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Some differential equations on Riemannian manifolds

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§ 1. Introduction.

Let (M, g) be a Riemannian manifold of dimension $m \geqq 2$ and let ∇ denote the Riemannian connection defined by g. In this paper we study the following system of differential equations of order three:

$$\nabla_h \nabla_j \nabla_i f + k(2\nabla_h f g_{ji} + \nabla_j f g_{ih} + \nabla_i f g_{hj}) = 0 \tag{1.1}$$

where k is a positive constant. Originally the differential equations (1.1) come from some study of the Laplacian on a Euclidean sphere $(S^m; k)$ of constant curvature k. The first eigenvalue of the Laplacian on $(S^m; k)$ is mk and each eigenfunction h corresponding to mk satisfies the following system of differential equations of order two:

$$\nabla_j \nabla_i h + khg_{ji} = 0. \tag{1.2}$$

The second eigenvalue is $2(m+1)k$ and each eigenfunction f corresponding to $2(m+1)k$ satisfies (1.1).

Assuming the existence of a non-constant function h satisfying (1.2) on a Riemannian manifold (M, g) many mathematicians studied differential geometric properties of (M, g) (cf. S. Ishihara and Y. Tashiro [**11**], M. Obata [**14**], [**15**], Y. Tashiro [**22**], etc.). In this case grad f is an infinitesimal conformal transformation.

Assume that there is a non-constant function f satisfying (1.1) on (M, g). Then grad f is an infinitesimal projective transformation and is a k-nullity vector field on (M, g). The converse is also true (cf. Proposition 2.1). This gives a geometric meaning of (1.1).

The system of differential equations (1.1) was first studied by M. Obata [**15**] and he announced the following.

THEOREM A. *Let (M, g) be a complete and simply connected Riemannian manifold. In order for (M, g) to admit a non-constant function f satisfying* (1.1)

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for some positive constant k, it is necessary and sufficient that (M, g) *is isometric to a Euclidean sphere* $(S^m; k)$.

However the outline of the proof given in [**15**] turned to be incomplete. The complete proof was first given by the present author [**21**]. Later D. Ferus [**8**] gave an elegant proof. Further, S. Gallot [**9**] announced his proof (but this proof is also incomplete, as we give a counter-example to his main lemma in § 6).

The purpose of this paper is to clarify the differential geometric implications of the existence of such a function f. In particular, we are concerned with the behavior of trajectories of grad f. Proof of Theorem A is given in § 5 and § 8. The mathematical essence of (1.1) will be seen in the next Theorem (cf. Theorem 5.1, Theorem 5.8).

THEOREM B. *Let* (M, g) *be a Riemannian manifold admitting a non-constant function* f *which satisfies* (1.1) *for some positive constant* k. *If* (M, g) *contains a whole trajectory* l *of* grad f *with its limit points, then* (M, g) *is constant curvature* k *at each point of the trajectory* l.

In § 7 we define the concept of t-connectedness. k-nullity theory and t-connectedness property enable us to state constancy of sectional curvature in local forms.

Kählerian analogues are also true.

Manifolds are assumed to be connected and of class C^∞. Functions and tensor fields are supposed to be class C^∞ unless otherwise stated.

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§ 2. Fundamental properties of f.

For a function f on a Riemannian manifold (M, g), by F we denote the gradient vector field of f: $F=\text{grad} f=(F^i)=(g^{ir}F_r)=(g^{ir}\nabla_r f)$. Here (g^{ir}) is the inverse of the matrix (g_{ji}). By $R=(R^i_{jhl})$ we denote the Riemannian curvature tensor of (M, g). A vector field X on (M, g) is called a k-nullity vector field on (M, g), if X satisfies

$$X_i R^i_{jhl}=k(X_h g_{jl}-X_l g_{jh}) \tag{2.1}$$

for a constant k (for more details, see § 4, and [**5**], [**6**], [**7**], etc.).

PROPOSITION 2.1. *Let* f *be a function on* (M, g). f *satisfies* (1.1) *for a constant* k, *if and only if*

(i) F *is an infinitesimal projective transformation, and*

(ii) F *is a* k*-nullity vector field on* (M, g).

PROOF. First we assume that f satisfies (1.1) for a constant k. By the

Ricci identity for $\nabla_l \nabla_h F_j - \nabla_h \nabla_l F_j$ and by (1.1) we get (2.1) with $X=F$. This proves (ii). Next in the classical relation (on the Lie derivative of the Christoffel's symbols):

$$L_F \Gamma^i_{jh} = \nabla_h \nabla_j F^i - R^i_{jhl} F^l , \tag{2.2}$$

we apply (1.1) and (2.1) with $X=F$, to get

$$L_F \Gamma^i_{jh} = -2k(F_h \delta^i_j + F_j \delta^i_h) . \tag{2.3}$$

This shows that F is an infinitesimal projective transformation on (M, g).

Conversely, let f be a function on (M, g) with properties (i) and (ii). By (i) there is a function θ on M such that

$$L_F \Gamma^i_{jh} = \theta_h \delta^i_j + \theta_j \delta^i_h , \tag{2.4}$$

where $\theta_h = \nabla_h \theta$. By (2.2), (2.4) and (2.1) with $X=F$, we obtain

$$\nabla_h \nabla_j F^i = k(F_j \delta^i_h - F^i g_{jh}) + \theta_h \delta^i_j + \theta_j \delta^i_h . \tag{2.5}$$

Lowering the index i and taking the symmetric part with respect to i and j, we obtain

$$2\nabla_h \nabla_j F_i = 2\theta_h g_{ij} + \theta_j g_{ih} + \theta_i g_{jh}, \tag{2.6}$$

where we have used $\nabla_j F_i = \nabla_i F_j$. Transvecting (2.5) [the index i being lowered] and (2.6) with g^{hj}, we obtain $\theta_i = -2kF_i$. Substituting this into (2.5) we get (1.1). Q. E. D.

From now on in this section we assume that (M, g) admits a non-constant function f satisfying (1.1) for some positive constant k.

Transvecting (1.1) with g^{ij}, we see that there is a constant c such that

$$\Delta(f-c) = -2(m+1)k(f-c) , \tag{2.7}$$

where Δ denotes the Laplacian on (M, g); $\Delta f = \nabla_r \nabla^r f$.

Let $\{x(s)\}$ be a geodesic in (M, g) with arc-length parameter s. We put $(c^i(s)) = (dx^i(s)/ds)$. Transvecting (1.1) with $c^i c^j c^h$, we see that the restriction $f(s)$ of f to $\{x(s)\}$ satisfies

$$f''' + 4kf' = 0$$

where the dash means the differentiation with respect to s. Solving the last equation we obtain

$$f(s) = (f''(0)/2k) \sin^2 \sqrt{k}\, s + (f'(0)/2\sqrt{k}) \sin^2 \sqrt{k}\, s + f(0) . \tag{2.8}$$

LEMMA 2.2. *Let x be a point of M and assume that M contains the closed $(\pi/2\sqrt{k})$-neighborhood U of x. Then there are points p and q in U where f takes its maximum value $b=f(p)$ and the minimum value $a=f(q)$.*

PROOF. Fixing $x=x(0)$ and changing the direction of geodesics, by (2.8) we see that f takes its maximum value b at some point p of M within the distance $\pi/2\sqrt{k}$ from $x(0)$. Similarly there is a point q where f takes its minimum value a within the distance $\pi/2\sqrt{k}$ from $x(0)$. Q. E. D.

LEMMA 2.3. *Let z be an arbitrary critical point of f. Let $\{x_v(s)=\mathrm{Exp}_z sv\}$ be a unit speed geodesic starting at z with the initial direction v. Then the restriction $f_v(s)$ of f to this geodesic is given by*

$$f_v(s)=f(z)+(1/2k)H(v, v)\sin^2\sqrt{k}\, s, \tag{2.9}$$

where H denotes the Hessian $(\nabla_j F_i)$ of f at z.

PROOF. This follows from (2.8). Q. E. D.

§ 3. The behavior of trajectories of F.

Let M^b be the subset of M of all critical points where f takes its maximum value b and let M^a be one of all critical points where f takes its minimum value a.

