähler, q.e.d.

The second part of Proposition 4.2 has the hypothesis $\mathfrak{B}_{\lambda\bar{\mu}\sigma\bar{\tau}} = 0$ i.e., in view of (4.7)

$$g_{\lambda\bar{\mu}}\, \partial_\sigma \bar{\omega}_{\tau} - g_{\lambda\bar{\tau}}\, \partial_\sigma \bar{\omega}_{\mu} = 0 \,. \tag{4.16}$$

By contracting with $g^{\bar{\mu}\lambda}$, this yields $d'\omega'' = 0$, whence also $d''\omega' = 0$.

Now, since the Bianchi identity implies (4.2), as in classical Riemannian geometry [9], we know that M is locally conformal Kähler. But we proved in Theorem 2.1 of [16] that a compact locally conformal Kähler manifold with $d''\omega' = 0$ is a Kähler manifold q.e.d.

Corollary 4.3. – *A compact Hermitian surface with a vanishing curvature $R_{(c)}$ is a flat Kähler surface.*

Remark. – As a matter of fact, formula (4.7) yields a stronger result, namely that a compact Hermitian surface M which satisfies the condition

$$R_{(c)}{}^{\bar{\mu}}{}_{\bar{\mu}\sigma\bar{\tau}} = R_{(c)}{}^{\bar{\mu}}{}_{\bar{\tau}\sigma\bar{\mu}} \tag{4.17}$$

is a Kähler surface. Indeed, (4.17) is equivalent to

$$g^{\bar{\mu}\lambda}\, \mathfrak{B}_{\lambda\bar{\mu}\sigma\bar{\tau}} = 0 \,,$$

which means $d'\omega'' = 0$, hence $d''\omega' = 0$ as well. But, by Theorem 1 of [10, p. 754] every holomorphic 1-form of a compact surface is closed, whence $d\omega' = 0$. Then, we easily deduce $d\omega = 0$, and we see that M is locally conformal Kähler with $d''\omega' = 0$, i.e. M is Kähler as in [16, Theorem 2.1], q.e.d.

5. – Curvature and plurigenera.

In this section, we shall consider some relations between the curvature, the Chern numbers, and the plurigenera of a compact Hermitian surface M.

Theorem 5.1. – *Assume that M is a compact Hermitian surface with non-negative but not identical zero unitary holomorphic bisectional curvature. Then the first Chern class $c_1(M)$ is nonnegative, M has vanishing plurigenera and no exceptional curve.*

Proof. – The stated curvature hypothesis means

$$R_{(c)}(X, JX, Y, JY) \geqslant 0 \quad \text{and} \ \not\equiv 0 \,, \tag{5.1}$$

whence, by (2.15), $C_{(c)}(X, Y)$ is a nonnegative (1, 1)-form. Hence $c_1(M) \geqslant 0$, and M enters into the class of surfaces classified by Theorem 3 of [18, p. 224].

Moreover, from (5.1) and (2.20) we get that $k_M \geqslant 0$, $k_M \not\equiv 0$, where k_M is the Ricci scalar of the contravariant canonical bundle $K^*(M)$. Since the Ricci scalar of $(K^*(M))^m$ is mk_M, M has vanishing plurigenera by [2, Corollary 2, p. 124].

Let us also note that $c_1(M) \geqslant 0$ implies $c_1^2(M) \geqslant 0$. More precisely, if we set $\mathcal{C}_{(c)}(X, Y) = C_{(c)}(X, JY)$, we get a positive semidefinite Hermitian form. Then, there is a basis $\{E_\alpha, E_{\alpha^*}\}$ which diagonalizes $\mathcal{C}_{(c)}$, and with respect to which we have [6, Theorem 3.1, p. 467]

$$C_{(c)}^2(E_1, E_{1^*}, E_2, E_{2^*}) = 2C_{(c)}(E_1, E_{1^*})C_{(c)}(E_2, E_{2^*}) \geqslant 0 . \tag{5.2}$$

Finally, the non-existence of exceptional curves can be proven exactly as in the Kählerian case [7, Lemma 6, p. 495], with the only difference that the Gauss equation (7) of [7, p. 494] should be replaced as follows: Let C be a regular curve in M, and X, Y tangent vector fields of C. Then we have a decomposition

$$\nabla_{(c)X} Y = \nabla'_X Y + b(X, Y)$$

into a tangent and a normal part with respect to C, and it easy to see that ∇' is nothing else than the Levi-Civita connection of C. This decomposition yields

$$R_{(c)}(X, JX, X, JX) = K_C + 2|b(X, X)|^2 ,$$

where K_C is the Gaussian curvature of C, and this is the formula which we shall use instead of [7, (7), p. 494].

