**FILOMAT 17 (2003), 1–8**

# SOME NEW DIFFERENCE SEQUENCES SPACES DEFINED BY AN ORLICZ FUNCTION

**TUNAY BILGIN**

Abstract. **In this paper we introduce some new difference sequence spaces combining lacunary sequences and Orlicz functions. We establish some inclusion relations between these spaces.**

## 1. Introduction

Let $\ell_\infty$ and $c$denote the Banach spaces of real bounded and convergent sequences x = ( $x_i$) normed by $\|x\| = \sup_i |x_i|$, respectively.

A sequence of positive integers $\theta = (\mathrm{k}_r)$ is called "lacunary" if $\mathrm{k}_0 = 0$,

$0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as r $\to \infty$ .The intervals determined by $\theta$ will be denoted by $\mathrm{I}_r = (\mathrm{k}_{r-1}, \mathrm{k}_r)$ and $\mathrm{q}_r = \mathrm{k}_r/\mathrm{k}_{r-1}$. The space of lacunary strongly convergent sequence $N_\theta$ was defined by Freedman et al [5] as:

$N_\theta = \{$ x : $\lim_{r\to\infty} h_r^{-1} \sum\limits_{i\in I_r} |x_i - s| = 0$, for some s $\}$

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non- decreasing and convex with $M(0) = 0$ , $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of $M$ is replaced by subadditivity, then this function is called a modulus functions (see, Ruckle [13] ).

Let w be the spaces of all real or complex sequence x = ( $x_i$) . Lindentrauss and Tzafriri [8] used the idea of Orlicz function to defined the following sequence spaces.

$l_M \quad = \{$ x : $\sum\limits_{i=1}^{\infty} M\left(\frac{|x_i|}{\rho}\right) < \infty, \rho > 0\}$

which is called an Orlicz sequence spaces $l_M$ is a Banach space with the norm,

$$\|x\| \quad = \quad \inf\{\rho > 0 : \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{\rho}\right) \leq 1\}.$$

---

1991 *Mathematics Subject Classification.* 40A05, 40F05.

*Key words and phrases.* Difference sequence, Lacunary sequence, Orlicz function.

Strongly almost convergent sequence was introduced and studied by Maddox [10] and also independently by Freedman et al [5].

Parashar and Chaudhary [12] have introduced and examined some properties of the sequence spaces defined by using an Orlicz function $M$, which generalized the well-known Orlicz sequence spaces $[c,1,p]$ , $[c,1,p]_0$ and $[c,1,p]_\infty$. It may be noted here that the spaces of strongly summable sequences were discussed by Maddox [9].

Kızmaz [6] was defined the sequence spaces

$l_\infty(\Delta) = \{$ x= $(x_i) : \sup|\Delta x_i| < \infty$ $\}$,

c$(\Delta) = \{$ x =$(x_i) : \lim_i |\Delta x_i - s| = 0$ for some s $\}$,

c$_o$ $(\Delta) = \{$ x =$(x_i) : \lim_i |\Delta x_i| = 0\}$, where $\Delta x_i = (x_i - x_{i+1})$. Subsequently difference sequence spaces has been discussed in Bilgin[2], Ahmad and Mursaleen [1] , Malkowsky and Parashar[11] Et and Başarir [3], Et and Çolak [4]and others. The purpose of this paper is to introduce and study a concept of lacunary $\Delta$ -convergence using Orlicz function and to examine inclusion relations among new spaces in the same way that c$(\Delta)$ is related to c.

