

Some Difference Double Sequence Spaces Defined By Orlicz Function

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ABSTRACT. In this article we introduce some difference sequence spaces defined by Orlicz function and study different properties of these spaces like completeness, solidity, symmetricity etc. We establish some inclusion results among them.

1. Introduction

Throughout, a double sequence is denoted by $A = \langle a_{ij} \rangle$. A double sequence is a double infinite array of elements $a_{ij} \in R$ for all $i, j \in N$ and ${}_2w$ will denote the class of all double sequences. The initial works on double sequences is found in Bromwich [2]. Later on it was studied by Hardy [6], Moricz [12], Moricz and Rhoades [13], Tripathy ([16], [17]), Basarir and Sonalcan [1] and many others. Hardy [6] introduced the notion of regular convergence for double sequences. The concept of paranormed sequences was studied by Nakano [14] and Simmons [15] at the initial stage. Later on it was studied by many others. The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [8] as follows:

$$Z(\Delta) = \{(x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞ , where $\Delta x_k = x_k - x_{k+1}$, for all $k \in N$.

Received August 14, 2006.

2000 Mathematics Subject Classification: 40A05, 40B05, 46E30.

Key words and phrases: Orlicz function, difference space, completeness, solid space, symmetric space etc.

The above spaces are Banach spaces, normed by

$$\|(x_k)\| = |x_1| + \sup_{k \geq 1} |\Delta x_k|.$$

The notion was further investigated by many workers on sequence spaces.

2. Definitions and preliminaries

An Orlicz function M is a mapping $M : [0, \infty) \rightarrow [0, \infty)$ such that it is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \rightarrow \infty$, as $x \rightarrow \infty$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to construct the sequence space

$$\ell^M = \left\{ (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

which is a Banach space normed by

$$\|(x_k)\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

The space ℓ^M is closely related to the space ℓ^p , which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

An Orlicz function M is said to satisfy the Δ_2 -condition for all values of u , if there exists a constant $K > 0$, such that $M(2u) \leq K(Mu)$, $u \geq 0$.

Remark 1. Let $0 < \lambda < 1$, then $M(\lambda x) \leq \lambda M(x)$, for all $x \geq 0$.

Let $p = (p_k)$ be a positive sequence of real numbers. If $0 < p_k \leq \sup p_k = H$ and $D = \max(1, 2^{H-1})$, then for $a_k, b_k \in C$ for all $k \in N$, we have

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}.$$

Definition 2.1. A double sequence space E is said to be *solid* if $\langle \alpha_{ij} a_{ij} \rangle \in E$ whenever $\langle a_{ij} \rangle \in E$, for all double sequences $\langle \alpha_{ij} \rangle$ of scalars with $|\alpha_{ij}| \leq 1$, for all $i, j \in N$.

Definition 2.2. Let $K = \{(n_i, k_j) : i, j \in N; n_1 < n_2 < n_3 < \dots \text{ and } k_1 < k_2 < k_3 < \dots\} \subseteq N \times N$ and E be a double sequence space. A *K-step space* of E is a sequence space

$$\lambda_K^E = \{ \langle a_{n_i k_j} \rangle \in E : \langle a_{nk} \rangle \in E \}.$$

A *canonical pre-image* of a sequence $\langle a_{n_i k_j} \rangle \in E$ is a sequence $\langle b_{nk} \rangle \in E$ defined as follows:

$$b_{nk} = \begin{cases} a_{nk}, & \text{if } (n, k) \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A *canonical pre-image of a step space* λ_K^E is a set of canonical pre-images of all elements in λ_K^E .

Definition 2.3. A double sequence space E is said to be *monotone* if it contains the canonical pre-images of all its step spaces.

Remark 2. From the above notions, it follows that “If a sequence space E is solid then E is monotone”.

Definition 2.4. A double sequence space E is said to be *symmetric* if $\langle a_{ij} \rangle \in E$ implies $\langle a_{\pi(i)\pi(j)} \rangle \in E$, where π is a permutation of N .