Theorem 5.1 is thereby completely proven.

REMARKS. – 1) It follows that the surfaces which satisfy the hypotheses of Theorem 5.1 should be divided into two classes:

a) Algebraic surfaces. These surfaces will be found among the algebraic surfaces without exceptional curves enumerated by Theorem 3, p. 224 of [18]. In particular, here we have the compact Hermitian surfaces with strictly positive holomorphic bisectional unitary curvature. Indeed, in this case we have $c_1^2(M) > 0$, and M is an algebraic surface by [10, Theorem 9, p. 757]. Moreover, in this case $c_1(M) > 0$, and, since M has no exceptional curve, we get from Theorem 4 of [18, p. 225] that M is biholomorphically equivalent either to CP^2 or to $CP^1 \times CP^1$. Of course, CP^2 satisfies the hypotheses of Theorem 5.1, but it is an open question to know whether $CP^1 \times CP^1$ admits a Hermitian metric of positive holomorphic bisectional curvature. Note that, if it admits one, this cannot be a Kähler metric since $CP^1 \times CP^1$ is not equivalent to CP^2. Note also the parallelism between this open question and the famous unsolved Hopf problem as to whether $S^2 \times S^2$ carries a Riemann metric of positive sectional curvature.

b) Nonalgebraic surfaces. Then the surface belongs to the class VII of the Kodaira classification [10], and it has necessarilly $c_1^2(M) = 0$, $b_1(M) = 1$, $\chi(M) = 0$ [18,

p. 221]. For example, the Hopf surface $S^1 \times S^3$ with its natural locally conformal Kähler metric [3, 13] can be seen to have a nonnegative and nonidentical zero holomorphic bisectional unitary curvature. (This follows easily from the expression of its unitary curvature given in [3, p. 167].)

2) If M is a locally conformal Kähler manifold of an arbitrary dimension, $\tilde{C}$ of (2.18) belongs to the conformal local Kähler metrics and, hence, it has the type $(1, 1)$. Then, the same proof of [6] for Formula (5.2) can be applied to $\tilde{C}$, and we deduce as in [6]:

Proposition 5.2. – *If M is a compact m-dimensional locally conformal Kähler manifold, and if its local conformal Kähler metrics have nonnegative (nonpositive) holomorphic bisectional curvature, the Chern number $c_1^n(M)$, $((-1)^n c_1^n(M))$ is nonnegative.*

Next, in order to relate plurigenera and Riemannian curvature, we shall prove

Theorem 5.3. – *The relation*

$$k_M = \tfrac{1}{2} r + \tfrac{1}{4} |\omega|^2 - \tfrac{1}{2} \delta\omega \tag{5.3}$$

holds good on any surface M.

Proof. – Formula (2.11) yields for the basic vectors

$$\begin{aligned} R_{(c)}(E_\alpha, E_{\alpha^*}, E_\beta, E_{\beta^*}) = R(E_\alpha, E_{\alpha^*}, E_\beta, E_{\beta^*}) + \\ \tfrac{1}{2} \delta_{\alpha\beta} \left[L(E_\beta, E_\alpha) + L(E_{\beta^*}, E_{\alpha^*}) - \frac{|\omega|^2}{2} \right] + \tfrac{1}{2} d\theta(E_\beta, E_{\beta^*}). \end{aligned} \tag{5.4}$$

Using this result in (2.20), we obtain

$$\begin{aligned} k_M = \sum_{\alpha,\beta} R(E_\alpha, E_{\alpha^*}, E_\beta, E_{\beta^*}) + \sum_\alpha d\theta(E_\alpha, E_{\alpha^*}) + \\ + \frac{1}{2} \sum_\alpha [L(E_\alpha, E_\alpha) + L(E_{\alpha^*}, E_{\alpha^*})] - \frac{|\omega|^2}{2}. \end{aligned} \tag{5.5}$$

The first term here is $r^*/2$ in view of (3.7).
The second term can be computed as follows. Since $E_{\alpha^*} = JE_\alpha$, the vectors

$$F_\alpha = \frac{1}{\sqrt{2}} (E_\alpha - \sqrt{-1}\, E_{\alpha^*}), \qquad F_{\bar{\alpha}} = \frac{1}{\sqrt{2}} (E_\alpha + \sqrt{-1}\, E_{\alpha^*})$$

define a basis of elements of the type $(1, 0)$ and $(0, 1)$, and we have

$$\sum_\alpha d\theta(F_\alpha, F_{\bar{\alpha}}) = \sqrt{-1} \sum_\alpha d\theta(E_\alpha, E_{\alpha^*}).$$

Now, if we go over to the basis $\partial/\partial z^\alpha$, $\partial/\partial \bar{z}^\alpha$, we get

$$\sum_\alpha d\theta(E_\alpha, E_{\alpha^*}) = -\sqrt{-1}\, g^{\mu\bar{\lambda}}(d\theta)_{\lambda\bar{\mu}} = -\Lambda\, d\theta \tag{5.6}$$

where, as usual $\Lambda = i(\Omega)$.