LEMMA 3.1. *Each connected component of M^b is a totally geodesic submanifold with respect to the induced metric from (M, g).*

PROOF. Let p be an arbitrary point of M^b. Then for a unit tangent vector v at p we have (2.9) (with $p=z$) along the geodesic $\{\mathrm{Exp}_p sv\}$. It is clear that the Hessian H at p is negative semi-definite. If v is not an eigenvector corresponding to the eigenvalue zero of H, $H(v, v)<0$ holds and $f_v(s)<f(p)$ holds for all s; $0<s$ $(<\pi/2\sqrt{k})$. If v is an eigenvector corresponding to zero, then $f_v(s)=f(p)$ holds for all s (for which $\mathrm{Exp}_p sv$ is defined). Therefore $\{\mathrm{Exp}_p sv\}$ is contained in M^b. Q. E. D.

REMARK 3.2. M^b and M^a have corresponding properties as seen by considering a function $b+a-f$. So it suffices to state propositions only on M^b.

For a curve $l=\{x(s)\}$ we use the following notations:

$$l[r]=\{x(s)\,;\ 0\leqq s\leqq r\},$$

$$l[r)=\{x(s)\,;\ 0\leqq s<r\},$$

$$l(r)=\{x(s)\,;\ 0<s<r\}.$$

LEMMA 3.3. *Let p be a point of M^b and let v be a unit eigenvector corresponding to a non-zero eigenvalue ν of the Hessian H of f at p. For the geodesic $l=\{x(s)=\mathrm{Exp}_p sv\}$ we have*

(i) *if $0<r<\pi/2\sqrt{k}$ and $l(r)\subset M$, then $l(r)$ is a part of a trajectory of F,*

(ii) *if $l(\pi/2\sqrt{k})\subset M$, then it is a whole trajectory of F,*

(iii) *if $l[\pi/2\sqrt{k}]\subset M$, then $x(\pi/2\sqrt{k})$ is a critical point of f.*

PROOF. In proofs of (ii) and (iii), the proof of (i) is contained. So we assume that $l[\pi/2\sqrt{k}]\subset M$. If $l[\pi/2\sqrt{k}]$ has no conjugate point of $p=x(0)$, let s_0 be an arbitrary real number such that $0<s_0<\pi/2\sqrt{k}$. If $l[\pi/2\sqrt{k})$ has conjugate points, let $x(s_1)$ be the first conjugate point of p and let s_0 be an arbitrary real number such that $0<s_0<s_1<\pi/2\sqrt{k}$.

Let $(0\geqq)\,\nu_1\geqq\nu_2\geqq\cdots\geqq\nu_m$ be the eigenvalues of H and assume $\nu=\nu_i$. Let j be any integer such that $j\neq i$ and $1\leqq j\leqq m$. We define a curve $\{w_j(\theta);\ -\pi<\theta<\pi\}$ in the tangent space M_p at p by

$$w_j(\theta)=\cos\theta(s_0 v)+\sin\theta(s_0 v_j)\,, \tag{3.1}$$

where v_j denotes a unit eigenvector corresponding to ν_j so that $\{v_1, v_2, \cdots, v_i=v, \cdots, v_m\}$ is an orthonormal base of M_p such that $H(v_r,\cdot)=\nu_r g(v_r,\cdot)$ $(r=1, \cdots, m)$. Next we define a curve $\{z_j(\theta)\}$ in M by

$$z_j(\theta)=\mathrm{Exp}_p\, w_j(\theta)\,.$$

Then $\{z_j(\theta);\ -\varepsilon<\theta<\varepsilon\}$ is a C^∞-curve passing through $x(s_0)$ for sufficiently small ε. Then

$$Z_j=(dz_j/d\theta)(0)$$

is a non-zero tangent vector at $x(s_0)$. Z_j is orthogonal to the geodesic $\{x(s)\}$ at $x(s_0)$ by the well known Gauss lemma. Next we show that Z_j and F are orthogonal at $x(s_0)$. For this purpose we define $f_j(\theta)$ by $f_j(\theta)=f(z_j(\theta))$. Then $g(Z_j, F)=0$ at $x(s_0)$ is equivalent to

$$(df_j/d\theta)(0)=0\,. \tag{3.2}$$

By (2.9) we obtain

$$\begin{aligned} f_j(\theta)&=b+(1/2ks_0^2)H(w_j(\theta), w_j(\theta))\sin^2\sqrt{k}\,s_0 \\ &=b+(1/2k)(\nu\cos^2\theta+\nu_j\sin^2\theta)\sin^2\sqrt{k}\,s_0\,, \end{aligned}$$

from which (3.2) follows. Therefore F is orthogonal to all Z_j $(j\neq i)$ at $x(s_0)$. Since the geodesic l is also orthogonal to all Z_j $(j\neq i)$, F is tangent to l at $x(s_0)$. Thus F is tangent to l at each point $x(s)$ for s; $0<s<\pi/2\sqrt{k}$ or $0<s<s_1$.

In the case where $x(s_1)$ is the first conjugate point of p, the geodesic $l(s_1)$ is a part of a trajectory of F. By Proposition 2.1 the sectional curvature for each 2-plane which contains F is equal to k. Hence $x(s_1)$ can not be a conjugate point of p unless $s_1\geqq\pi/\sqrt{k}$. This contradicts $s_1<\pi/2\sqrt{k}$.

Therefore in any case, $l(\pi/2\sqrt{k})$ is a part of a trajectory of F. By (2.9) we see that $g(F, F)$ tends to zero both when $x(s)\to x(0)$ and $x(s)\to x(\pi/2\sqrt{k})$, and hence $l(\pi/2\sqrt{k})$ is a whole trajectory of F and $x(\pi/2\sqrt{k})$ is a critical point of f. Q.E.D.

COROLLARY 3.4. *If $p\in M^b$ and $q\in M^a$ are joined by a geodesic* $\{y(s);$

$0 \leqq s \leqq \pi/2\sqrt{k}\}$ *in* M *with* $p=y(0)$ *and* $q=y(\pi/2\sqrt{k})$, *then* $u=(dy/ds)(0)$ *is an eigenvector corresponding to the minimum eigenvalue of* H *at* p, *and* $\{y(s)$; $0<s<\pi/2\sqrt{k}\}$ *is a whole trajectory of* F.

PROOF. By (2.9) $H(u, u)$ must be the minimum eigenvalue of H, and Corollary 3.4 follows from Lemma 3.3. Q.E.D.

COROLLARY 3.5. *Each unit eigenvector* v *corresponding to a non-zero eigenvalue of* H *at* p *of* M^b *belongs to the k-nullity space at* p. *In particular, the normal space to* M^b *in* M *at* p *is contained in the k-nullity space at* p.

PROOF. This follows from Lemma 3.3 (i) and Proposition 2.1. Q.E.D.

From now on in this section we assume that (M, g) contains some complete connected component $^*M^b$ of M^b and its closed $(\pi/2\sqrt{k})$-neighborhood $W(^*M^b)$:

$$W(^*M^b)=\{w \in M\,;\ \text{distance}\ (w,\ ^*M^b) \leqq \pi/2\sqrt{k}\}. \tag{3.3}$$

By $W_0(^*M^b)$ we denote the subset of $W(^*M^b)$ defined by the inequality in (3.3). By the boundary of $W(^*M^b)$ we mean $\partial W(^*M^b)=W(^*M^b)-W_0(^*M^b)$.

LEMMA 3.6. *There is no critical point of* f *in* $W_0(^*M^b)-{}^*M^b$.

PROOF. Let w be an arbitrary point of $W_0(^*M^b)-{}^*M^b$. w can be joined to $^*M^b$ by a shortest geodesic. The length of this geodesic is smaller than $\pi/2\sqrt{k}$. Therefore the derivative of f along this geodesic cannot vanish at w by (2.9), and w can not be a critical point of f. Q.E.D.

LEMMA 3.7. *For each critical point* z *in* $W(^*M^b)-{}^*M^b$ *the distance between* z *and each point of* $^*M^b$ *is equal to* $\pi/2\sqrt{k}$.