Let M be an Orlicz function and $p = \langle p_{ij} \rangle$ be a double sequence of strictly positive real numbers. We introduce the following sequence spaces.

$${}_2W(M, \Delta, p) = \left\{ \langle a_{ij} \rangle \in {}_2w : \lim_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|\Delta a_{ij} - L|}{\rho} \right) \right)^{p_{ij}} = 0, \right. \\ \left. \text{for some } \rho > 0 \text{ and } L \right\}.$$

$${}_2W_0(M, \Delta, p) = \left\{ \langle a_{ij} \rangle \in {}_2w : \lim_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|\Delta a_{ij}|}{\rho} \right) \right)^{p_{ij}} = 0, \right. \\ \left. \text{for some } \rho > 0 \right\}.$$

$${}_2W_\infty(M, \Delta, p) = \left\{ \langle a_{ij} \rangle \in {}_2w : \sup_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|\Delta a_{ij}|}{\rho} \right) \right)^{p_{ij}} < \infty, \right. \\ \left. \text{for some } \rho > 0 \right\}.$$

3. Main results

Theorem 3.1. Let $p = \langle p_{ij} \rangle$ be bounded. The classes of sequences ${}_2W(M, \Delta, p)$, ${}_2W_0(M, \Delta, p)$ and ${}_2W_\infty(M, \Delta, p)$ are linear spaces.

Theorem 3.2. The sequence spaces ${}_2W(M, \Delta, p)$, ${}_2W_0(M, \Delta, p)$ and ${}_2W_\infty(M, \Delta, p)$ are paranormed sequence spaces, paranormed by

$$g(\langle a_{ij} \rangle) = \sup_i |a_{i1}| + \sup_j |a_{1j}| + \inf \left\{ \rho^{\frac{p_{ij}}{J}} > 0 : \sup_{i,j} \left(M \left(\frac{|\Delta a_{ij}|}{\rho} \right) \right) \leq 1 \right\},$$

where $J = \max(1, H)$, $H = \sup_{i,j} p_{ij}$.

Proof. Clearly $g(0) = 0$, $g(-\langle a_{ij} \rangle) = g(\langle a_{ij} \rangle)$. Let $\langle a_{ij} \rangle, \langle b_{ij} \rangle \in {}_2W_\infty(M, \Delta, p)$.

Then there exist some $\rho_1, \rho_2 > 0$ such that

$$\sup_{i,j} \left(M \left(\frac{|\Delta a_{ij}|}{\rho_1} \right) \right) \leq 1 \quad \text{and} \quad \sup_{i,j} \left(M \left(\frac{|\Delta b_{ij}|}{\rho_2} \right) \right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then we have,

$$\begin{aligned} & \sup_{i,j} \left(M \left(\frac{|\Delta a_{ij} + \Delta b_{ij}|}{\rho} \right) \right) \\ & \leq \frac{\rho_1}{\rho_1 + \rho_2} \sup_{i,j} \left(M \left(\frac{|\Delta a_{ij}|}{\rho_1} \right) \right) + \frac{\rho_2}{\rho_1 + \rho_2} \sup_{i,j} \left(M \left(\frac{|\Delta b_{ij}|}{\rho_2} \right) \right) \leq 1. \end{aligned}$$

Now

$$\begin{aligned} g(\langle a_{ij} \rangle + \langle b_{ij} \rangle) &= |a_{i1} + b_{i1}| + |a_{1j} + b_{1j}| \\ &+ \inf \left\{ (\rho_1 + \rho_2)^{\frac{p_{ij}}{J}} > 0 : \sup_{i,j} \left(M \left(\frac{|\Delta a_{ij} + \Delta b_{ij}|}{\rho_1 + \rho_2} \right) \right) \leq 1 \right\} \\ &\leq |a_{i1}| + |b_{i1}| + \inf \left\{ \rho_1^{\frac{p_{ij}}{J}} > 0 : \sup_{i,j} \left(M \left(\frac{|\Delta a_{ij}|}{\rho_1} \right) \right) \leq 1 \right\} \\ &+ |a_{1j}| + |b_{1j}| + \inf \left\{ \rho_2^{\frac{p_{ij}}{J}} > 0 : \sup_{i,j} \left(M \left(\frac{|\Delta b_{ij}|}{\rho_2} \right) \right) \leq 1 \right\} \\ &= g(\langle a_{ij} \rangle) + g(\langle b_{ij} \rangle). \end{aligned}$$

Let $\eta \in C$, then the continuity of the product follows from the following equality.