At this point, we shall use a commutation formula, which we established for locally conformal Kähler manifolds in [14], and which is true in the same way for surfaces as well:

$$[\Lambda, d] = \delta^C + (n-p)\varepsilon^C + e\Lambda\,. \tag{5.7}$$

Here $e(\cdot) = \omega\wedge\cdot$, $\varepsilon = i(\omega)$, C is the usual operator of Hermitian geometry [17], $n = \dim_C M$, and $p = \deg\alpha$, where α is the form on which (5.7) is acting. (Note also a change of the sign with respect to [14], explained by the present sign convention for Ω.)

By applying (5.7) to (5.6) we get

$$\sum_\alpha d\theta(E_\alpha, E_{\alpha^*}) = \delta\,\omega + |\omega|^2\,. \tag{5.8}$$

Finally, the third term of (5.5) will be computed by (2.8), which yields

$$\sum_\alpha [L(E_\alpha, E_\alpha) + L(E_{\alpha^*}, E_{\alpha^*})] = \tfrac{1}{2}|\omega|^2 - \delta\omega\,. \tag{5.9}$$

Now, Theorem 5.3 follows from the formulas (5.5), (5.8), (5.9), and (3.2).

Formula (5.3) yields a generalization of a theorem proved by YAU [18] for Kähler surfaces:

THEOREM 5.4. – *Let M be a compact complex surface which admits a distinguished non-Kähler Hermitian metric with a non-negative integral Riemannian scalar curvature. Then the surface M has vanishing plurigenera.*

PROOF. – Recall from Section 1 that the Hermitian metric g is called *distinguished* if $\delta\omega = 0$. If this is the case, *the eccentricity function* of g is $f_0 = 1$ [2], and we get from (5.3) that the *fundamental constant* [2] of the contravariant canonical bundle is

$$\int_M k_M = \frac{1}{2}\int_M \left(r + \frac{|\omega|^2}{2}\right) > 0\,.$$

Then, the stated result follows from Gauduchon's Plurigenera Theorem [2, p. 136].

Let us note that a Kähler metric has $\omega = 0$ and is therefore distinguished. In this case, YAU [18] proved that if $\int r \geqslant 0$ then either M has vanishing plurigenera or the integral Chern class $c_1(M)$ is a torsion class.

COROLLARY 5.5. – *A compact Hermitian surface whose Lee vector field B is a non-zero Killing vector field, and which has a non-negative integral Riemannian scalar curvature is a surface with vanishing plurigenera.*

PROOF. – B Killing implies $\delta\omega = 0$, and the result follows from Theorem 5.4.
It is also worth noting the following more simple consequence of Theorem 5.3:

THEOREM 5.6. – *If a compact non-Kähler Hermitian surface M satisfies the condition $r + r^* \geqslant 0$, then M has vanishing plurigenera.*

PROOF. – By combining the formulas (5.3) and (3.2), we obtain

$$k_M = \tfrac{1}{4}(r + r^*) + \tfrac{1}{2}|\omega|^2 , \tag{5.10}$$

whence the result follows from [2, Corollary 2, p. 124].

The above mentioned results are relating Riemannian curvature and plurigenera. But it is also possible to relate it with $c_1^2(M)$. Indeed, by (5.2) $c_1^2(M)$ is given by the integration of a sum of products of unitary bisectional curvatures like those defined by (5.4).

Now, for $\alpha \neq \beta$, (5.4) gives

$$R_{(c)}(E_\alpha, E_{\alpha^*}, E_\beta, E_{\beta^*}) = R(E_\alpha, E_{\alpha^*}, E_\beta, E_{\beta^*}) + \tfrac{1}{2}\, d\theta(E_\beta, E_{\beta^*}) , \tag{5.11}$$

and for $\alpha = \beta$, (5.4) gives

$$\begin{aligned} R_{(c)}(E_\alpha, E_{\alpha^*}, E_\alpha, E_{\alpha^*}) &= R(E_\alpha, E_{\alpha^*}, E_\alpha, E_{\alpha^*}) + \\ &+ \frac{1}{2}\left\{L(E_\alpha, E_\alpha) + L(E_{\alpha^*}, E_{\alpha^*}) + d\theta(E_\alpha, E_{\alpha^*}) - \frac{|\omega|^2}{2}\right\}. \end{aligned} \tag{5.12}$$

