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SOME *E*-OPTIMAL BLOCK DESIGNS

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When a BIBD or a Group Divisible design with $\lambda_2 = \lambda_1 + 1$ is extended by certain disjoint and binary blocks the resulting structure is proved E-optimal. A BIBD abridged by a certain number of such blocks is also shown E-optimal. These optimality results hold over the class of all block designs (with the respective sets of parameters). Proofs rely mainly on averaging information matrices, which proves useful in many settings related to design optimality.

1. Introduction. We are given v varieties labeled $1, 2, \cdots, v$. These v varieties are to be compared via b blocks of size k each, with $k < v$. Any arrangement of the v varieties in the b blocks is called a design. A design can also be thought of as a $k \times b$ array, d, with varieties as entries and blocks as columns.

Let $\Omega_{v,b,k}$ denote the collection of all designs with parameters v, b, and k. The usual additive model specifies the expectation of an observation on variety i in block j as $\alpha_i + \beta_j$, where α_i is the unknown effect of the ith variety and β_j is the (unknown) effect of the jth block. The kb observations are assumed uncorrelated with common variance σ^2 (usually unknown).

The information matrix for the variety effects, when the design d is used, is known to be

$$C_d = \operatorname{diag}(r_{d1}, \cdots, r_{dv}) - k^{-1} N_d N_d'$$

where r_{di} is the number of replications of variety i in d and $N_d = (n_{dij})$, with n_{dij} signifying the number of times variety i appears in block j. For convenience we denote $\sum_{u=1}^{b} n_{diu} n_{dju}$ by λ_{dij}. The $v \times v$ matrix C_d contains all the relevant statistical information relating to the variety effects. It is well-known that C_d is nonnegative definite, has nonpositive off-diagonal elements and has row sums zero for all $d \in \Omega_{v,b,k}$.

For a design $d \in \Omega_{v,b,k}$ let $0 = \mu_{d0} \leq \mu_{d1} \leq \cdots \leq \mu_{d,v-1}$ denote the eigenvalues of its information matrix C_d. We call a design d connected if its information matrix C_d has rank $v - 1$. A design d is called equireplicated if $r_{d1} = \cdots = r_{dv}$. A block of d is said to be binary if it consists of distinct varieties; d is said to be binary if all its blocks are binary.

A design $d^* \in \Omega_{v,b,k}$ is called E-optimal if $\mu_{d^*1} \geq \mu_{d1}$ for all designs $d \in \Omega_{v,b,k}$. The following well-known lemma (see Kiefer (1959) and Ehrenfeld (1955)) gives statistical meaning to an E-optimal design.

LEMMA 1.1. *A design d^* is E-optimal if and only if the maximal variance among all best linear unbiased estimators of normalized linear contrasts is minimal under d^*.*

2. On averaging and convexity. Due to Lemma 1.1 our aim becomes that of finding a design $d^* \in \Omega_{v,b,k}$ for which $\mu_{d^*1} \geq \mu_{d1}$ holds, where d ranges over all of $\Omega_{v,b,k}$. Because of the large variety of information matrices this comparison becomes difficult, even though a choice for d^* is usually available. To make the comparison possible we rely on an

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intermediate device, an averaged version $\bar{C}_d$ of the information matrix C_d, which usually shrinks the spectrum of C_d and allows a successful comparison. Before we show how $\bar{C}_d$ is derived from C_d we need another lemma.

Let A be a $v \times v$ nonnegative definite matrix with zero row sums and $\{\sigma_i\}$ a collection of n permutations on the symbols $1, 2, \cdots, v$. Define $\bar{A} = \frac{1}{n} \sum_{i=1}^{n} A^{\sigma_i}$, where $A^{\sigma_i} = P_i A P_i'$, with P_i representing the $v \times v$ matrix representation of σ_i. Denote by $\mu_{v-1} \geq \cdots \geq \mu_1 \geq \mu_0 = 0$ and $\bar{\mu}_{v-1} \geq \cdots \geq \bar{\mu}_1 \geq \bar{\mu}_0 = 0$ the eigenvalues of A and $\bar{A}$ respectively.

LEMMA 2.1. *$\bar{A}$ is nonnegative definite with zero row sums,* Trace $\bar{A}$ = Trace A, $\mu_1 \leq \bar{\mu}_1$ *and* $\mu_{v-1} \geq \bar{\mu}_{v-1}$.