$$\begin{aligned} & g(\eta \langle a_{ij} \rangle) \\ &= |\eta a_{i1}| + |\eta a_{i1}| + \inf \left\{ \rho^{\frac{p_{ij}}{J}} : \sup_{i,j} \left(M \left(\frac{|\Delta \eta a_{ij}|}{\rho} \right) \right) \leq 1, \rho > 0 \right\} \\ &= |\eta| |a_{i1}| + |\eta| |a_{i1}| + \inf \left\{ (|\eta| r)^{\frac{p_{ij}}{J}} : \sup_{i,j} \left(M \left(\frac{|\Delta \eta a_{ij}|}{r} \right) \right) \leq 1, r > 0 \right\}, \end{aligned}$$

where $\frac{1}{r} = \frac{|\eta|}{\rho}$. □

Theorem 3.3. *The sequence spaces ${}_2W(M, \Delta, p)$, ${}_2W_0(M, \Delta, p)$ and ${}_2W_\infty(M, \Delta, p)$ are complete paranormed spaces, paranormed by g .*

Proof. Let $\langle a_{ij}^s \rangle$ be a Cauchy sequence in ${}_2W_\infty(M, \Delta, p)$. Then $g(\langle a_{ij}^s - a_{ij}^t \rangle) \rightarrow 0$ as $s, t \rightarrow \infty$.

For a given $\epsilon > 0$, choose $r > 0$ and $x_0 > 0$ be such that $\frac{\epsilon}{rx_0} > 0$ and $M\left(\frac{rx_0}{2}\right) \geq 1$.

Now $g(\langle a_{ij}^s - a_{ij}^t \rangle) \rightarrow 0$ as $s, t \rightarrow \infty$ implies that there exists $m_0 \in N$ such that

$$g(\langle a_{ij}^s - a_{ij}^t \rangle) < \frac{\epsilon}{rx_0} \quad \text{for all } s, t \geq m_0.$$

$$\Rightarrow \sup_i |a_{i1}^s - a_{i1}^t| + \sup_j |a_{1j}^s - a_{1j}^t| +$$

$$\inf \left\{ \frac{p_{ij}}{\rho^J} : \sup_{i,j} \left(M \left(\frac{|\Delta a_{ij}^s - \Delta a_{ij}^t|}{\rho} \right) \right) \leq 1 \right\} < \frac{\epsilon}{rx_0} \quad \dots \quad (1)$$

This shows that $\langle a_{i1}^s \rangle, \langle a_{1j}^s \rangle$ are Cauchy sequences of real numbers. As the set of real numbers is complete so there exists real numbers a_{i1}, a_{1j} such that

$$\lim_{s \rightarrow \infty} a_{i1}^s = a_{i1}, \quad \lim_{s \rightarrow \infty} a_{1j}^s = a_{1j}.$$

Now from (1) we have,

$$\begin{aligned} & \left(M \left(\frac{|\Delta a_{ij}^s - \Delta a_{ij}^t|}{\rho} \right) \right) \leq 1 \\ \Rightarrow & \sup_{i,j} \left(M \left(\frac{|\Delta a_{ij}^s - \Delta a_{ij}^t|}{\rho} \right) \right) \leq 1 \leq M \left(\frac{rx_0}{2} \right) \\ \Rightarrow & \frac{|\Delta a_{ij}^s - \Delta a_{ij}^t|}{g(\langle a_{ij}^s - a_{ij}^t \rangle)} \leq \frac{rx_0}{2} \\ \Rightarrow & |\Delta a_{ij}^s - \Delta a_{ij}^t| < \frac{rx_0}{2} \cdot \frac{\epsilon}{rx_0} = \frac{\epsilon}{2}. \end{aligned}$$

This implies $\langle \Delta a_{ij}^s \rangle$ is a Cauchy sequence of real numbers. Let $\lim_{s \rightarrow \infty} \Delta a_{ij}^s = y_{ij}$ for all $i, j \in N$. Now $\Delta a_{11}^s = a_{11}^s - a_{12}^s - a_{21}^s + a_{22}^s$ and so

$$\lim_{s \rightarrow \infty} a_{22}^s = \lim_{s \rightarrow \infty} (\Delta a_{11}^s - a_{11}^s + a_{12}^s + a_{21}^s) = y_{11} - a_{11} + a_{12} + a_{21}.$$

