SOME ELEMENTARY EXAMPLES OF UNIRATIONAL VARIETIES WHICH ARE NOT RATIONAL

By M. ARTIN *and* D. MUMFORD†

[Received 8 November 1971]

An outstanding problem in the algebraic geometry of varieties of dimension $n \geqslant 3$ over an algebraically closed field k has been whether there exist unirational varieties which are not rational. Here V is *unirational* if it has the equivalent properties:

(a) there exists a rational surjective map $f: \mathbf{P}^n \to V$,

or, there exists an embedding $k(V) \subset k(X_1, \ldots, X_n)$;

while V *rational* means equivalently:

(b) there exists a birational map $f: \mathbf{P}^n \to V$,

or, there exists an isomorphism $k(V) \cong k(X_1, \ldots, X_n)$.

For $n = 1$, these are equivalent (Lüroth's theorem). For $n = 2$ they are equivalent in characteristic 0 (Castelnuovo's theorem) or if the map f in (a) is assumed separable (Zariski's extension of Castelnuovo's theorem). In 1959 ([**13**]), Serre clarified classical work on this problem for $n = 3$. It has been generally accepted since then that none of the examples proposed by Fano or Roth had been correctly proved irrational.

In the past year, two solutions of this problem have been found: Clemens and Griffiths ([**6**]) showed that all non-singular *cubic* hypersurfaces in $\mathbf{P}^4$ are irrational, and Iskovskikh and Manin ([**16**]) showed that all non-singular *quartic* hypersurfaces in $\mathbf{P}^4$ are irrational. Some are unirational (Segre ([**11**])).

Both of these solutions are quite deep and it seems worth while to have an elementary example as well, even if our method applies to a very special kind of variety. Ramanujam suggested using torsion in H^3 and this led us to the examples presented here. We construct varieties, of all dimensions $n \geqslant 3$ and all characteristics $p \neq 2$, which are unirational and which have 2-torsion in H^3. With the present state of resolution of singularities, we can show that such a V cannot be rational if the characteristic is 0 or if the characteristic is not 2 and $n = 3$.

† Both authors would like to thank the Mathematics Institute of the University of Warwick for its warm hospitality and generous support at the time when this research was done. We also acknowledge gratefully the support of the National Science Foundation and the Nuffield Foundation respectively.

An outline of this paper is as follows. In § 1, we prove the torsion criterion for distinguishing between rational and irrational varieties. In § 2, we construct an example and prove that it has 2-torsion when $k = \mathbf{C}$ using simplicial methods. In § 3, we digress to prove a theorem on the structure of the Brauer group of a function field in two variables. We use this in § 4 to construct a whole class of examples including the particular one given in § 2, and prove that in suitable circumstances they have 2-torsion in their 2-adic étale cohomology groups.

We would like to point out that our examples belong to a general class—conic bundles over rational surfaces—which have been much studied classically, and that our theory has many points of contact with classical work: cf. Roth ([**10**], Ch. 4, §§ 4–7).

1. The criterion

Serre ([**13**]) showed that over the complex field $\mathbf{C}$ almost all cohomological properties enjoyed by non-singular projective *rational* 3-folds hold for non-singular projective *unirational* 3-folds as well. One small possible difference escaped though. To be precise, let V be a non-singular projective 3-fold over the complex field $\mathbf{C}$. Applying Poincaré duality and the universal coefficient theorem, its integral cohomology has the form in the left-hand column

$$\begin{aligned}
H^0(V) &\cong \mathbf{Z}, && = \mathbf{Z},\\
H^1(V) &\cong \mathbf{Z}^{B_1}, && = 0,\\
H^2(V) &\cong \mathbf{Z}^{B_2} + T_1, && = \mathbf{Z}^{B_2},\\
H^3(V) &\cong \mathbf{Z}^{B_3} + T_2, && = \mathbf{Z}^{B_3} + T_2,\\
H^4(V) &\cong \mathbf{Z}^{B_2} + T_2, && = \mathbf{Z}^{B_2} + T_2,\\
H^5(V) &\cong \mathbf{Z}^{B_1} + T_1 \cong H_1(V), && = 0,\\
H^6(V) &\cong \mathbf{Z}, && = \mathbf{Z},
\end{aligned}$$

for suitable integers B_1, B_2, B_3 and finite groups T_1, T_2. Moreover, its complex cohomology admits the canonical decomposition given on the left:

$$\begin{aligned}
H^1(V) \otimes \mathbf{C} &\cong H^{0,1} + H^{1,0}, && = 0,\\
H^2(V) \otimes \mathbf{C} &\cong H^{0,2} + H^{1,1} + H^{2,0}, && = H^{1,1},\\
H^3(V) \otimes \mathbf{C} &\cong H^{0,3} + H^{1,2} + H^{2,1} + H^{3,0}, && = H^{1,2} + H^{2,1},\\
H^4(V) \otimes \mathbf{C} &\cong H^{1,3} + H^{2,2} + H^{3,1}, && = H^{2,2}\\
H^5(V) \otimes \mathbf{C} &\cong H^{2,3} + H^{3,2}, && = 0.
\end{aligned}$$