PROOF. The first two statements are immediate. The last two inequalities follow easily after observing that $\mu_1 = \min_x x'Ax/x'x$, where the components of x sum up to zero, and that $\mu_{v-1} = \max_x x'Ax/x'x$, with no restriction on x. This concludes the proof.

$\bar{A}$ is called an average version of A. Schur-convex functions defined on the nonzero eigenvalues of A and $\bar{A}$ satisfy inequalities as above (see Magda (now Constantine) (1979), Lemma 3.2.1).

In order to show that a design d^* is E-optimal, by Lemma 2.1 it is enough to show that the minimal eigenvalue of some averaged version of C_d does not exceed the minimal eigenvalue of C_{d^*}. Kiefer (1975, Proposition 1) relied on averaging information matrices to prove the universal optimality of a design with a completely symmetric information matrix of maximal trace. For certain optimal candidate d^* the matrix C_{d^*} consists of diagonal blocks of completely symmetric matrices (i.e., matrices with their diagonal entries equal and all the off-diagonal entries equal) and off-diagonal blocks consisting of matrices with equal entries. We would like our averaged versions of C_d to have the same shape for an easy comparison (for details see Magda (1979) and Constantine (1980)).

Let $i_1, i_2, \cdots, i_p$ be a subset of the varieties $1, 2, \cdots, v$. We say that we average C_d over the varieties $i_1, i_2, \cdots, i_p$ if we take in Lemma 2.1 $A = C_d$ and P_i ($i = 1, 2, \cdots, p!$) the $v \times v$ matrix representation of the symmetric group on the symbols $i_1, i_2, \cdots, i_p$ extended by identity to the rest of the varieties. We usually average separately over two (and sometimes more) disjoint subsets of $1, 2, \cdots, v$. If we average separately over m such disjoint subsets of sizes p_i, $1 \leq i \leq m$, we would have $p_1! p_2! \cdots p_m!$ permutation matrices to use in Lemma 2.1. They would correspond to the matrix representation of the direct sum of the m symmetric groups of the disjoint subsets under consideration. When the m subsets partition the set of the v varieties, this averaging process transforms C_d into a matrix $\bar{C}_d$ which consists of completely symmetric blocks along the diagonal and blocks with equal entries elsewhere. Computing the eigenvalues of $\bar{C}_d$ is often a tractable task and in view of Lemma 2.1 these eigenvalues help us relate a (hopefully E-optimal) design d^* to an arbitrary design d. We now find out what they are.

Whenever the dimensions are clear from the context we simply write I for the identity matrix and J for a (not necessarily square) matrix with all its entries 1. The vector with all its entries 1 is denoted by $\mathbf{1}$.

LEMMA 2.2. *The matrix $\bar{C}_d$ obtained by averaging an information matrix C_d separately over varieties $1, 2, \cdots, p$ and $p+1, p+2, \cdots, v$ is of the form*

$$\bar{C}_d = \frac{1}{k} \begin{bmatrix} (\bar{a} + \bar{\alpha})I - \bar{\alpha}J & -\bar{\beta}J \\ -\bar{\beta}J & (\bar{b} + \bar{\gamma})I - \bar{\gamma}J \end{bmatrix}$$

with the submatrix in the upper left-hand corner of dimension $p \times p$. $k\bar{C}_d$ has eigenval-

ues: $\bar{a} + \bar{\alpha}$ *of multiplicity* $p - 1$, $\bar{b} + \bar{\gamma}$ *of multiplicity* $v - p - 1$, $v\bar{\beta}$ *of multiplicity* 1 *and* 0 *of multiplicity* $1 (1 \le p \le v - 1)$.

The proof becomes straightforward after adding $\bar{\beta}J$ to $k\bar{C}_d$.

3. Some E-optimal block designs.

3.1 *E-optimality of extended and abridged* BIBD's. Suppose that the parameters v, b and k satisfy the divisibility conditions necessary for the existence of a BIBD. Let $\lambda = bk(k-1)/v(v-1)$ and $r = bk/v$. We are interested in the optimal block designs based on v varieties and $b + m$ blocks of size k ($< v$). For $m < v/k$ the answer pertaining to E-optimality is comprised in the following:

THEOREM 3.1. *Let* $0 \le m < v/k$ *be an integer. Whenever a balanced incomplete block design extended by m disjoint binary blocks exists it is E-optimal among all designs.*