PROOF. z is in the boundary $\partial W(^*M^b)$ of $W(^*M^b)$ by Lemma 3.6. So there is a point p in $^*M^b$ such that the distance between z and p is equal to $\pi/2\sqrt{k}$. Let y be a point in $^*M^b$ near p, and join y to z by a shortest geodesic. Then, considering (2.9) along this geodesic we see that the distance between y and z is equal to $\pi/2\sqrt{k}$. Since $^*M^b$ is connected, by continuity of the distance function from z we get Lemma 3.7.

COROLLARY 3.8. $^*M^b$ *and* $W(^*M^b)$ *are compact.*

LEMMA 3.9. *Let* w *be a point in* $W_0(^*M^b)$ *and let* $\{x(t)\}$ *be a trajectory of* F *passing through* $w=x(0)$. *Then the distance function* $\rho(t)$ *between* $x(t)$ *and* $^*M^b$ *is strongly monotone decreasing for* $t \geqq 0$ *and* $\lim \rho(t)=0$ *as* $t \to \infty$.

PROOF. Let $l=\{y(s)\,;\ 0 \leqq s \leqq s_0\}$ be a shortest geodesic of length $s_0<\pi/2\sqrt{k}$ connecting $x(0)=w=y(s_0)$ and some point $y(0)$ of $^*M^b$. Since the tangent component of F to l is not zero by (2.9), there are two real numbers $s_1<s_0$ and $\varepsilon>0$, such that for any δ $(\varepsilon>\delta>0)$ the distance between $x(\delta)$ and $y(s_1)$ is smaller than s_0-s_1. Thus the distance between $x(\delta)$ and $^*M^b$ is smaller than s_0. Continuing this process and applying Lemma 3.6, we have Lemma 3.9.

LEMMA 3.10. *If* $\dim {}^*M^b \geqq 1$, *then* $W(^*M^b)=M$ *and* $^*M^b=M^b$.

PROOF. Let p be a point in $^*M^b$. Let v be a unit eigenvector corresponding to some non-zero eigenvalue of the Hessian at p. Then $z:=\mathrm{Exp}_p(\pi/2\sqrt{k})v$ is a critical point in $\partial W(^*M^b)$. Since $\dim {}^*M^b\geqq 1$, we have linearly independent two vectors e_1 and e_2 of length $\pi/2\sqrt{k}$ at z such that

$$\mathrm{Exp}_z\, e_1=p\,, \qquad \mathrm{Exp}_z\, e_2\in {}^*M^b\,.$$

e_1 and e_2 are eigenvectors of the Hessian at z corresponding to the maximum eigenvalue by (2.9). By completeness of $^*M^b$ we see that $\mathrm{Exp}_z u\in {}^*M^b$ for each vector u of length $\pi/2\sqrt{k}$ in the 2-plane determined by e_1 and e_2. In particular, $\mathrm{Exp}_z(-e_1)\in {}^*M^b$. This shows that $\{\mathrm{Exp}_p\, sv\,;\, 0\leqq s\leqq \pi/\sqrt{k}\}$ is contained in $W(^*M^b)$.

Next let $V(p)$ be the normal space at p to $^*M^b$ and let $^*S(p)$ be the hypersphere of radius $\pi/\sqrt{k}$ in $V(p)$. Applying the continuity argument from $\mathrm{Exp}_p(\pi/\sqrt{k})v\in {}^*M^b$, we see that $\mathrm{Exp}_p(^*S(p))$ is contained in $^*M^b$. Thus the closed $(\pi/\sqrt{k})$-disk of $V(p)$ is mapped into $W(^*M^b)$ by Exp_p. Since p is an arbitrary point of $^*M^b$, we see that $\mathrm{Exp}_q V(q)$ is contained in $W(^*M^b)$ for each q of $^*M^b$. Q. E. D.

LEMMA 3.11. *Assume that* $\dim {}^*M^b=0$ *and* M^b *is composed of one point* p. *If* (M, g) *is complete, then* $W(p)=M$.

PROOF. For any unit tangent vector v at p, we have $\mathrm{Exp}_p(\pi/\sqrt{k})v=p$ by (2.9). Therefore $W(p)=M$.

LEMMA 3.12. *Assume that* $\dim {}^*M^b=0$ *and* M^b *has at least two points* p, q *with distance* $\pi/\sqrt{k}$. *If* $W(p)$ *and* $W(q)$ *are the closed* $(\pi/2\sqrt{k})$*-neighborhoods of* p, q *in* M *then* $M=W(p)\cup W(q)$ *and* M^b *is composed of only two points* p, q.

PROOF. Let $\{\mathrm{Exp}_p\, sv\,;\, 0\leqq s\leqq \pi/\sqrt{k}\}$ be a geodesic connecting p and $q=\mathrm{Exp}_p(\pi/\sqrt{k})v$. By the method similar to that in the proof of Lemma 3.10 we obtain $\mathrm{Exp}_p(^*S(p))=q$. Conversely, $\mathrm{Exp}_q(^*S(q))=p$. Since $V(p)$ is the same as the tangent space M_p at p in this case, $\mathrm{Exp}_p\, V(p)=W(p)\cup W(q)=M$. Q. E. D.

By Lemmas 3.10~3.12, if (M, g) is complete then M is compact. The only case where $W(^*M^b)$ is different from M is possible for $M^b=\{p, q\}$.

LEMMA 3.13. *Assume that* $M^b=\{p, q\}$ *and* $M=W(p)\cup W(q)$. *Let* $^*T(p)$ *be the hypersphere of radius* $\pi/2\sqrt{k}$ *in* M_p. *Then* $\mathrm{Exp}_p(^*T(p))=\partial W(p)$ *and* $\mathrm{Exp}_p|^*T(p)$ *is a diffeomorphism.*

PROOF. Let $^*D_0(p)$ be the open $(\pi/\sqrt{k})$-disk of M_p. We show that $\mathrm{Exp}_p|^*D_0(p)$ is a diffeomorphism of $^*D_0(p)$ onto $M-q$. Suppose that there are two geodesics

$$\{\mathrm{Exp}_p\, sv\,;\, 0\leqq s\leqq \pi/\sqrt{k}\}, \qquad \{\mathrm{Exp}_p\, tu\,;\, 0\leqq t\leqq \pi/\sqrt{k}\}$$

such that $\mathrm{Exp}_p\, s_1v=\mathrm{Exp}_p\, t_1u$ for some s_1, t_1; $0<s_1$, $t_1<\pi/\sqrt{k}$, where v and u are unit vectors at p. Since

$$\mathrm{Exp}_p(\pi/\sqrt{k})v=\mathrm{Exp}_p(\pi/\sqrt{k})u=q$$

and M is compact, the distance between p and q must be smaller than $\pi/\sqrt{k}$.

This contradicts (2.9). Q. E. D.

COROLLARY 3.14. *Under the same situation as in Lemma* 3.13, $W(p)$ *is closed with respect to trajectories of* F. *Every trajectory passing through a point in* $W_0(p)$ *stays in* $W_0(p)$, *and every trajectory passing through a point of the boundary* $\partial W(p)$ *stays in the boundary.*

PROOF. Since $\partial W(p)=\mathrm{Exp}_p(*T(p))$, F is tangent to $\partial W(p)$ by (2.9). Therefore every trajectory of F passing through a point of $\partial W(p)$ stays in $\partial W(p)$. Consequently every trajectory of F passing through a point in $W_0(p)-p$ can not touch $\partial W(p)$ and stays in $W_0(p)$. Q. E. D.

Let $(*M_j^a; j=1, \cdots, u)$ be connected components of $M^a \cap W(*M^b)$. For each j we define $W_0(*M_j^a)$ by

$$W_0(*M_j^a)=\{w \in W(*M^b); \text{ distance } (w, *M_j^a)<\pi/2\sqrt{k}\}.$$

COROLLARY 3.15. *Let* $W(*M^b)$ *be one of these considered in Lemmas* 3.10~3.12. *For each* j $(=1, \cdots, u)$ *and for each* w *in* $W_0(*M_j^a) \cap W_0(*M^b)$ *the trajectory of* F *passing through* w *comes from some point of* $*M_j^a$ *and tends to some point of* $*M^b$.