Hence $\lim_{s \rightarrow \infty} a_{22}^s$ exists. Proceeding in this way we conclude that $\lim_{s \rightarrow \infty} a_{ij}^s$ exists and thus $\lim_{s \rightarrow \infty} \Delta a_{ij}^s$ exists. Using continuity of M , we have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \sup_{i,j} \left(M \left(\frac{|\Delta a_{ij}^s - \Delta a_{ij}^t|}{\rho} \right) \right) \leq 1 \\ \Rightarrow & \limsup_{t \rightarrow \infty} \sup_{i,j} \left(M \left(\frac{|\Delta a_{ij}^s - \Delta a_{ij}^t|}{\rho} \right) \right) \leq 1. \end{aligned}$$

Let $s \geq m_0$, then taking infimum of such ρ 's we have $g(\langle a_{ij}^s - a_{ij} \rangle) < \epsilon$.

Thus $\langle a_{ij}^s - a_{ij} \rangle \in {}_2W_\infty(M, \Delta, p)$. By linearity of the space ${}_2W_\infty(M, \Delta, p)$ we have $\langle a_{ij} \rangle \in {}_2W_\infty(M, \Delta, p)$. Hence ${}_2W_\infty(M, \Delta, p)$ is complete. \square

Proposition 3.4.

$$(i) \quad {}_2W(M, \Delta, p) \subset {}_2W_\infty(M, \Delta, p),$$

$$(ii) \quad {}_2W_0(M, \Delta, p) \subset {}_2W_\infty(M, \Delta, p).$$

The inclusions are strict.

Proof. The proof is easy, so omitted.

Theorem 3.5. *The spaces ${}_2W(M, \Delta, p)$ and ${}_2W_0(M, \Delta, p)$ are nowhere dense subset of ${}_2W_\infty(M, \Delta, p)$.*

Proof. The proof is obvious in view of Theorem 3.3 and Theorem 3.4. \square

Theorem 3.6. *If $\sup \frac{p_{ij}}{p_{uv}} < \infty$, for all $i \geq u, j \geq v$, then ${}_2W(M, p) \subset {}_2W_0(M, \Delta, p)$.*

The inclusion is strict.

Proof. Let $\langle a_{ij} \rangle \in {}_2W(M, p)$. Then

$$\lim_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|a_{ij} - L|}{\rho} \right) \right)^{p_{ij}} = 0, \text{ for some } \rho > 0 \text{ and } L \dots \quad (2)$$

Since $\sup \frac{p_{ij}}{p_{uv}} < \infty$, so there exists $K > 0$ such that $p_{ij} < K \cdot p_{uv}$ for all $i \geq u, j \geq v$.

Thus from (2) we have,

$$\begin{aligned} \lim_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|a_{ij} - L|}{\rho} \right) \right)^{p_{i,j+1}} &= 0, \\ \lim_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|a_{ij} - L|}{\rho} \right) \right)^{p_{i+1,j}} &= 0, \\ \text{and } \lim_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|a_{ij} - L|}{\rho} \right) \right)^{p_{i+1,j+1}} &= 0, \end{aligned}$$

Now for $|\Delta a_{ij}| = |a_{ij} - a_{i,j+1} - a_{i+1,j} + a_{i+1,j+1} + L - L + L - L|$ we have,

$$\begin{aligned} &\lim_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|\Delta a_{ij}|}{\rho} \right) \right)^{p_{i,j}} \\ &\leq \lim_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|a_{ij} - L|}{\rho} + \frac{|a_{i+1,j} - L|}{\rho} + \frac{|a_{i,j+1} - L|}{\rho} \right. \right. \\ &\quad \left. \left. + \frac{|a_{i+1,j+1} - L|}{\rho} \right) \right)^{p_{i,j}} \end{aligned}$$