PROOF. Let d be an arbitrary design in $\Omega_{v,b+m,k}$. We are to compare μ_{d1} with μ_{d^*1}, where d^* is a BIBD extended by m disjoint and binary blocks. By adding λJ to kC_{d^*} we facilitate the finding of its eigenvalues; the minimal nonzero eigenvalue of C_{d^*} turns out to be

$$\mu_{d^*1} = \frac{r(k-1)+\lambda}{k} \qquad \left(= \frac{v\lambda}{k}\right).$$

This quantity does not depend on which BIBD is used or which disjoint binary blocks are added. A direct attack on comparing μ_{d1} and μ_{d^*1} yields little success because of the large number of possibilities for C_d. This is why we resort to an averaged version $\bar{C}_d$ of C_d and use it as a convenient intermediate comparison.

Since $mk < v$ there exists at least one variety in d which is replicated at most r times. By relabeling if necessary, we can assume that $r_{d1} \le r$. Average C_d over all the varieties with the exception of 1. Denote the resulting matrix by $\bar{C}_d$. We know from Lemma 2.1 that $\bar{C}_d$ is nonnegative definite and has row sums zero. Let the eigenvalues of $\bar{C}_d$ be $\bar{\mu}_{d,v-1} \ge \cdots \ge \bar{\mu}_{d1} \ge \bar{\mu}_{d0} = 0$. The same lemma states that $\mu_{d1} \le \bar{\mu}_{d1}$. The (first) row having $r_{d1}(k-1)/k$ on the diagonal of $\bar{C}_d$, has the remaining entries equal to $-r_{d1}(k-1)/(k(v-1))$. As we pointed out in Lemma 2.2 with $p = 1$, $\bar{C}_d$ has $vr_{d1}(k-1)/(k(v-1))$ as an eigenvalue. We can now see that

$$\mu_{d1} \le \bar{\mu}_{d1} \le \frac{vr_{d1}(k-1)}{k(v-1)} \le \frac{vr(k-1)}{k(v-1)}$$
$$= \frac{r(k-1)+\lambda}{k} = \mu_{d^*1}.$$

This concludes the proof of the theorem.

The above theorem relates to instances when addition of disjoint binary blocks seems to affect the E-optimality. It explains why a BIBD on 7 varieties and 7 blocks of size 3 remains E-optimal when extended with two disjoint binary blocks (because $m = 2 < 7/3 = v/k$) but a BIBD on 8 varieties and 14 blocks of size 4 loses this property when two such blocks are added. In this latter case $m = 2 = 8/4 = v/k$, a boundary situation not included in the theorem. The strict inequality $m < v/k$ is hence required. That a BIBD with $v = 8$, $b = 14$ and $k = 4$ is not E-optimal when extended by two disjoint binary blocks was pointed out in Cheng (1979). Our Theorem 3.1 is related to Theorem 3.2 and the content of Section 5 of Jacroux (1980). His method of proof extends that of Takeuchi (1961).

The next result concerns bounds for the minimal eigenvalue of the information matrix of a design. These bounds are expressed in terms of convenient parameters of the design.

They often provide ways of detecting which designs are bad and sometimes suffice to establish the optimal designs. We shall point out some such instances later.

In order to simplify the notation in the theorem that follows, let us relabel the varieties in a given design d so that after relabeling $r_{d1} \le r_{d2} \le \cdots \le r_{dv}$. Throughout this section we assume that the replication numbers are ordered this way unless we specify otherwise.

THEOREM 3.2. *In any block design $d \in \Omega_{v,b,k}$ the following inequalities hold*:

$$\mu_{d1} \le \frac{v}{(v-1)} \min_{1\le i\le v} \left(r_{di} - \frac{1}{k}\sum_{j=1}^{b} n_{dij}^2\right) \tag{3.1}$$

$$\mu_{d1} \le \frac{(k-1)vr_{d1}}{k(v-1)} \tag{3.2}$$

$$\mu_{d1} \le \min_{2\le n\le v} \left[\frac{(k-1)}{kn}\sum_{i=1}^{n} r_{di} + \frac{2}{kn(n-1)}\sum_{1\le i<j\le n} \lambda_{dij}\right]. \tag{3.3}$$

PROOF. We obtain (3.1) by averaging C_d repeatedly over all but one variety and using Lemma 2.2 with $p = 1$. The upper bound in (3.2) is in general less efficient than (3.1) and is obtained from (3.1) by minimizing over integers $\sum_{j=1}^{b} n_{dij}^2$ subject to $\sum_{j=1}^{b} n_{dij} = r_{di}$. The last upper bound, which is usually sharper than (3.2), follows from Lemma 2.2 with $p = n$ after averaging separately over $1, 2, \cdots, n$ and $n+1, n+2, \cdots, v$. This ends the proof.