PROOF. We apply Lemma 3.9 to the trajectory $\{x(t)\}$ $(x(0)=w)$ of F for $t \geqq 0$. For $t \leqq 0$ consider $b+a-f$ with respect to $*M_j^a$. Q. E. D.

The behavior of trajectories of F in $W(*M^b)$ is as follows. Since $W(*M^b)$ is compact and $W(*M^b)$ is closed with respect to trajectories of F, every trajectory of F in $W(*M^b)$ is written as

$$\{x(t)\}=\{\varphi_t x(0); -\infty<t<\infty\}, \tag{3.4}$$

where $\{\varphi_t\}$ is a 1-parameter group of (local) transformations generated by F. Let ν_* and ν_m be the non-zero maximum eigenvalue and the minimum eigenvalue of the Hessian H at a point of $*M^b$. ν_* and ν_m are independent of the choice of points in $*M^b$, because H is parallel along $*M^b$ by (1.1).

For each point w in $W_0(*M_j^a) \cap W_0(*M^b)$, let (3.4) be the trajectory of F passing through $w=x(0)$. We put

$$x(-\infty)=\lim_{t \to -\infty} x(t), \tag{3.5}$$

$$x(\infty)=\lim_{t \to \infty} x(t). \tag{3.6}$$

Then $x(-\infty) \in *M_j^a$ and $x(\infty) \in *M^b$. We put

$$v=\lim_{t \to \infty}(F/|F|)(x(t)), \tag{3.7}$$

where $|F|^2=g(F, F)$. If $\{x(t)\}$ is geodesic then v is an eigenvector corresponding to ν_m. If $\{x(t)\}$ is not geodesic, then v is an eigenvector corresponding to $\nu_* \neq \nu_m$.

To verify these it is convenient to study the case where the normal space V to $^*M^b$ in M at p is 2-dimensional, as a simple model. Let e_1 and e_2 be unit eigenvectors in V corresponding to ν_* and ν_m, respectively. Then $f(u_1, u_2) = f(\mathrm{Exp}_p(u_1e_1+u_2e_2))$ is given by

$$f(u_1, u_2)=b+(1/2ks^2)(\nu_*u_1^2+\nu_m u_2^2)\sin^2\sqrt{k}\, s\,,$$

where $s^2=u_1^2+u_2^2$. Thus each level curve $L(c)$ corresponding $f=c=$constant in V or $\mathrm{Exp}_p V$ is given by

$$\nu_*u_1^2+\nu_m u_2^2=(c-b)2ks^2/\sin^2\sqrt{k}\, s\,.$$

Since $\sqrt{k}\, s \doteqdot \sin\sqrt{k}\, s$ for $s \doteqdot 0$, this level curve $L(c)$ is approximately equal to an ellipse

$$(-\nu_*)u_1^2+(-\nu_m)u_2^2=2(b-c)\,.$$

Thus we have a shrinking family of homothetic ellipses parametrized by $c\to b$ in V or in $\mathrm{Exp}_p V$. Therefore each orthogonal trajectory to this family [which is not the u_2-axis curve] tends to be tangent to the u_1-axis curve as $s\to 0$.

§ 4. A proposition on nullity distributions.

Let T be a curvature-like tensor field on (M, g). By definition T is of type (1.3) and satisfies the same algebraic relations satisfied by the Riemannian curvature tensor and the second Bianchi identity:

$$(4.1)\qquad (\nabla_X T)(W, V)+(\nabla_W T)(V, X)+(\nabla_V T)(X, W)=0\,,$$

where X, V and W are vector fields on M.

The nullity space $N_T(p)$ with respect to T at a point p of M is defined by

$$N_T(p)=\{X\in M_p\,;\ T(X, Y)=0 \quad \text{for any} \quad Y\in M_p\}\,.$$

The nullity index function $\mu_T: p\to \mu_T(p)=\dim N_T(p)$ is upper semi-continuous on M. The distribution $N_T: p\to N_T(p)$ is called the nullity distribution with respect to T. If μ_T is constant on an open set G of M, then the distribution N_T is of class C^∞ and involutive on G, and each integral submanifold of N_T is totally geodesic in G. We need a generalization of this fact. A vector field X on (M, g) is called a nullity vector field with respect to T, if X belongs to $N_T(p)$ at each point p of M.

PROPOSITION 4.1. *If X and Y are nullity vector fields with respect to a curvature-like tensor field T on (M, g), then also $\nabla_X Y$ and $\nabla_Y X$ are nullity vector fields with respect to T.*

PROOF. Let V, W, Z be arbitrary vector fields on M. By (4.1) and $X, Y \in N_T$ we obtain

$$\begin{aligned}0&=g(Y, (\nabla_X T)(Z, W)V+(\nabla_Z T)(W, X)V+(\nabla_W T)(X, Z)V)\\&=g(Y, \nabla_X(T(Z, W)V)+\nabla_Z(T(W, X)V)+\nabla_W(T(X, Z)V))\\&=g(Y, \nabla_X(T(Z, W)V))\\&=-g(\nabla_X Y, T(Z, W)V).\end{aligned}$$

Therefore $\nabla_X Y\in N_T$. Q. E. D.

COROLLARY 4.2. *Let* $X\in N_T$ *and put* $A=(\nabla_j X^i)$. *Then* $AX\in N_T$, $A^2X\in N_T$, *etc.*

PROOF. This follows from $AX=\nabla_X X$, $A^2X=\nabla_{AX}X$, etc.

REMARK 4.3. A k-nullity vector field (we are working) is a nullity vector field with respect to the following curvature-like tensor field Z_k:

$$(Z_k)^i_{jhl}=R^i_{jhl}-k(\delta^i_h g_{jl}-\delta^i_l g_{jh}).$$

§ 5. (M, g) containing a whole trajectory of F.

In this section we prove the following

THEOREM 5.1. *Let* (M, g) *be a Riemannian manifold admitting a non-constant function* f *satisfying* (1.1) *for some positive constant* k. *If* (M, g) *contains a whole trajectory* l *of* F *with its limit points in some critical submanifolds of* f, *then* (M, g) *is of constant curvature* k *at each point of* l.

Let $\{\varphi_t\}$ be a (local) 1-parameter group of (local) transformations generated by F. We put $l=\{x(t);\ -\infty<t<\infty\}$, where $x(t)=\varphi_t x(0)$ for an arbitrary point $x(0)$ of l. We define $x(-\infty)$ and $x(\infty)$ by (3.5) and (3.6). We define a (1,1)-tensor field A by ∇F. Then by (1.1) we obtain

$$L_F A^i_j=-2k((Ff)\delta^i_j+F^iF_j), \tag{5.1}$$

where L_F denotes the Lie derivation with respect to F.

There is an integer r such that

$$F,\ AF,\ \cdots,\ A^{r-1}F$$

are linearly independent at $x(0)$, and $F, AF, \cdots, A^rF$ are not linearly independent at $x(0)$.

LEMMA 5.2. *There are* C^∞-*vector fields* $\{e_\alpha;\ \alpha=1, \cdots, r\}$ *along* l *such that*

(i) *each* e_α *is invariant by* φ_t,

(ii) *each* e_α *is a linear combination of* $F, AF, \cdots, A^{\alpha-1}F$ *with functions along* l *as coefficients* (*the coefficient of* $A^{\alpha-1}F$ *being* 1).

PROOF. Since $L_F F=[F, F]=0$, we can put $e_1=F$ along l. By (5.1) and $L_F F=0$, we get

$$L_F(AF+4kfF)=0,$$

because $L_F f = Ff = g(F, F)$. Therefore $e_2 = AF + 4kfF$ is invariant by φ_t.