Now we introduce the following sequence spaces:

**Definition 1.1** Let M be an Orlicz function and p = $(p_i$ ) be any bounded sequence of strictly positive real numbers. We have

$w_0^\theta(M,p)_\Delta$ =$\{$ x : $\lim_{r\to\infty} h_r^{-1} \sum\limits_{i\in I_r} M\left(\frac{|\Delta x_i|}{\rho}\right)^{p_i} = 0,\ \rho > 0\}$

$w^\theta(M,p)_\Delta$ =$\{$ x : $\lim_{r\to\infty} h_r^{-1} \sum\limits_{i\in I_r} M\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} = 0$,for some s, $\rho > 0\}$

$w_\infty^\theta(M,p)_\Delta$ =$\{$ x : $\sup_r h_r^{-1} \sum\limits_{i\in I_r} M\left(\frac{|\Delta x_i|}{\rho}\right)^{p_i} < \infty,\ \rho > 0\}$,

where for convenince, we put $M\left(\frac{|\Delta x_i|}{\rho}\right)^{p_i}$instead of $\left[M\left(\frac{|\Delta x_i|}{\rho}\right)\right]^{p_i}$. If x $\in$ $w^\theta(M,p)_\Delta$, we say that x is lacunary $\Delta$ -convergence to s with respect to the Orlicz function M.

When $M$(x) = x, then we write $w_0^\theta(p)_\Delta$, $w^\theta(p)_\Delta$ and $w_\infty^\theta(p)_\Delta$ for the spaces $w_0^\theta(M,p)_\Delta$, $w^\theta(M,p)_\Delta$ and $w_\infty^\theta(M,p)_\Delta$, respectively. If $p_i = 1$ for all i, then $w_0^\theta(M,p)_\Delta$, $w^\theta(M,p)_\Delta$ and $w_\infty^\theta(M,p)_\Delta$ reduce to $w_0^\theta(M)_\Delta$, $w^\theta(M)_\Delta$ and $w_\infty^\theta(M)_\Delta$,respectively.

The following inequality will be used troughout the paper;

$$|a_i + b_i|^{p_i} \le C(|a_i|^{p_i} + |b_i|^{p_i}) \tag{1.1}$$

where $a_i$ and $b_i$ are complex numbers , C = $\max(1, 2^{H-1})$, and H = sup $p_i < \infty$

## 2. Inclusion theorems

By using (1), it is easy to prove the following theorem.

**Theorem 2.1.** Let M be an Orlicz function and p =($p_i$ ) be a bounded sequence of strictly positive real numbers. Then $w_0^\theta(M,p)_\Delta$, $w^\theta(M,p)_\Delta$ and $w_\infty^\theta(M,p)_\Delta$ are linear spaces over the set of complex numbers.

**Theorem 2.2** Let M be an Orlicz function. If $\sup_i(M(x))^{p_i} < \infty$ for all fixed $x > 0$ then

$$w^\theta(M,p)_\Delta \subset w_\infty^\theta(M,p)_\Delta.$$

**Proof.** Let x $\in w^\theta(M,p)_\Delta$.There exists some positive $\rho_1$ such that

$$\lim_{r\to\infty} h_r^{-1}\sum_{i\in I_r} M\left(\frac{|\Delta x_i - s|}{\rho_1}\right)^{p_i} = 0.$$

Define $\rho = 2\rho_1$. Since M is non decreasing and convex ,by using (1.1), we have

$$\sup_r h_r^{-1}\sum_{i\in I_r} M\left(\frac{|\Delta x_i|}{\rho}\right)^{p_i} = \sup_r h_r^{-1}\sum_{i\in I_r} M\left(\frac{|\Delta x_i - s + s|}{\rho}\right)^{p_i}$$

$\leq$C$\{\sup_r h_r^{-1}\sum_{i\in I_r}\frac{1}{2^{p_i}}M\left(\frac{|\Delta x_i - s|}{\rho_1}\right)^{p_i} + \sup_r h_r^{-1}\sum_{i\in I_r}\frac{1}{2^{p_i}}M\left(\frac{|s|}{\rho_1}\right)^{p_i}\}$

$< C\{\sup_r h_r^{-1}\sum_{i\in I_r} M\left(\frac{|\Delta x_i - s|}{\rho_1}\right)^{p_i} + \sup_r h_r^{-1}\sum_{i\in I_r} M\left(\frac{|s|}{\rho_1}\right)^{p_i}\} < \infty.$

Hence $x \in w_\infty^\theta(M,p)_\Delta$. This completes the proof.