The inequality in (3.3) is related to a slightly more general inequality much the same way (3.2) is related to (3.1). In the stated form it is however much simpler and readily applies to the bearings that will follow; especially the case with $n = 2$, which gives:

$$\mu_{d1} \le \frac{(k-1)}{2k}(r_{d1} + r_{d2}) + \frac{\lambda_{d12}}{k}. \tag{3.4}$$

Let once again, v, b, r, k and λ satisfy the necessary conditions for the existence of a BIBD. Our first use of these inequalities is in the proof of the following result:

THEOREM 3.3. *When $m(v/k^2 \le m \le v/k)$ disjoint blocks are deleted from a balanced incomplete block design, the resulting structure is E-optimal over all designs.*

PROOF. Let $v/k^2 \le m \le v/k$ be an integer. Denote by d^* a BIBD with m disjoint blocks deleted. Then d^* has mk varieties replicated $r-1$ times and the rest replicated r times. By adding λJ to kC_{d^*}, or otherwise, we obtain

$$k\mu_{d^*1} = v\lambda - k \;(= (r-1)(k-1) + \lambda - 1).$$

Let us partition the collection of designs $\Omega_{v,b-m,k}$ in the following way:

$$S_1 = \{d : r_{d1} \le r-2\}$$

$$S_2 = \{d : r_{di} \ge r-1 \quad \text{for all} \quad i \quad \text{and there exist} \quad r_{d1}, r_{d2} = r-1 \quad \text{with} \quad \lambda_{d12} \le \lambda - 1\}$$

$$S_3 = \{d : r_{di} \ge r-1 \quad \text{for all} \quad i \quad \text{and whenever} \quad r_{di}, r_{dj} = r-1 \quad \text{we have} \quad \lambda_{dij} \ge \lambda\}.$$

It is easy to check that the three S_i's partition $\Omega_{v,b-m,k}$. Moreover in any design $d \in S_2 \cup S_3$ at least mk varieties are replicated exactly $r-1$ times (or else we would have $r_{d1} \le r-2$).

Let d be a design in S_1. Using (3.2) we have

$$\mu_{d1} \le \frac{(k-1)vr_{d1}}{k(v-1)} \le \frac{(k-1)v}{k(v-1)}(r-2)$$
$$= \frac{(k-1)vr}{k(v-1)} - 2\frac{(k-1)v}{k(v-1)} < \frac{v\lambda}{k} - 2\frac{(k-1)}{k} \le \frac{v\lambda}{k} - 1$$
$$= \mu_{d^*1}, \qquad \text{since} \quad 2\frac{(k-1)}{k} \ge 1 \quad \text{for} \quad k \ge 2.$$

This shows that d^* is E- better than any design in S_1. Suppose d is a design in S_2. Relying on (3.4) we obtain

$$k\mu_{d1} \le \frac{(k-1)}{2}(r_{d1} + r_{d2}) + \lambda_{d12} \le (k-1)(r-1) + \lambda - 1 = k\mu_{d^*1}.$$

Finally, for a design $d \in S_3$ we use (3.3) with $n = mk$ and the fact that $v/k^2 \le m \le v/k$ as follows. Average C_d separately over mk varieties replicated exactly $r-1$ times and their complement, to obtain

$$\bar{C}_d = \frac{1}{k}\begin{bmatrix} (\bar{a} + \bar{\alpha})I - \bar{\alpha}J & -\bar{\beta}J \\ -\bar{\beta}J & (\bar{b} + \bar{\gamma})I - \bar{\gamma}J \end{bmatrix}$$

where the upper diagonal block is $mk \times mk$. The mk varieties replicated $r-1$ times (that we averaged over) have been lined up in the first mk positions in C_d. Since each diagonal entry in the $mk \times mk$ upper diagonal block in kC_d is at most $(r-1)(k-1)$ we have $\bar{a} \le (r-1)(k-1)$. On account of the fact that $d \in S_3$ we easily conclude that $\bar{\alpha} \ge \lambda$. The row sums are zero in $\bar{C}_d$, so $(r-1)(k-1) \ge \bar{a} = (v-mk)\bar{\beta} + (mk-1)\bar{\alpha} \ge (v-mk)\bar{\beta} + (mk-1)\lambda$. Rewriting, we obtain

$$\bar{\beta} \le ((r-1)(k-1) - (mk-1)\lambda)(v-mk)^{-1}$$
$$= \lambda - (k-1)(v-mk)^{-1}.$$