Assuming that there are $e_1, e_2, \cdots, e_n$ with properties (i) and (ii), we construct e_{n+1}. By (5.1) and $L_F e_n = 0$ we get

$$L_F(Ae_n) = -2k(Ff)e_n - 2k\, g(F, e_n)F .$$

We define a function $h = h(t)$ on l by

$$h(t) = \int_0^t 2k\, g(F, e_n)(x(t))dt .$$

Then e_{n+1} defined by

$$e_{n+1} = Ae_n + 2kfe_n + hF$$

is what we wanted. Therefore we obtain $\{e_\alpha ; \alpha = 1, \cdots, r\}$ along l with properties (i) and (ii). Q. E. D.

REMARK 5.3. The construction of $\{e_\alpha\}$ in Lemma 5.2 shows that the integer r is independent of the choice of point $x(0)$. In particular, $A^r F$ is expressed as a linear combination of $F, AF, \cdots A^{r-1}F$ at each point of l.

REMARK 5.4. $\{e_\alpha\}$ defines an r-dimensional distribution D along l such that D is invariant by φ_t and A. By Corollary 4.2 and Proposition 2.1, D is contained in the k-nullity space at each point of l.

LEMMA 5.5. *The distribution $D^\perp$ along l orthocomplementary to D is also invariant by φ_t and A.*

PROOF. Since $A = (\nabla_j \nabla^i f)$ is symmetric with respect to g, $D^\perp$ is also invariant by A. To show that $D^\perp$ is invariant by φ_t, first we show $L_F Y \in D^\perp$ for each $Y \in D^\perp$. Operating L_F to $g(e_\alpha, Y)$ and noticing that $L_F g = (2\nabla_j F_i)$, we get

$$2g(Ae_\alpha, Y) + g(e_\alpha, L_F Y) = 0 .$$

Since $Ae_\alpha \in D$, we get $L_F Y \in D^\perp$. Next, let $Z_{x(0)}$ be an arbitrary tangent vector which belongs to $D^\perp_{x(0)}$. Define a vector field Z along l by $Z_{x(t)} = \varphi_t Z_{x(0)}$, where φ_t also denotes its differential. Let

$$Z = Z_1 + Z_2 \in D + D^\perp$$

be the decomposition of Z. Since $L_F Z = 0$, we get

$$L_F Z_1 + L_F Z_2 = 0 .$$

Since $L_F Z_1 \in D$ and $L_F Z_2 \in D^\perp$, we get $L_F Z_1 = 0$. Since Z_1 vanishes at $x(0)$, $Z_1 = 0$ along l. Thus $Z = Z_2 \in D^\perp$, and $D^\perp$ is invariant by φ_t. Q. E. D.

LEMMA 5.6. *There is a field of orthogonal basis $\{e_u ; u = r+1, \cdots, m\}$ of $D^\perp$ such that*

(i) *each e_u is invariant by φ_t,*

(ii) *for each e_u there is a constant c_u satisfying*

$$Ae_u = -2k(c_u + f)e_u , \tag{5.2}$$

(iii) $\{e_u\}$ *is orthonormal at* $x(0)$.

PROOF. Let C_u $(u=r+1, \cdots, m)$ be eigenvalues of A restricted to $D^{\perp}$ at $x(0)$ and let $\{(e_u)_{x(0)}\}$ be an orthonormal base of $D^{\perp}$ at $x(0)$ such that

$$A(e_u)_{x(0)}=C_u(e_u)_{x(0)}.$$

For each u we define a constant c_u by $C_u=-2k(c_u+f(x(0)))$, and e_u by $(e_u)_{x(t)}=\varphi_t(e_u)_{x(0)}$. By (5.1) we get

$$L_F(Ae_u+2k(c_u+f)e_u)=0,$$

because $g(F, e_u)=0$. Therefore $Ae_u+2k(c_u+f)e_u$ is invariant by φ_t. Since it vanishes at $x(0)$, it vanishes at each point of l. Thus we get (ii). Finally we show that $\{e_u\}$ is orthogonal. We operate L_F to $g(e_u, e_v)$, where $u\neq v$ and $r+1\leqq u, v\leqq m$. Then

$$L_F(g(e_u, e_v))=2g(Ae_u, e_v)$$
$$=-4k(c_u+f)g(e_u, e_v).$$

This is an ordinary differential equation with respect to $g(e_u, e_v)$. Since $g(e_u, e_v)$ vanishes at $x(0)$, the uniqueness of the solution implies that $g(e_u, e_v)=0$ along l.

Q. E. D.

Now we have obtained a field of φ_t-invariant frames along l;

$$\{e_i\}=\{e_\alpha, e_u;\ 1\leqq\alpha\leqq r,\ r+1\leqq u\leqq m\}.$$

Let $\{w^i\}$ be the field of dual frames of $\{e_i\}$ along l;

$$w^i(e_j)=\delta^i_j.$$

By operating L_F to the both sides of the last equation, we see that each 1-form w^i along l is also invariant by φ_t.

Let P be the Weyl projective curvature tensor of (M, g). By (P^i_{jhl}) we denote the components of P with respect to $\{e_i\}$ along l;

$$P^i_{jhl}=w^i(P(e_j, e_h, e_l)).$$

Since φ_t is projective (cf. Proposition 2.1), P is invariant by φ_t. Since e_i and w^i are also invariant by φ_t, P^i_{jhl}'s are constant along l.

LEMMA 5.7. $P^u_{vwz}=0$ *for* $r+1\leqq u, v, w, z\leqq m$.

PROOF. We define E_u and W^u, $u=r+1, \cdots, m$, by

$$E_u=e_u/|e_u|,$$
$$W^u=|e_u|w^u.$$

Then $\{E_u\}$ is field of orthonormal basis of $D^{\perp}$ along l, and $\{W^u\}$ is its dual. We assume that there are u, v, w, z such that $P^u_{vwz}\neq 0$ and we consider

$$(5.3)\qquad W^u(P(E_v,\ E_w,\ E_z))=\frac{|e_u|}{|e_v|\,|e_w|\,|e_z|}P^u_{vwz}$$

to induce a contradiction. First we claim that the left hand side of (5.3) is bounded on l. By $|P|^2$ we denote the square of the norm of P. Then $|P|^2 =\sum(P^i_{jhl})^2$ for the components of P with respect to an arbitrary orthonormal frame at a point where we consider $|P|^2$. Since P is a tensor field on (M, g) and $x(-\infty)\cup l\cup x(\infty)$ is compact, $|P|^2$ is bounded on l. Since

$$(W^u(P(E_v,\ E_w,\ E_z)))^2\leqq|P|^2,$$

the left hand side of (5.3) is bounded on l.

Therefore if we show that

$$(5.4)\qquad \lim Q(t)=\infty \qquad (\text{as } t\to\infty \text{ or } t\to-\infty)$$

for $Q=|e_u|^2|e_v|^{-2}|e_w|^{-2}|e_z|^{-2}$, then (5.3) gives a contradiction. Since

$$L_F|e_u|^2=2g(Ae_u,\ e_u)$$
$$=-4k(c_u+f)|e_u|^2,$$

etc., we obtain

$$(5.5)\qquad L_FQ=dQ/dt=4k(2f-c_u+c_v+c_w+c_z)Q\,.$$

By b_0 and a_0 we denote the critical value of f; $f(x(\infty))=b_0$ and $f(x(-\infty))=a_0$. As the first case we assume

$$4(2b_0-c_u+c_v+c_w+c_z)>0\,.$$

Then we have some positive numbers ε and t_1 such that

$$4k(2f-c_u+c_v+c_w+c_z)>\varepsilon$$

holds for all $t>t_1$, since $f(t)$ is increasing and $f(t)\to b_0$ as $t\to\infty$. Therefore

$$(L_FQ)/Q>\varepsilon$$

holds for all $t>t_1$, and

$$Q(t)>(\text{non-zero constant})e^{\varepsilon t}\,.$$

This means that $Q(t)\to\infty$ as $t\to\infty$.