**Theorem 2.3.** Let M be an Orlicz function and $0 < h = \inf p_i$. Then $w_\infty^\theta(M,p)_\Delta \subset w_0^\theta(p)_\Delta$ if and only if

$$\lim_{r\to\infty} h_r^{-1}\sum_{i\in I_r} M\,(t)^{p_i} = \infty \tag{1.2}$$

for some $t > 0$.

**Proof.** Let $w_\infty^\theta(M,p)_\Delta \subset w_0^\theta(p)_\Delta$. Suppose that (2) does not hold. Therefore there are a subinterval $I_{r(m)}$ of the set of interval $I_r$ and a number $t_0 > 0$, where $t_0 = \frac{|\Delta x_i|}{\rho}$ for all i, such that

$$h_{r(m)}^{-1}\sum_{i\in I_{r(m)}} M\,(t_0)^{p_i} \leq K < \infty, m = 1,2,3,... \tag{1.3}$$

Let us define x = $(x_i)$ as following

$$\Delta x_i = \begin{cases} \rho t_0 & ; \quad i \in I_{r(m)} \\ 0 & ; \quad i \notin I_{r(m)} \end{cases}$$

Thus by (3), x$\in w_\infty^\theta(M,p)_\Delta$. But x $\notin w_0^\theta(p)_\Delta$. Hence (2) must hold.

Conversely, suppose that (2) holds and that x$\in w_\infty^\theta(M,p)_\Delta$. Then, for each r

$$h_r^{-1}\sum_{i\in I_r} M\left(\frac{|\Delta x_i|}{\rho}\right)^{p_i} \le K < \infty, \tag{1.4}$$

Suppose that x $\notin w_0^\theta(p)_\Delta$.Then, for some number $1 > \varepsilon > 0$, there is a number $i_0$ such that , for a subinterval $I_{r_1}$ of the set of interval $I_r$, $\frac{|\Delta x_i|}{\rho} > \varepsilon$ for i$\geq i_0$. From properties of the Orlicz function, we can write

M$\left(\frac{|\Delta x_i|}{\rho}\right)^{p_i} \geq$ M$(\varepsilon)^{p_i}$

which contradicts (2), by using (4). Hence we get $w_\infty^\theta(M,p)_\Delta \subset w_0^\theta(p)_\Delta$. This completes the proof.

**Definition 2.1** An Orlicz function $M$ is said to satisfy the $\Delta_2$-condition for all values of $u$, if there exists a constant $L > 0$ such that M(2u) $\leq$LM(u), u $\geq$ 0.

It is also easy to see that always $L > 2$. The $\Delta_2$- condition equivalent to the satisfaction of inequality M(Tu) $\leq$LTu M(u) for all values of $u$ and for all $T > 1$ ( see, Krasnoselskii and Rutitsky [7]).

**Theorem 2.4** Let $0 < h = \inf p_i \leq p_i \leq \sup p_i = H < \infty$. For an Orlicz function M which satisfies $\Delta_2$- condition , we have $w_0^\theta(p)_\Delta \subset w_0^\theta(M,p)_\Delta$,$w^\theta(p)_\Delta \subset w^\theta(M,p)_\Delta$ and $w_\infty^\theta(p)_\Delta \subset w_\infty^\theta(M,p)_\Delta$.

**Proof.** Let x $\in w^\theta(p)_\Delta$. Then we have

$h_r^{-1}\sum\limits_{i\in I_r}\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} \to 0$ as $r\to\infty$, for some s.