We have $v\bar{\beta} \le v\lambda - v(k-1)(v-mk)^{-1} \le v\lambda - k = k\mu_{d^*1}$ if and only if $v \le mk^2$. Hence if $v/k^2 \le m \le v/k$ we have $v\bar{\beta}/k \le v\lambda/k - 1 = \mu_{d^*1}$. Let $\bar{\mu}_{d1}$ denote the smallest eigenvalue of $\bar{C}_d$ (apart from the eigenvalue 0 associated with the eigenvector **1**). Since $v\bar{\beta}/k$ is an eigenvalue of $\bar{C}_d$ (see Lemma 2.2) we can utilize Lemma 2.1 to conclude the proof as follows:

$$\mu_{d1} \le \bar{\mu}_{d1} \le \frac{v\bar{\beta}}{k} \le \mu_{d^*1}.$$

Let us point out that a BIBD with any number of disjoint blocks deleted is always E-optimal over $S_1 \cup S_2$. The first part of the above proof accounts for this. It is known that v/k disjoint blocks can be deleted from any resolvable BIBD. Hence the upper bound on m in the theorem can be attained.

When for some set of parameters more than one nonisomorphic BIBD's exist, and chances of loss of blocks are considerable, a BIBD with as many disjoint blocks as possible ought to be chosen for the experiment. In case a certain number of disjoint blocks are lost, the remaining structure is still E-optimal.

3.2 *E-optimality of some extended group divisible designs.* Suppose now that the parameters v, b and k are such that a group divisible design d^* exists, with the intragroup parameter λ_2 exceeding the intergroup parameter λ_1 by one. It is known (and not very hard to show) that $k\mu_{d^*1} = r(k-1) + \lambda_1$, where r is the replication number of any variety in d^*. Let $d \in \Omega_{v,b,k}$ be any design. In case d is not equireplicated, an easy computation based on (3.2) shows that d^* is E-better than d. For an equireplicated design d the bound in (3.4) can be used to reach the same conclusion. This shows that a group divisible design with $\lambda_2 = \lambda_1 + 1$ is E-optimal among all designs with the same parameters v, b and k. This is also a consequence of our next theorem. The result was first proved by Takeuchi (1961) using a different technique. It has since then been extended by Cheng (1978) to a large class of optimality criteria when d^* has two groups. We shall also extend Takeuchi's result but in a different direction. First we need to introduce some new terminology.

Given a partition of the v varieties and a set of blocks, we say that the set of blocks is compatible with the partition if none of the blocks contains varieties from two (or more) elements of the partition; i.e., if the partition is $P = \{P_1, P_2, \cdots, P_t\}$ and the set of blocks is $\Gamma = \{B_1, B_2, \cdots, B_u\}$, then Γ and P are said to be compatible if $P_i \cap B_j = \phi$ or $P_i \cap B_j = B_j$ for all $1 \le i \le t$ and $1 \le j \le u$. We are now ready to enunciate the following theorem:

THEOREM 3.4. *A group divisible design with m groups and $\lambda_2 = \lambda_1 + 1$ extended by less than $(v - m)/k$ disjoint and binary blocks compatible with the partition provided by the groups is E-optimal among all designs.*

PROOF. The reason we need compatibility is to keep the off-diagonal entries of kC_{d^*}, where d^* is the conjectured E-optimal design, differ at most by 1. This is important since μ_{d1} is dependent upon the off-diagonal entries of C_d, as (3.3), (3.4) and Lemma 2.2 show; it is also in agreement with a conjecture of John and Mitchell (1977) regarding a possible description of complete classes of optimal designs.

Let $d^* \in \Omega_{v,b,k}$ be a group divisible design with m groups of size n and $\lambda_2 = \lambda_1 + 1$ extended by a set of s ($s < (v - m)/k$) disjoint and binary blocks compatible with the partition provided by the groups. Denote by d° the group divisible design to which these blocks have been added and let r be the replication number of any variety in d°.