Finally we assume

$$4(2b_0-c_u+c_v+c_w+c_z)\leqq 0\,.$$

Then

$$-4(2a_0-c_u+c_v+c_w+c_z)\geqq 8(b_0-a_0)\,.$$

In this case we change the parameter $t\to{}'t=-t$. Then in (5.5) only $(dt\to d't)$ changes sign and hence

$$dQ('t)/d't=-4k(2f('t)-c_u+c_v+c_w+c_z)Q('t)\,.$$

As $'t\to\infty$, $f('t)$ is decreasing and $f('t)\to a_0$. Therefore we have some positive numbers ε $(<8(b_0-a_0))$ and t_2 such that

$$\begin{aligned}-4(2f('t)-c_u+c_v+c_w+c_z)&>-4(2a_0-c_u+c_v+c_w+c_z)-\varepsilon\\&\geqq 8(b_0-a_0)-\varepsilon\end{aligned}$$

holds for all $'t>t_2$. Therefore $Q('t)\to\infty$ as $'t\to\infty$ or $t\to-\infty$. Thus we obtain (5.4), and this completes the proof.

PROOF OF THEOREM 5.1. Let R^i_{jhl} be the components of the Riemannian curvanture tensor R with respect to $\{e_i\}=\{e_\alpha, e_u\}$ along l. Since each e_α belongs to the k-nullity distribution of (M, g) along l (cf. Remark 5.4), if at least one index (for example $h=\alpha$) of i, j, h, l is smaller than $r+1$, then

$$R^i_{j\alpha l}=k(\delta^i_\alpha g_{jl}-\delta^i_l g_{j\alpha})\,. \tag{5.6}$$

In particular we obtain

$$\sum_{\alpha=1}^{r} R^\alpha_{v\alpha z}=rkg_{vz} \tag{5.7}$$

where $r+1\leqq v,\ z\leqq m$. On the other hand, $P^u_{vwz}=0$ implies

$$R^u_{vwz}=(1/(m-1))(\delta^u_w R_{vz}-\delta^u_z R_{vw})\,. \tag{5.8}$$

where (R_{jl}) denotes the Ricci tensor. Therefore

$$\sum_{u=r+1}^{m} R^u_{vuz}=(1/(m-1))(m-r-1)R_{vz}\,. \tag{5.9}$$

Adding (5.7) and (5.9) we obtain

$$R_{vz}=(1/(m-1))(m-r-1)R_{vz}+rkg_{vz}\,,$$

from which we obtain

$$R_{vz}=(m-1)kg_{vz}\,. \tag{5.10}$$

By (5.6), (5.8) and (5.10), we see that (M, g) is of constant curvature k at each point $x(t)$ of l.

THEOREM 5.8. *In Theorem 5.1, let $x(\infty)$ and $x(-\infty)$ be limit points of l. If f takes its maximum value at $x(\infty)$ and its minimum value at $x(-\infty)$, then (M, g) contains an open set W containing l so that (W, g) is of constant curvature k.*

PROOF. Let w_1 be a point of l near $x(\infty)$. Then there is an open neighborhood U_1 of w_1 such that $\{\varphi_t U_t\,;\,0<t<\infty\}$ is contained in M (cf. § 3). Similarly for a point w_2 of l near $x(-\infty)$, we have an open neighborhood U_2 of w_2 such that $\{\varphi_t U_2\,;\,-\infty<t<0\}$ is contained in M. The existence of such U_1 and U_2 shows that a trajectory of F passing through a point z near l lies near l and comes from some point of M^a near $x(-\infty)$ and tends to some point of M^b near $x(\infty)$. Therefore there is an open set W containing l so that (W, g) is of constant curvature k, by Theorem 5.1.

PROOF OF THEOREM A. If (M, g) is complete, by the behavior of trajectories of F studied in § 3 and by Theorem 5.1, Theorem A is verified.

§ 6. Examples.

Let $(B, {}^*g)$ be an $(m-1)$-dimensional Riemannian manifold and let $I=(-\pi/2, \pi/2)$ be an open interval of the real line. On $I\times B$ we define a warped product metric g by

$$ds^2=dt^2+\cos^2 t\, d^*s^2\,. \tag{6.1}$$

Then the function h on $I\times B$ defined by

$$h(t, x)=h(t)=\sin t \tag{6.2}$$

is a special concircular field on $(I\times B, g)$, that is, it satisfies

$$\nabla_j\nabla_i h=-h\, g_{ji} \tag{6.3}$$

(cf. for example, Y. Tashiro [22], p. 254). If we put $f=h^2$, then f satisfies (1.1) with $k=1$.

(i) Let $(S^{m-1}, {}^*g)$ be a totally geodesic sphere of a Euclidean sphere (S^m, g_0) of constant curvature 1. Denoting by N_0 and S_0 the north and south poles of S^m, we obtain

$$S^m-N_0-S_0=I\times S^{m-1}\,.$$

Notice that the metric g_0 on $S^m-N_0-S_0$ is the same as ds_0^2 defined by the right hand side of (6.1). Define a function h on S^m by $h=\sin t$ on $I\times S^{m-1}$ and $h(N_0)=1$, $h(S_0)=-1$. h is of class C^∞ and satisfies (6.3) on (S^m, g_0).

Let U be a sufficiently small simple open set in S^{m-1}, and let α be a non-constant positive function on S^{m-1} such that α takes value 1 outside U. By $Cl\ U$ we denote the closure of U.

Removing $[-\pi/3, -\pi/6]\times Cl\ U$ and $[\pi/6, \pi/3]\times Cl\ U$ from S^m and replacing the metric ds_0^2 on $((-\pi/6, \pi/6)\times U, ds_0^2)$ by $dt^2+(\cos^2 t)\alpha d^*s^2$, we get a Riemannian manifold (M, g) of dimension m. By the same letter h we denote the restriction of h on S^m to M. Then h satisfies (6.3) also on (M, g). Summarizing the properties of (M, g) we get

(i-1) (M, g) admits a non-constant function $f=h^2$ satisfying (1.1) with $k=1$,

(i-2) there is a point z in S^{m-1} such that (M, g) contains the closed $(\pi/2)$-neighborhood of z in M,

(i-3) (M, g) is not of constant curvature k (in $(-\pi/6, \pi/6)\times U$).

REMARK 6.1. Example (i) is a counter-example to the lemma of a paper [9] by S. Gallot.

(ii) In example (i), consider an open submanifold

$$(S^m-[-\pi/3, \pi/3]\times Cl\ U, g_0=g)$$

of (M, g). Then each trajectory of grad f in this manifold has N_0 or S_0 as its limit point. This property is generalized to the concept of t-connectedness.

§ 7. t-connectedness.

DEFINITION 7.1. Let X be a vector field on a manifold M. M is called to be *t-connected* (i.e., trajectory-connected) *with respect to* X, if for any two different points x and y of M, there is a piecewise C^∞-curve $l(x, y)$ joining x and y such that

(i) except a finite number of points $(p_1, \cdots, p_j)$ of $l(x, y)$, $l(x, y)$ is composed of trajectories of X,

(ii) $p_1, \cdots, p_j$ are singular points (i.e., vanishing points) of X, and hence they are limit points of the trajectories of X in $l(x, y)$.

REMARK 7.2. Let f be a function on a Riemannian manifold (M, g) and let q be an isolated singular point of grad f. If f takes a local maximum (or local minimum) at q, then some neighborhood of q in M is t-connected with respect to grad f.

DEFINITION 7.3. Let $X_1, \cdots, X_a$ be vector fields on M. M is called *t-connected with respect to* $(X_1, \cdots, X_a)$, if for any two different points x and y of M, there is a piecewise C^∞-curve $l(x, y)$ joining x and y such that

(i) except a finite number of points $(p_1, \cdots, p_j, q_1, \cdots, q_h)$ of $l(x, y)$, $l(x, y)$ is composed of some trajectories of $X_1, \cdots, X_a$,

(ii) each of $p_1, \cdots, p_j$ is a singular point of some of $X_1, \cdots, X_a$,

(iii) each of $q_1, \cdots, q_h$ is the intersection of some two trajectories of $X_1, \cdots, X_a$.

We prepare about nullity theory for the proof of the main Theorem in this section (Theorem 7.5). Let N_T be the nullity distribution with respect to a curvature-like tensor field T on (M, g) (cf. § 4) and let μ_T be the index function of nullity of T. The minimum value μ_T^0 of μ_T on (M, g) is called the index of nullity of T on (M, g). The subset M^0 of M composed of all points where $\mu_T=\mu_T^0$ holds is called the nullity set of T. Since μ_T is upper semi-continuous, M^0 is open in M. Each leaf (maximal integral submanifold) of N_T is totally geodesic in M^0.