$$\begin{aligned} &\leq D^2 \cdot \lim_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left\{ \left(M \left(\frac{|a_{ij} - L|}{\rho} \right) \right)^{p_{ij}} + \left(M \left(\frac{|a_{i+1,j} - L|}{\rho} \right) \right)^{p_{ij}} \right. \\ &\quad \left. + \left(M \left(\frac{|a_{i,j+1} - L|}{\rho} \right) \right)^{p_{ij}} + \left(M \left(\frac{|a_{i+1,j+1} - L|}{\rho} \right) \right)^{p_{ij}} \right\} \\ &\leq D^2 \cdot \lim_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left\{ \left(M \left(\frac{|a_{ij} - L|}{\rho} \right) \right)^{p_{ij}} + \left(M \left(\frac{|a_{i+1,j} - L|}{\rho} \right) \right)^{p_{i+1,j}} \right. \\ &\quad \left. + \left(M \left(\frac{|a_{i,j+1} - L|}{\rho} \right) \right)^{p_{i,j+1}} + \left(M \left(\frac{|a_{i+1,j+1} - L|}{\rho} \right) \right)^{p_{i+1,j+1}} \right\} = 0. \end{aligned}$$

Thus $\langle a_{ij} \rangle \in {}_2W_0(M, \Delta, p)$ and hence ${}_2W(M, p) \subset {}_2W_0(M, \Delta, p)$. □

The inclusion is strict follows from the following example.

Example 3.1. Let $M(x) = x^p$, $p_{ij} = 1$ for all i odd and for all $j \in N$ and $p_{ij} = 2$ otherwise.

Consider the sequence $\langle a_{ij} \rangle$ defined by $a_{ij} = i + j$ for all $i, j \in N$. We have $\Delta a_{ij} = 0$ for all $i, j \in N$. Hence $\langle a_{ij} \rangle \in {}_2W(M, \Delta, p)$ but $\langle a_{ij} \rangle \notin {}_2W(M, p)$.

Theorem 3.7.

- (i) If $0 < \inf p_{ij} \leq p_{ij} < 1$, then ${}_2W(M, \Delta, p) \subseteq {}_2W(M, \Delta)$.
- (ii) If $1 < p_{ij} \leq \sup p_{ij} < \infty$, then ${}_2W(M, \Delta) \subseteq {}_2W(M, \Delta, p)$.

Proof. The first part of the result follows from the inequality

$$\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|\Delta a_{ij} - L|}{\rho} \right) \right) \leq \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|\Delta a_{ij} - L|}{\rho} \right) \right)^{p_{i,j}},$$

and the second part of the result follows from the inequality

$$\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|\Delta a_{ij} - L|}{\rho} \right) \right)^{p_{i,j}} \leq \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|\Delta a_{ij} - L|}{\rho} \right) \right).$$

□

Theorem 3.8. Let M_1 and M_2 be two Orlicz functions. Then

$${}_2W(M_1, \Delta, p) \cap {}_2W(M_2, \Delta, q) \subseteq {}_2W(M_1 + M_2, \Delta, q).$$

Proof. Let $\langle a_{ij} \rangle \in {}_2W(M_1, \Delta, p) \cap {}_2W(M_2, \Delta, q)$. Then

$$\lim_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M_1 \left(\frac{|\Delta a_{ij} - L|}{\rho_1} \right) \right)^{p_{i,j}} = 0, \text{ for some } \rho_1 > 0$$

$$\lim_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M_2 \left(\frac{|\Delta a_{ij} - L|}{\rho_2} \right) \right)^{p_{i,j}} = 0, \text{ for some } \rho_2 > 0.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The result follows from the following inequality.

$$\begin{aligned} & \sum_{i=1}^m \sum_{j=1}^n \left((M_1 + M_2) \left(\frac{|\Delta a_{ij} - L|}{\rho} \right) \right)^{p_{i,j}} \\ & \leq D \left\{ \sum_{i=1}^m \sum_{j=1}^n \left(M_1 \left(\frac{|\Delta a_{ij} - L|}{\rho_1} \right) \right)^{p_{i,j}} + \sum_{i=1}^m \sum_{j=1}^n \left(M_2 \left(\frac{|\Delta a_{ij} - L|}{\rho_2} \right) \right)^{p_{i,j}} \right\}. \end{aligned}$$

□

Theorem 3.9. *The sequence space ${}_2W_\infty(M, \Delta, p)$ is solid and hence monotone.*