Let $h^{p,q} = \dim H^{p,q}$. Serre showed that if V was unirational, then

(a) $\pi_1(V) = (0)$, hence $H_1(V) = 0$, i.e.

$$B_1 = h^{1,0} = h^{0,1} = 0 \quad \text{and} \quad T_1 = 0.$$

(b) $h^{p,0} = 0$, hence $h^{0,p} = 0$ too.

This reduces the cohomology to the form in the right-hand column. For a rational variety, the numbers B_2 and B_3 do not seem to satisfy any particularly useful further restrictions (except of course B_3 even).† However, two things are left:

(c) The Hodge decomposition on H^3 gives an abelian variety:

$$J(V) = H^3(V) \otimes \mathbf{C}/(\operatorname{Im} H^3(V, \mathbf{Z}) + H^{1,2})$$

—the '*intermediate jacobian*' of Weil ([**15**]). Via cup product, $J(V)$ carries a *canonical principal polarization* Θ, and Clemens and Griffiths have shown that for rational 3-folds:

$$(*) \qquad (J(V), \Theta) \cong \prod_i (J(C_i), \Theta_i),$$

where C_i are non-singular curves, $J(C_i)$ their jacobians, and Θ_i are the usual theta-polarizations on $J(C_i)$. On the other hand, they have shown that no non-singular cubic hypersurface in $\mathbf{P}^4$ satisfies (*), although these hypersurfaces are unirational.

(d) The torsion T_2—concerning this, we have:

Proposition 1. *The torsion subgroup $T_2 \subset H^3(V, \mathbf{Z})$ is a birational invariant of a complete non-singular complex variety V of any dimension n. In particular, $T_2 = 0$ if V is rational.*

Proof. The last assertion is of course clear since $T_2(\mathbf{P}) = 0$. Let $f\colon V' \to V$ be a morphism of smooth complete varieties which is birational. It induces maps

$$H^q(V', \mathbf{Z}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} H^q(V, \mathbf{Z}),$$

the lower arrow being the Gysin map obtained via Poincaré duality. Since f is birational, $f_* f^*$ is identity.‡ Thus

$$(1.1) \qquad H^q(V', \mathbf{Z}) \approx H^q(V, \mathbf{Z}) + K^q$$

for suitable K^q. In particular,

$$T_2(V) \subset T_2(V').$$

† It is quite possible that, for rational varieties with $B_2 = 1$, B_3 can take only a few small values. But if so, this is quite likely very hard to prove.

‡ By the identity $f_*(x.f^*(y)) = f_*(x).y$, it suffices to prove that $f_*f^*1 = 1$, where is the canonical generator of $H^0(V, \mathbf{Z})$. This is proved in [**4**], § 4.15.

Suppose furthermore that f is the blow-up of a smooth subvariety $Y \subset V$, say of codimension $r+1$. Then the fibres of f above Y are isomorphic to $\mathbf{P}^r$, and so the direct image $R^q f_* \mathbf{Z}$ is $\mathbf{Z}$ if $q = 0$, is the extension of $\mathbf{Z}$ by zero outside Y if $q = 2i$ $(1 \leqslant i \leqslant r)$, and is zero for other values of q. Thus the Leray spectral sequence for the map f yields an exact sequence

$$H^0(Y,\mathbf{Z}) \to H^3(V,\mathbf{Z}) \to H^3(V',\mathbf{Z}) \to H^1(Y,\mathbf{Z}) \to H^4(V,\mathbf{Z}) \to H^4(V',\mathbf{Z}).$$

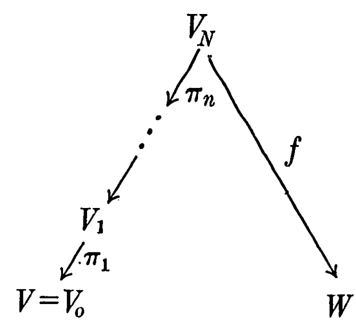
By (1.1) this sequence splits, i.e.

$$H^3(V',\mathbf{Z}) \approx H^3(V,\mathbf{Z}) \oplus H^1(Y,\mathbf{Z}).$$

Since $H^1(Y,\mathbf{Z})$ is torsion-free for any Y, we have $T_2(V) \approx T_2(V')$ in this case.