Let $\varepsilon > 0$ **a**nd choose $\delta$ with $0 < \delta < 1$such that $M(t) < \varepsilon$ for $0 \leq t \leq \delta$. We can write

$$h_r^{-1}\sum_{i\in I_r} M\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} = h_r^{-1}\sum_{\substack{i\in I_r \\ |\Delta x_i - s|/\rho \leq \delta}} M\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} + $$
$$+ h_r^{-1}\sum_{\substack{i\in I_r \\ |\Delta x_i - s|/\rho > \delta}} M\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i}$$

For the first summation above, we immediately write

$h_r^{-1}\sum\limits_{\substack{i\in I_r \\ |\Delta x_i - s|/\rho \leq \delta}} M\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} <\max(\varepsilon,\varepsilon^h)$

by using continuity of M. For the second summation, we will make following procedure. We have

$$\left(\frac{|\Delta x_i - s|}{\rho}\right) < \quad 1 + \left(\frac{|\Delta x_i - s|}{\rho}\right)/\delta.$$

Since M is non decreasing and convex, it follows that
$M\left(\frac{|\Delta x_i - s|}{\rho}\right) < M\{1 + \left(\frac{|\Delta x_i - s|}{\rho}\right)/\delta\} \leq \frac{1}{2}\text{M}(2) + \frac{1}{2}M\{2\left(\frac{|\Delta x_i - s|}{\rho}\right)/\delta\}$
Since M satisfies $\Delta_2$-condition , we can write
$M\left(\frac{|\Delta x_i - s|}{\rho}\right) \leq \frac{1}{2}L\{\left(\frac{|\Delta x_i - s|}{\rho}\right)/\delta\}\text{M}(2) + \frac{1}{2}L\{\left(\frac{|\Delta x_i - s|}{\rho}\right)/\delta\}\text{M}(2)$
$= L\{\left(\frac{|\Delta x_i - s|}{\rho}\right)/\delta\}\text{M}(2)$
In this way, we write

$$h_r^{-1}\sum_{i\in I_r} M\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} \leq \max(\varepsilon, \varepsilon^h) +$$
$$+ \max\{1, [LM(2)/\delta]^H\}h_r^{-1}\sum_{i\in I_r}\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i}$$

Taking the limit as $\varepsilon \to 0$ and $r \to \infty$, it follows that $x \in w^\theta(M,p)_\Delta$.

Following similar arguments we can prove that $w_0^\theta(p)_\Delta \subset w_0^\theta(M,p)_\Delta$ and $w_\infty^\theta(p)_\Delta \subset w_\infty^\theta(M,p)_\Delta$.

After step of this section, different inclusion relations among these sequence spaces are going to be studied. Now we have

**Theorem 2.5.** Let M be an Orlicz function. Then the following statements are equivalent.
i) $w_\infty^\theta(p)_\Delta \subset w_\infty^\theta(M,p)_\Delta$
ii) $w_0^\theta(p)_\Delta \subset w_\infty^\theta(M,p)_\Delta$
iii) $\sup_r h_r^{-1}\sum_{i\in I_r} M(t)^{p_i} < \infty$ for all $t > 0$.

**Proof.** *i)* $\Rightarrow$*ii)*: Let (i) holds. To verify (ii), it is enough to prove $w_0^\theta(p)_\Delta \subset w_\infty^\theta(p)_\Delta$. Let x $\in w_0^\theta(p)_\Delta$. Then, there exist $r \geq r_0$, for $\varepsilon > 0$, such that

$$h_r^{-1}\sum_{i\in I_r}\left(\frac{|\Delta x_i|}{\rho}\right)^{p_i} < \varepsilon.$$

Hence there exists $K > 0$ such that

$$\sup_r h_r^{-1}\sum_{i\in I_r}\left(\frac{|\Delta x_i|}{\rho}\right)^{p_i} < K$$

So, we get x $\in w_\infty^\theta(p)_\Delta$

*ii)*$\Rightarrow$*iii)*: Let (ii) holds. Suppose that (iii) does not holds. Then for some $t > 0$

$$\sup_r h_r^{-1}\sum_{i\in I_r} M(t)^{p_i} = \infty$$

and therefore we can find a subinterval $I_{r(m)}$ of the set of interval $I_r$ such that

$$h_{r(m)}^{-1} \sum_{i \in I_{r(m)}} M\left(\frac{1}{m}\right)^{p_i} > m, m = 1, 2, 3, ... \tag{1.5}$$

Let us define x = $(x_i)$ as following

$$\Delta x_i = \begin{cases} \frac{\rho}{m} & ; \quad i \in I_{r(m)} \\ 0 & ; \quad i \notin I_{r(m)} \end{cases}$$

Then $x \in w_0^\theta(p)_\Delta$ but by (5), x $\notin w_\infty^\theta(M,p)_\Delta$, which contradicts (ii). Hence (iii) must holds.