Here, a term of the form $L(X, X) + L(JX, JX)$ can be computed as in [15]. Namely, if we set

$$\lambda(X, Y) = L(X, JY) - L(JX, Y) ,$$

if we explicitate this by (2.8), and use (3.11), we obtain

$$\lambda(X, Y) = |\omega|^2 \Omega(X, Y) - \tfrac{1}{2}(\omega \wedge \theta)(X, Y) + d\theta(X, Y) - d\omega(JX, Y) .$$

This yields

$$\begin{aligned} L(X, X) + L(JX, JX) &= -\lambda(X, JX) = \\ &= |\omega|^2 |X|^2 - \tfrac{1}{2}\omega^2(X) - \tfrac{1}{2}\theta^2(X) - d\theta(X, JX) , \end{aligned} \tag{5.13}$$

and from (5.12) we obtain

$$R_{(c)}(E_\alpha, E_{\alpha^*}, E_\alpha, E_{\alpha^*}) = R(E_\alpha, E_{\alpha^*}, E_\alpha, E_{\alpha^*}) + \tfrac{1}{4}[|\omega|^2 - \omega^2(E_\alpha) - \omega^2(E_{\alpha^*})] . \tag{5.14}$$

Furthermore, from (5.11), (5.13) and (3.9) we get

$$R_{(c)}(E_\alpha, E_{\alpha^*}, E_\beta, E_{\beta^*}) = R(E_\alpha, E_\beta, E_\alpha, E_\beta) + R(E_\alpha, E_{\beta^*}, E_\alpha, E_{\beta^*}) + d\theta(E_\beta, E_{\beta^*}) + L(E_\alpha, E_\alpha) - |\omega|^2 + \tfrac{1}{4}\omega^2(E_\beta) + \tfrac{1}{4}\omega^2(E_{\beta^*}) \quad (\alpha \neq \beta)\,. \tag{5.15}$$

Now, (5.14) and (5.15) give us the relation

$$2\pi C_{(c)}(E_\beta, E_{\beta^*}) = R(E_\beta, E_{\beta^*}, E_\beta, E_{\beta^*}) + R(E_\alpha, E_\beta, E_\alpha, E_\beta) + R(E_\alpha, E_{\beta^*}, E_\alpha, E_{\beta^*}) - \tfrac{3}{4}|\omega|^2 + d\theta(E_\beta, E_{\beta^*}) + (\nabla_{E_\alpha}\omega)(E_\alpha) + \tfrac{1}{2}\omega^2(E_\alpha)\,, \tag{5.16}$$

where $\alpha \neq \beta$, and $C_{(c)}(E_\beta, E_{\beta^*})$ are the factors of the formula (5.2).
As a consequence of these computation we obtain

PROPOSITION 5.7. – *Let M be a complete Hermitian surface which satisfies the following conditions*: i) $d\theta\ (X, JX) \geqslant 0$ *and* $(\nabla_X\omega)(X) \geqslant 0$ *for every tangent vector* X; ii) *the sectional curvature of M is* $\geqslant |\omega|^2/4 + \delta$ *for some* $\delta > 0$. *Then M is a compact simply connected algebraic surface of vanishing plurigenera and with* $c_1^2(M) > 0$.

PROOF. – Indeed, M is compact and simply connected in view of Theorem 3.6, and it has $k_M > 0$, $c_1^2(M) > 0$ because of the formulas (2.20), (5.2), (5.16) and the hypotheses i) and ii). All the conclusions follow then easily. (In particular, M is algebraic by Theorem 9, p. 758 of [10].)

REMARKS. – 1) One can see that hypothesis i) holds, in particular, if the Lee field B is Killing and analytic. Indeed, B Killing implies $(\nabla_X\omega)(X) = 0$. On the other hand, B Killing and analytic implies the vanishing of the Lie derivative $L_B\Omega = 0$, and, since $L_B = i(B)d + di(B)$, this means

$$|\omega|^2\Omega - \omega\wedge\theta + d\theta = 0\,,$$

whence $d\theta(X, JX) = |\omega|^2|X|^2 - \omega^2(X) - \omega^2(JX) \geqslant 0$.

2) Let us also note that the metric of M of Proposition 5.7 is necessarily distinguished since $(\nabla_X\omega)(X) \geqslant 0$ yields $\delta\omega \geqslant 0$, which implies $\delta\omega = 0$ by an integration over M.

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