Now let V, W be birationally equivalent and non-singular. According to the results of Hironaka, there is a diagram of birational morphisms

(1.2)



where π_i is the blow-up of a smooth $Y_{i-1} \subset V_{i-1}$. Thus the above remarks show that $T_2(W) \subset T_2(V_N) \approx T_2(V)$. By symmetry, $T_2(W) \approx T_2(V)$, as required.

Moreover, in characteristic $p \neq 0$ we have

Proposition 1*. *The torsion subgroup of the étale l-adic cohomology group $H^3(V, \mathbf{Z}_l)$ is a birational invariant of a complete non-singular 3-fold V over k, where k is algebraically closed and $l \neq$* char k. *In particular, this group is torsion-free if V is rational.*

Proof. By the results of Abhyankar ([**1**]), we can again find a diagram (1.2). Using the results of [**2**], Exposé 18, the proof goes through as before.

Note that we use the hypothesis $\dim X = 3$ only because the resolution theorem that we need is not known in higher dimension.

2. A double space with quartic branch locus

To start off, we will work over any algebraically closed field of characteristic different from 2. Let

$$A \subset \mathbf{P}^2$$

be a non-singular conic, defined by a homogeneous quadratic equation:

$$\alpha(X_0, X_1, X_2) = 0.$$

Let

$$D_1, D_2 \subset \mathbf{P}^2$$

be non-singular cubics defined by equations $\delta_1 = 0, \delta_2 = 0$ such that

(a) D_1 and D_2 meet A tangentially at six distinct points:

$$D_1 \cap A = \{P_1^{(1)}, P_2^{(1)}, P_3^{(1)}\},$$
$$D_2 \cap A = \{P_1^{(2)}, P_2^{(2)}, P_3^{(2)}\},$$

(b) D_1 meets D_2 transversally at nine distinct points $O_1, \ldots, O_9$.

It is easy to check that such cubics exist.

Next, $(D_1 + D_2)\cdot A$, as a cycle on A, equals $2\mathfrak{A}$, where $\mathfrak{A} = \sum_{i=1}^{2} \sum_{j=1}^{3} P_j^{(i)}$. Since curves of degree 3 cut out a complete system on A, we have

$$\mathfrak{A} = B\cdot A$$

for some third cubic curve B. In homogeneous equations, this means that

$$\alpha \mid \delta_1\delta_2 - \beta^2,$$

where $\beta = 0$ is a suitable equation of B, hence

$$\delta_1\delta_2 = \beta^2 - 4\alpha\gamma$$

for some γ of degree 4.

Let

$$K \subset \mathbf{P}^3$$

be the quartic surface with homogeneous equation:

$$\alpha(X_0, X_1, X_2)X_3{}^2 + \beta(X_0, X_1, X_2)X_3 + \gamma(X_0, X_1, X_2) = 0.$$

If P_0 is the point $(0, 0, 0, 1)$, then P_0 is a node of K (i.e. a double point with non-singular tangent cone); projecting from P_0, K is a double cover of $\mathbf{P}^2$ ramified along the curve $\beta^2 - 4\alpha\gamma = 0$, i.e. along $D_1 \cup D_2$. Therefore K has 10 nodes in all—P_0, plus one more point P_i ($1 \leqslant i \leqslant 9$) over each point O_i of $D_1 \cap D_2$—and no other singularities.

Next, let V_0 be the double covering of $\mathbf{P}^3$ (the 'double space') branched in K. V_0 has the weighted homogeneous equation:

$$X_4{}^2 = \alpha X_3{}^2 + \beta X_3 + \gamma; \quad \deg X_0 = \ldots = \deg X_3 = 1, \quad \deg X_4 = 2.$$

Moreover, V_0 has a node Q_i over each node P_i of K, and no other singularities. Finally, let V be the desingularization of V_0 obtained by blowing up all the Q_i to exceptional divisors E_i (cf. Figs. 1 and 2).

First of all, it is clear that V is *unirational.* In suitable affine coordinates, V_0 is just

$$X_4^2 = (X_1^2 - X_2)X_3^2 + \beta(X_1, X_2)X_3 + \gamma(X_1, X_2).$$

Consider the double covering W of this affine 3-fold defined by

$$X_5 = \sqrt{(X_1^2 - X_2)}$$

If we eliminate X_2 by the relation $X_2 = X_1^2 - X_5^2$, the new 3-fold has the equation:

$$X_4^2 = X_5^2X_3^2 + \beta(X_1, X_1^2 - X_5^2)X_3 + \gamma(X_1, X_1^2 - X_5^2). \tag{2.1}$$

This is a rational variety, via the birational map:

$$W \xrightarrow{\ f\ } \mathbf{A}^3 \quad (\text{coordinates } Y_1, Y_2, Y_3)$$

given by

$$Y_1 = X_1,$$
$$Y_2 = X_5$$
$$Y_3 = X_4 - X_5X_3.$$

In fact, to compute the fibre of $f^{-1}(a_1, a_2, a_3)$, put $X_1 = a_1$, $X_5 = a_2$, and $X_4 = a_3 + a_2X_3$ in equation (2.1). This leads to

$$a_3^2 + 2a_2a_3X_3 = \beta(a_1, a_1^2 - a_2^2)X_3 + \gamma(a_1, a_1^2 - a_2^2),$$

which almost always has a unique solution.