*iii)⇒ i)*: Let (iii) hold and $x \in w_\infty^\theta(p)_\Delta$. Suppose that x $\notin w_\infty^\theta(M,p)_\Delta$. Then for $x \in w_\infty^\theta(p)_\Delta$

$$\sup_r h_r^{-1} \sum_{i \in I_r} M\left(\frac{|\Delta x_i|}{\rho}\right)^{p_i} = \infty \tag{1.6}$$

Let t = $\frac{|\Delta x_i|}{\rho}$ for each i, then by (6)

$$\sup_r h_r^{-1} \sum_{i \in I_r} M(t)^{p_i} = \infty$$

which contradicts (iii). Hence (i) must holds.

**Theorem 2.6.** Let M be an Orlicz function. Then the following statements are equivalent.

i) $w_0^\theta(M,p)_\Delta \subset w_0^\theta(p)_\Delta$

ii) $w_0^\theta(M,p)_\Delta \subset w_\infty^\theta(p)_\Delta$

iii) $\inf_r h_r^{-1} \sum_{i \in I_r} M(t)^{p_i} > 0$ for all $t > 0$.

**Proof.** *i) ⇒ii)*: It is obvious.

i*i) ⇒iii)*: Let (ii) holds. Suppose that (iii) does not holds. Then

$\inf_r h_r^{-1} \sum_{i \in I_r} M(t)^{p_i} = 0$for some $t > 0$,

and we can find a subinterval $I_{r(m)}$ of the set of interval $I_r$ such that

$$h_{r(m)}^{-1} \sum_{i \in I_{r(m)}} M(m)^{p_i} < \frac{1}{m}, m = 1, 2, 3, ... \tag{1.7}$$

Let us define x = $(x_i)$ as following

$$\Delta x_i = \begin{cases} \rho m & ; \quad i \in I_{r(m)} \\ 0 & ; \quad i \notin I_{r(m)} \end{cases}$$

Thus, by (7) $x \in w_0^\theta(M,p)_\Delta$ but x $\notin w_\infty^\theta(p)_\Delta$which contradicts (ii). Hence (iii) must holds.

ii$i$) $\Rightarrow$ $i$): Let (iii) holds. Suppose that $x \in w_0^\theta(M,p)_\Delta$. Therefore,

$$h_r^{-1} \sum_{i \in I_r} M\left(\frac{|\Delta x_i|}{\rho}\right)^{p_i} \to 0 \tag{1.8}$$

as $r \to \infty$. Again, suppose that x $\notin w_0^\theta(p)_\Delta$ for some number $\varepsilon > 0$ and a subinterval $I_{r(m)}$ of the set of interval $I_r$, we have $\frac{|\Delta x_i|}{\rho} \geq \varepsilon$ for all i. Then, from properties of the Orlicz function, we can write

M$\left(\frac{|\Delta x_i|}{\rho}\right)^{p_i} \geq$ M$(\varepsilon)^{p_i}$

Consequently, by (8) we have

$$\lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} M\left(\varepsilon\right)^{p_i} = 0$$

which contradicts (iii). Hence (i) must holds.

Finally, in this section, we consider that $(p_i)$ and $(q_i)$ are any bounded sequences of strictly positive real numbers. We are able to prove $w^\theta(M,q)_\Delta \subseteq w^\theta(M,p)_\Delta$ only under additional conditions.