Proof. Let $\langle a_{ij} \rangle \in {}_2W_\infty(M, \Delta, p)$ and $\langle \alpha_{ij} \rangle$ be a scalar sequence such that $|\alpha_{ij}| \leq 1$ for all $i, j \in N$. Now

$$\begin{aligned} & M \left(\frac{|\alpha_{ij} \Delta a_{ij}|}{\rho} \right) \leq M \left(\frac{|\Delta a_{ij}|}{\rho} \right) \\ \Rightarrow & \left(M \left(\frac{|\alpha_{ij} \Delta a_{ij}|}{\rho} \right) \right)^{p_{ij}} \leq \left(M \left(\frac{|\Delta a_{ij}|}{\rho} \right) \right)^{p_{ij}} \\ \Rightarrow & \sup_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|\alpha_{ij} \Delta a_{ij}|}{\rho} \right) \right)^{p_{ij}} \\ & \leq \sup_{m,n} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \left(M \left(\frac{|\Delta a_{ij}|}{\rho} \right) \right)^{p_{ij}} < \infty. \end{aligned}$$

□

Result 3.10. *The sequence spaces ${}_2W(M, \Delta, p)$ and ${}_2W_0(M, \Delta, p)$ are not monotone and hence are not solid.*

Proof. The result follows from the following example. □

Example 3.2. Let $M(x) = x^p$, $p \geq 1$. Then the double sequence $\langle a_{ij} \rangle$ defined by $a_{ij} = 1$, for all $i, j \in N$ belongs to ${}_2W(M, \Delta, p)$ and ${}_2W_0(M, \Delta, p)$. Consider its pre-image $\langle b_{ij} \rangle$ defined as

$$b_{ij} = \begin{cases} a_{ij}, & \text{if } i + j \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\langle b_{ij} \rangle$ belongs neither to ${}_2W(M, \Delta, p)$ nor to ${}_2W_0(M, \Delta, p)$. Hence the spaces ${}_2W(M, \Delta, p)$ and ${}_2W_0(M, \Delta, p)$ are not monotone and by Remark 3 these are not solid also.

Result 3.11. *The sequence spaces ${}_2W(M, \Delta, p)$, ${}_2W_0(M, \Delta, p)$ and ${}_2W_\infty(M, \Delta, p)$ are not symmetric.*

Proof. To prove the results consider the following examples. □

Example 3.3. Let $M(x) = x^2$. Consider the sequence $\langle a_{ij} \rangle$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if } i \text{ is odd for all } j \in N, \\ -1, & \text{otherwise} \end{cases}$$

Then $\Delta a_{ij} = 0$ for all $i, j \in N$. Let $\langle b_{ij} \rangle$ be a rearrangement of the sequence $\langle a_{ij} \rangle$ defined by

$$b_{ij} = \begin{cases} -1, & \text{if } i + j \text{ is even,} \\ 1, & \text{otherwise.} \end{cases}$$

Then

$$\Delta b_{ij} = \begin{cases} -4, & \text{if } i + j \text{ is even,} \\ 4, & \text{otherwise.} \end{cases}$$

Here $\langle a_{ij} \rangle \in {}_2W_0(M, \Delta, p) \subseteq {}_2W(M, \Delta, p)$, but $\langle b_{ij} \rangle \notin {}_2W(M, \Delta, p)$.

Example 3.4. Let $M(x) = x^p$, $p \geq 1$, $p_{ij} = 2$ for all $i, j \in N$. Consider the sequence $\langle a_{ij} \rangle$ defined by

$$a_{ij} = \begin{cases} 0, & \text{if } i \text{ is even, for all } j \in N, \\ i, & \text{otherwise.} \end{cases}$$

Then $\Delta a_{ij} = 0$ for all $i, j \in N$.

Let $\langle b_{ij} \rangle$ be a rearrangement of the sequence $\langle a_{ij} \rangle$ defined by

$$b_{ij} = \begin{cases} 0, & \text{if } i + j \text{ is odd,} \\ i, & \text{otherwise.} \end{cases}$$

$$\text{Then } \Delta b_{ij} = \begin{cases} 2i + 1, & \text{if } i + j \text{ is even,} \\ -2i - 1, & \text{otherwise.} \end{cases}$$

Here $\langle a_{ij} \rangle \in {}_2W_\infty(M, \Delta, p)$, but $\langle b_{ij} \rangle \notin {}_2W_\infty(M, \Delta, p)$.

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