Let us find $\mu_{d^*1} \cdot C_{d^*}$ can be obtained from C_{d° (the information matrix of d°) by adding to it the information matrix of the set of disjoint and binary blocks that we have added to d°. As any information matrix, the matrix that we add is nonnegative definite. Hence $C_{d^*} \ge C_{d^\circ}$, i.e., $C_{d^*} - C_{d^\circ}$ is nonnegative definite, and therefore (see Bellman (1970)) $\mu_{d^*1} \ge (r(k - 1) + \lambda_1)/k = \mu_{d^\circ 1}$. On the other hand, since we add less than $(v - m)/k$ disjoint binary blocks to d°, there are at least two varieties replicated r times in d^* and contained together in λ_1 blocks. By averaging separately over two such varieties (without loss 1 and 2) and using (3.4) we obtain.

$$k\mu_{d^*1} \le \frac{(k-1)}{2}(r_{d1} + r_{d2}) + \lambda_{d12} = r(k - 1) + \lambda_1.$$

Hence $\mu_{d^*1} = (r(k - 1) + \lambda_1)/k$.

Let S_1, S_2, and S_3 denote the subsets of $\Omega_{v,b,k}$ defined as follows:

$$S_1 = \{d : r_{d1} \le r - 1\},$$

$$S_2 = \{d : r_{di} \ge r \quad \text{for all} \quad i \quad \text{and there exist} \quad r_{d1}, r_{d2} = r \quad \text{with} \quad \lambda_{d12} \le \lambda_1\},$$

$$S_3 = \{d : r_{di} \ge r \quad \text{for all} \quad i \quad \text{and whenever} \quad r_{di}, r_{dj} = r \quad \text{we have} \quad \lambda_{dij} \ge \lambda_2\}.$$

$\Omega_{v,b,k}$ is the disjoint union of S_1, S_2 and S_3.

We show that d^* is E-better than any design d in S_1 by using (3.2) as follows:

$$\begin{aligned} k\mu_{d1} &\le \frac{(k-1)vr_{d1}}{(v-1)} \le \frac{(k-1)v}{(v-1)}(r-1) \\ &= (r-1)(k-1)(1 + (v-1)^{-1}) \le r(k-1) + \lambda_1 \\ &= k\mu_{d^*1}. \end{aligned}$$

Let d be any design in S_2. By (3.4) we immediately conclude that

$$\begin{aligned} k\mu_{d1} &\le \frac{(k-1)}{2}(r_{d1} + r_{d2}) + \lambda_{d12} \le r(k-1) + \lambda_1 \\ &= k\mu_{d^*1}. \end{aligned}$$

In case $d \in S_3$, d contains at least $v - sk$ varieties replicated r times. Average C_d separately over $v - sk$ varieties replicated r times (lined up in the first $v - sk$ positions of C_d) and their complement. Denote the averaged version so obtained by $\bar{C}_d$. Then

$$\bar{C}_d = \frac{1}{k}\begin{bmatrix} (\bar{a} + \bar{\alpha})I - \bar{\alpha}J & -\bar{\beta}J \\ -\bar{\beta}J & (\bar{b} + \bar{\gamma})I - \bar{\gamma}J \end{bmatrix}$$

with the upper diagonal block of dimension $(v - sk) \times (v - sk)$. By the fact that $d \in S_3$ we know that $\bar{\alpha} \geq \lambda_2$ and $\bar{a} \leq r(k - 1)$. Wherefore,

$$r(k-1) \geq \bar{a} = (v - sk - 1)\bar{\alpha} + sk\bar{\beta}$$
$$\geq (v - sk - 1)\lambda_2 + sk\bar{\beta},$$

which gives

$$v\bar{\beta} \leq \frac{v}{sk}(r(k-1) - (v - sk - 1)\lambda_2) = v\lambda_2 - \frac{v(n-1)}{sk}.$$

Since $s < (v - m)/k = v(n-1)/kn$ we have

$$v\lambda_2 - \frac{v(n-1)}{sk} \leq v\lambda_2 - n = r(k-1) + \lambda_1 = k\mu_{d^*1}$$

and hence (by Lemma 2.1 and Lemma 2.2)

$$\mu_{d1} \leq \frac{v\bar{\beta}}{k} \leq \mu_{d^*1}.$$

This concludes our demonstration.

Arguments similar to the ones applied thus far lead us to Theorem 3.1 of Cheng (1980).

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