**Theorem2.7.** i) If $0 < \inf p_i \leq p_i \leq 1$ for all k, then $w^\theta(M)_\Delta \subseteq w^\theta(M,p)_\Delta$

i i) $1 \leq p_i \leq \sup p_i = H < \infty$, then $w^\theta(M,p)_\Delta \subseteq w^\theta(M)_\Delta$

**Proof.** i) Let x $\in w^\theta(M,p)_\Delta$since $0 < inf p_i \leq p_i \leq 1$ we get

$$h_r^{-1} \sum_{i \in I_r} M\left(\frac{|\Delta x_i - s|}{\rho}\right) \leq h_r^{-1} \sum_{i \in I_r} M\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i}$$

and hence x $\in w^\theta(M)_\Delta$.

Let $1 \leq p_i \leq \sup p_i = H < \infty$, and x $\in w^\theta(M)_\Delta$. Then for each $0 < \varepsilon < 1$ there exists a positive integer $r_0$ such that

$$h_r^{-1} \sum_{i \in I_r} M\left(\frac{|\Delta x_i - s|}{\rho}\right) \leq \varepsilon < 1$$

for all r $\geq r_0$. This implies that

$$h_r^{-1} \sum_{i \in I_r} M\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} \leq h_r^{-1} \sum_{i \in I_r} M\left(\frac{|\Delta x_i - s|}{\rho}\right).$$

Therefore x $\in w^\theta(M,p)_\Delta$.

Using the same technique as in Theorem 2 in [14], it is easy to prove the following theorem.

**Theorem 2.8.** Let $0 < p_i \leq q_i$ for all i and let $(q_i \ / \ p_i)$ be bounded . Then

$$w^\theta(M,q)_\Delta \subseteq w^\theta(M,p)_\Delta$$

## References


[1] Z. U. Ahmad, Mursaleen,Köthe-Toeplitz duals of some new sequence spaces and their matrix maps, *Pub. Inst. Math. Beograd* **42(56)** (1987), 57–61.

[2] T. Bilgin, The lacunary strong $\Delta$-convergence in Banach spaces, *to appear*

[3] M. Et, M. Başarir, On some new generalized difference sequence spaces , *Periodica Math. Hung.* **35,(3)** (1997), 169–176.

[4] M. Et, R. Çolak, On some generalized difference sequence spaces , *Soochow J. Math.,21* **21(4)** (1995), 377–386.

[5] A. R. Freedman, J. J. Sember, M. Raphael, Some Cesaro- type summability spaces, *Proc. London Math. Soc.* **37(3)** (1978), 508–520.

[6] H.,Kızmaz, On certain sequence spaces, *Canad.Math.Bull.* **24;2** (1981),169–176.

[7] M. A. Krasnoselskii, Y. B. Rutitsky, Convex function and Orlicz spaces, Groningen, The Netherlands, 1961.

[8] J. Lindenstrauss, L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.* **10(3)** (1971), 379–390.

[9] I. J. Maddox, Spaces of strongly summable sequences, *Quart. J. Math. Oxford* **18(2)** (1967), 345–355.

[10] I. J. Maddox, On strong almost convergence, *Math. Proc. Camb. Phil. Soc.* **85** (1979), 345–350.

[11] E. Malkowsky, S. D. Parashar, Matrix transformations in spaces of bounded and convergent difference sequences of order m, *Analysis* **17** (1997), 87–97.

[12] S. D. Parashar, B.Chaudhary, Sequence spaces defined by Orlicz functions, *Indian J. Pure Appl. Math.* **25** (1994), 419–428.

[13] W. H. Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, *Can. J. Math.* **25** (1973), 973–978.

[14] [14] E.,Öztürk , T.,Bilgin. Strongly summable sequence spaces defined by a modulus, *Indian J. Pure Appl. Math.* **25(6)** (1994), 621-625.



Yüzüncü Yıl University, Faculty of Education, Department of Mathematics, Van, Turkey

*E-mail address*: tbilgin@yyu.edu.tr