The completeness theorem of nullity foliations by N_T is stated as follows: If (M, g) is complete, then each leaf of N_T on M^0 is also complete (cf. K. Abe [**1**], Y.H. Clifton and R. Maltz [**5**], D. Ferus [**7**], etc.).

What is proved in this completeness theorem is the following.

THEOREM 7.4 (*Local form of completeness theorem*). *Let* $\{x(s);\ c\leqq s\leqq b\}$ *be a geodesic in* (M, g) *with arc-length parameter* s, *such that* $\{x(s);\ c\leqq s<b\}$ *is*

contained in a leaf L of N_T on M^0. Then $x(b)\in L$, too.

We apply this to the following.

THEOREM 7.5. *Let X be a nullity vector field of a curvature-like tensor field T on (M, g). If some open set U in M is t-connected with respect to X, $T=0$ holds on U.*

In particular, if $T=Z_k$, (U, g) is of constant curvature k.

PROOF. Let μ^0 be the index of nullity of T on U and let U^0 be the nullity set of T in (U, g). Since U is t-connected with respect to X and since U^0 is open, we get $\mu^0\geqq 1$. Let x be an arbitrary point of U^0 such that X does not vanish at x, and let L be the leaf of the nullity distribution N_T passing through x. We claim that $L=U^0=U$.

Let y be an arbitrary point of U. By t-connectedness of U, we have a piecewise C^∞-curve $l(x, y)$ joining x and y in U, which is composed of trajectories of X except a finite number of points $p_1, \cdots, p_j$. We show that $l(x, y)$ is contained in L. By our choice of x, we get $x\neq p_1$. We denote the portion of $l(x, y)$ from x to p_1 by $[x\ p_1]$. By $[x\ p_1)$ we mean $[x\ p_1]-p_1$. $[x\ p_1)$ is a part of a trajectory of X. Since $X\in N_T$, the connected component $[x\ z)$ of $[x\ p_1)\cap U^0$ containing x is contained in L. We prove $z\in L$.

(1) If $[x\ p_1)$ is geodesic, $z\in L$ follows from Theorem 7.4.

(2) If $[x\ z)$ is not geodesic, then $\mu^0\geqq 2$. Let $B_\varepsilon(z)$ be an ε-ball neighborhood of z in M, where ε is sufficiently small so that $B_\varepsilon(z)$ is convex. Each geodesic in $L\cap B_\varepsilon(z)$ can be prolonged to a geodesic in $B_\varepsilon(z)$, which has the limit points in the boundary of $B_\varepsilon(z)$. By Theorem 7.4 again, this prolonged geodesic is contained in L. This means that L has no boundary points in $B_\varepsilon(z)$. In particular $z\in L$.

Consequently, we obtain $z=p_1$ and $p_1\in L$. Since U^0 is open in M some neighborhood of p_1 is contained in U^0 and hence some part of $(p_1\ p_2)$ is contained in L. Continuing the above argument we see that $[p_1\ p_2]$ is contained in L. And finally we see that $l(x, y)$ is contained in L. Thus, $U=L$ and $T=0$ holds on U.

THEOREM 7.6. *Let $X_1, \cdots, X_a$ be nullity vector fields of a curvature-like tensor field T on (M, g). If some open set U in M is t-connected with respect to $X_1, \cdots, X_a$, then $T=0$ holds on U.*

Proof is given by a slight modification of that of Theorem 7.5.

§ 8. Local theorems on (1.1).

By Theorem 7.5 we obtain

COROLLARY 8.1. *Let (M, g) be a Riemannian manifold admitting a function f satisfying* (1.1) *for some positive constant k. If M (or an open subset U of M) is t-connected with respect to* grad f, *then (M, g) (or (U, g), resp.) is of* *constant curvature k.*

SECOND PROOF OF THEOREM A. Assume that a complete Riemannian manifold (M, g) admits a non-constant function f satisfying (1.1) for some positive constant k. Then M is compact as was shown in § 3 and M is expressed as $M=W(^*M^b)$ or $M=W(p)\cup W(q)$ under the notations in § 3. Since the limit points of each trajectory of $F=\text{grad } f$ are critical points of f, it is easy to see that M is t-connected with respect to F. This gives the second proof of Theorem A.

THEOREM 8.3. *Let (M, g) be a Riemannian manifold admitting a non-constant function f satisfying* (1.1) *for some positive constant k. Assume that there is a point of M where f takes its maximum value b. Let M^b be the subset of M of all critical points of f where $f=b$ holds and let $^*M^b$ be a connected component of M^b. If* $\dim {}^*M^b \leqq 1$ *then there is an open set U containing $^*M^b$ such that (U, g) is of constant curvature k.*

PROOF. Since the set of all critical points of f is of measure zero and $F=\text{grad } f$ is a k-nullity vector field on (M, g), the index of k-nullity of (M, g) is greater than or equal to one.

Let y be an arbitrary point of $^*M^b$. Since the normal space to $^*M^b$ at y is contained in the k-nullity space (cf. Corollary 3.5), the index of k-nullity at y is equal to $m-\dim {}^*M^b \geqq m-1$. This means that the index of k-nullity at each point of $^*M^b$ is equal to m. Since there is no critical points near $^*M^b$ (except points of $^*M^b$), there is an open set U in M containing $^*M^b$ such that for each point z in U the trajectory of F passing through z tends to some point of $^*M^b$. Let w be an arbitrary point which belongs to the k-nullity set U^0 of (U, g), and let L be the leaf of the k-nullity distribution on U^0 passing through w. Then we can show that L meets $^*M^b$ just by the same way as in the proof of Theorem 7.5. Therefore (U, g) is of constant curvature k.

§ 9. Applications.

(i) From Theorem A we obtain

THEOREM 9.1 (T. Nagano [13]). *Let (M, g) be a complete Einstein space of positive constant scalar curvature S. If (M, g) admits an infinitesimal non-affine projective transformation, then (M, g) is of constant curvature $k=S/m(m-1)$.*

Or more generally,

THEOREM 9.2. *Let (M, g) be a complete Riemannian manifold with positive constant scalar curvature $S=m(m-1)k$. If (M, g) admits an infinitesimal non-affine projective transformation which leaves the gravitational tensor field $G=(R_{ji}-(S/m)g_{ji})$ invariant, then (M, g) is of constant curvature k.*

This follows from the following.

PROPOSITION 9.3. *Assume that (M, g) has positive constant scalar curvature*

$S=m(m-1)k$. Then the existence of a non-constant function f satisfying (1.1) *on M is equivalent to the existence of an infinitesimal non-affine projective transformation X on (M, g) which leaves the gravitational tensor field G invariant.*

Proof is standard (S. Tanno [21]) and we omit it here. We only give the relation between f and X; $f \to X=\mathrm{grad} f$ and $X \to f=-\nabla_r X^r/2(m+1)$ (cf. also, K. Yano [23], p. 271).

(ii) A Killing vector field ξ of unit length on a Riemannian manifold (M, g) is called a Sasakian structure if it is a 1-nullity vector field on (M, g). (M, g) admitting a Sasakian structure is called a Sasakian manifolds.

THEOREM 9.4 (S. Tachibana and W. N. Yu [17]). *If a complete Riemannian manifold (M, g) admits two Sasakian structure ξ and η such that $f=g(\xi, \eta)$ is not constant, then f satisfies* (1.1) *with $k=1$ and (M, g) is of constant curvature* 1.

This theorem is useful in the study of isometry groups of Sasakian manifolds, etc. (cf. S. Tanno [18], [19]).

§ 10. The case of Kählerian manifolds.

Let (M, J, g) be a Kählerian manifold of dimension $m=2n \geqq 4$. The structure tensors J (almost complex structure tensor) and g (Kählerian metric tensor) satisfy the following.

$$J^2X=-X, \qquad \nabla J=0,$$
$$g(JX, JY)=g(X, Y)$$

for all vector fields X and Y on M.

A Kählerian manifold (M, J, g) is of constant holomorphic sectional curvature β at x, if and only if

$$R^i_{jhl}-(\beta/4)(\delta^i_h g_{jl}-\delta^i_l g_{jh}-J^i_h J_{lj}+J^i_l J_{hj}+2J_{hl}J^i_j)=0 \tag{10.1}$$

holds at x, where $J_{hj}=g_{hr}J^r_j$.

For a positive constant β we define a tensor field E of type (1,3) by

$$E=(E^i_{jhl})=(\text{the left hand side of (10.1)}).$$

Then E is a curvature-like tensor field on (M, J, g), and it satisfies

$$E^i_{jhl}J^h_r J^l_s=E^i_{jrs}, \tag{10.2}$$

etc. The holomorphic β-nullity space HN_x at x, the holomorphic β-nullity distribution HN, etc. are naturally defined. By (10.2) NH_x is invariant by J. The holomorphic sectional curvature with respect to a non-zero $X \in HN_x$ is equal to β.

Let $(CP^n, J, g_0; \beta)$ be a complex n-dimensional projective space with the Fubini-Study metric of constant holomorphic sectional curvature β. Then the first eigenvalue of the Laplacian on $(CP^n, J, g_0; \beta)$ is $(n+1)\beta$ and each eigenfunction f corresponding to $(n+1)\beta$ satisfies

$$\nabla_h\nabla_j\nabla_i f+(\beta/4)(2\nabla_h f g_{ji}+\nabla_j f g_{ih}+\nabla_i f g_{jh} +(J_j^s J_i^r+J_i^s J_j^r)\nabla_r f g_{hs})=0\,. \tag{10.3}$$

The following theorem was announced by M. Obata [**15**].

THEOREM 10.1. *Let (M, J, g) be a complete Kählerian manifold. In order for (M, J, g) to admit a non-constant function f satisfying* (10.3) *for some positive constant β, it is necessary and sufficient that (M, J, g) is holomorphically isometric to a $(CP^n, J, g_0\,;\,\beta)$.*

REMARK 10.2. Restricting (10.3) to a geodesic $\{x(s)\}$ we get the differential equation

$$f'''+\beta f'=0\,.$$

The case $\beta=4$ corresponds to $k=1$ in the Riemannian case, and so the local behavior of trajectories of $F=\mathrm{grad}\,f$ is quite the same as in the Riemannian case (§ 2, § 3).

A vector field X on (M, J, g) is called holomorphically projective, if

$$L_X J_j^i=-\nabla_r X^i J_j^r+\nabla_j X^r J_r^i=0\,, \tag{10.4}$$

$$L_X \Gamma_{jh}^i=\rho_j\delta_h^i+\rho_h\delta_j^i-J_h^i J_j^r\rho_r-J_h^r J_j^i\rho_r \tag{10.5}$$

for some function ρ, where $\rho_j=\nabla_j\rho$.

PROPOSITION 10.3. *Let f be a function on a Kählerian manifold (M, J, g). f satisfies* (10.3) *for a non-zero constant β, if and only if*

(i) *$F=\mathrm{grad}\,f$ is holomorphically projective,*

(ii) *F is a holomorphic β-nullity vector field on (M, J, g).*

PROOF. First we assume that non-constant function f satisfies (10.3) for a constant $\beta\neq 0$. By the Ricci identity for $\nabla_l\nabla_h F_j-\nabla_h\nabla_l F_j$, we get

$$F_i E_{jhl}^i=0\,.$$

This proves (ii). Applying this to (2.2) we obtain

$$L_F\Gamma_{jh}^i=-(\beta/2)(F_j\delta_h^i+F_h\delta_j^i-J_h^i J_j^r F_r-J_h^r J_j^i F_r)\,. \tag{10.6}$$

This proves (10.5) with $\rho=-(\beta/2)f$. By (10.3) we can verify

$$J_j^r\nabla_h\nabla_r F_i+J_i^r\nabla_h\nabla_r F_j=0\,.$$

This means that $J_j^r\nabla_r F_i+J_i^r\nabla_r F_j$ is a parallel symmetric (0,2)-tensor field. The existence of a non-trivial β-nullity vector field F implies that (M, g) is irreducible. So $J_j^r\nabla_r F_i+J_i^r\nabla_r F_j$ is proportional to g_{ji}. Transvecting this (0,2)-tensor field by g^{ij}, we see that $J_j^r\nabla_r F_i+J_i^r\nabla_r F_j=0$. So we obtain (10.4) with $X=F$ and hence (i).

The converse is proved by the method similar to the proof in Proposition

2.1. Q. E. D.

REMARK 10.4. If (M, J, g) is complete and admits a non-constant function f satisfying (10.3) for some positive constant β, we see that M is t-connected with respect to $F=\operatorname{grad} f$ by Remark 10.2. Therefore, (M, J, g) is of constant holomorphic sectional curvature by Theorem 7.5 and Proposition 10.3. Since a complete (M, J, g) of positive constant holomorphic sectional curvature is simply connected, (M, J, g) is holomorphically isometric to a $(CP^n, J, g_0; \beta)$.

This proves Theorem 10.1.

THEOREM 10.5. *Let (M, J, g) be a Kählerian manifold admitting a non-constant function f satisfying* (10.3) *for some positive constant β. If (M, J, g) contains a whole trajectory l of $F=\operatorname{grad} f$ with its limit points, then (M, J, g) is of constant holomorphic sectional curvature β at each point of l.*

The analogy of Theorem 5.8 *is also true.*

Proof is quite similar to that of Theorem 5.1, and so we give only an outline of the proof. We write $l=\{x(t)=\varphi_t x(0), -\infty<t<\infty\}$ as in the proof of Theorem 5.1. We define A by ∇F. Then $AJ=JA$ holds by (10.4). Assume that

$$F, JF, AF, JAF, \cdots, A^{r-1}F, JA^{r-1}F$$

are linearly independent at $x(0)$ and $F, JF, \cdots, A^{r-1}F, JA^{r-1}F, A^rF$ are linearly dependent at $x(0)$. By (10.3) we obtain

$$L_F A_j^i = -(\beta/2)((Ff)\delta_j^i + F^i F_j + (JF)^i (JF)_j). \tag{10.7}$$

By (10.7) we can construct φ_t-invariant vector fields

$$e_1=F, Je_1, e_2, Je_2, \cdots, e_r, Je_r$$

along l. So we have a $(2r)$-dimensional distribution D along l, which is invariant by φ_t, A, and J. By Corollary 4.2 and (10.2), we see that D is contained in the holomorphic β-nullity distribution HN at each point of l.

By $D^\perp$ we denote the distribution along l orthocomplementary to D. $D^\perp$ is also invariant by φ_t, A, and J.

Since φ_t is holomorphically projective, it leaves the holomorphically projective curvature tensor $Q=(Q_{jhl}^i)$ invariant (cf. for example, K. Yano [24], Chapter 7);

$$\begin{aligned} Q_{jhl}^i = R_{jhl}^i - (1/2(n+1))(&\delta_h^i R_{jl} - \delta_l^i R_{jh} \\ &- J_h^i J_j^s R_{ls} + J_l^i J_j^s R_{hs} + J_l^s J_j^i R_{hs} - J_h^s J_j^i R_{ls}). \end{aligned} \tag{10.8}$$

$Q=0$ at x is equivalent to $E=0$ at x. The rest of the proof is given by the natural modification of the proof of Theorem 5.1.

COROLLARY 10.6. *Let (M, J, g) be a complete Kähler-Einstein space with positive constant scalar curvature $S=n(n+1)\beta$. In order for (M, J, g) to admit a non-affine holomorphically projective vector field X, it is necessary and sufficient* *that (M, J, g) is holomorphically isometric to a $(CP^n, J, g_0; \beta)$.*

PROOF. In fact, for a holomorphically projective vector field X on a Kähler-Einstein space, $\delta X=(-\nabla_r X^r)$ satisfies (10.3) (cf. S. Tachibana [**16**], p. 50). So Corollary 10.6 follows from Theorem 10.1.

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