The really remarkable thing about V, however, is that it has 2-torsion in H^3 and H^4. We shall prove this here when $k = \mathbf{C}$, and in § 4 in general. Assuming these results, it follows from the criteria in § 1 that

(i) in any characteristic other than 2, V is unirational but not rational,

(ii) if the characteristic is zero, $V \times \mathbf{P}^n$ is an $(n+3)$-dimensional variety which is unirational but not rational.

The easiest way to compute the cohomology of V is to use the morphism:

$$f\colon V \to \mathbf{P}^2$$

defined outside E_0 by the composition:

$$V - E_0 \longrightarrow V_0 - \{P_0\} \longrightarrow \mathbf{P}^3 - \{Q_0\} \xrightarrow{\text{projection}} \mathbf{P}^2.$$

Let V' denote the blow-up of P_0 in V_0. Then f clearly factors:

$$V \xrightarrow{\ \pi\ } V' \xrightarrow{\ f'\ } \mathbf{P}^2.$$

If $a = (a_0, a_1, a_2)$ is a point of $\mathbf{P}^2$, the fibre $f'^{-1}(a)$ is the inverse image in V' of the line

$$X_0 : X_1 : X_2 = a_0 : a_1 : a_2$$

in $\mathbf{P}^3$, i.e. it is the conic

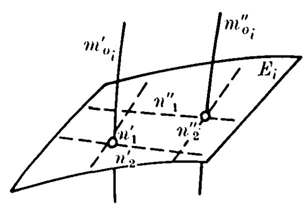
$$m_a\colon\ X_4^2 = \alpha(a_0, a_1, a_2)X_3^2 + \beta(a_0, a_1, a_2)X_3Z + \gamma(a_0, a_1, a_2)Z^2.$$

Now $a \in D_1 \cup D_2$ if and only if $\delta_1\delta_2(a) = (\beta^2 - 4\alpha\gamma)(a) = 0$, i.e. if and only if the conic m_a is singular. Moreover, m_a can be a double line only if α, β, γ all vanish at a. This implies that $a \in A$ and that a is a double point of $D_1 \cup D_2$. There is no such a, so we conclude that

if $a \notin D_1 \cup D_2$, then $m_a \cong \mathbf{P}^1$ (a conic);

if $a \in D_1 \cup D_2$, then $m_a \cong \mathbf{P}^1 \vee \mathbf{P}^1$ (2 copies of $\mathbf{P}^1$ meeting transversely at 1 point).

Now $f'(P_i) = O_i$, so the fibres of f itself are the same as those of f' except for the fibres $f^{-1}(O_i)$; and one sees easily by calculating in local coordinates that $f^{-1}(O_i)$ is just the quadric E_i plus two lines, like this:



m'_{0i}, m''_{0i} are the proper transforms by π of two components of $f'^{-1}(O_i)$.
n'_j, n''_j are the exceptional divisors in the blow-up induced by π on surface $f'^{-1}(D_j) \subset V_0$.

FIG. 1

When $a \in D_1 \cup D_2$, let m'_a and m''_a denote the two components of m_a. The essential point now is to examine for which loops in $D_1 \cup D_2$ the two components m'_a, m''_a are interchanged when one moves continuously around them, and for which loops the two components are not interchanged. Put another way, the set of pairs

$$D'_i = \{(a, m^*) \mid a \in D_i,\ m^* \text{ a component of } m_a\}$$

is a new curve which is an unramified double covering of D_i. Which covering is it? So long as $a \notin A$, the two components can be distinguished by whether their intersection with the line $Z = 0$ is the point

$$X_4 = +X_3\sqrt{\alpha(a_0, a_1, a_2)}, \quad Z = 0$$

or

$$X_4 = -X_3\sqrt{\alpha(a_0, a_1, a_2)}, \quad Z = 0.$$

Therefore D'_i is the normalization of D_i in the field obtained by adjoining $\sqrt{\alpha}$ (or more precisely, $\sqrt{(\alpha/l^2)}$, where l is a linear form). Now note that

(a) A has intersection multiplicity 2 with D_i whenever they meet, and therefore α vanishes everywhere to even order, and D'_i is everywhere unramified over D_i;

(b) the three intersections of A and D_i are *not* collinear, and so α is *not* congruent to a square $l(X_0, X_1, X_2)^2 \bmod \delta_i$; that is, D'_i does not break up into two copies of D_i.

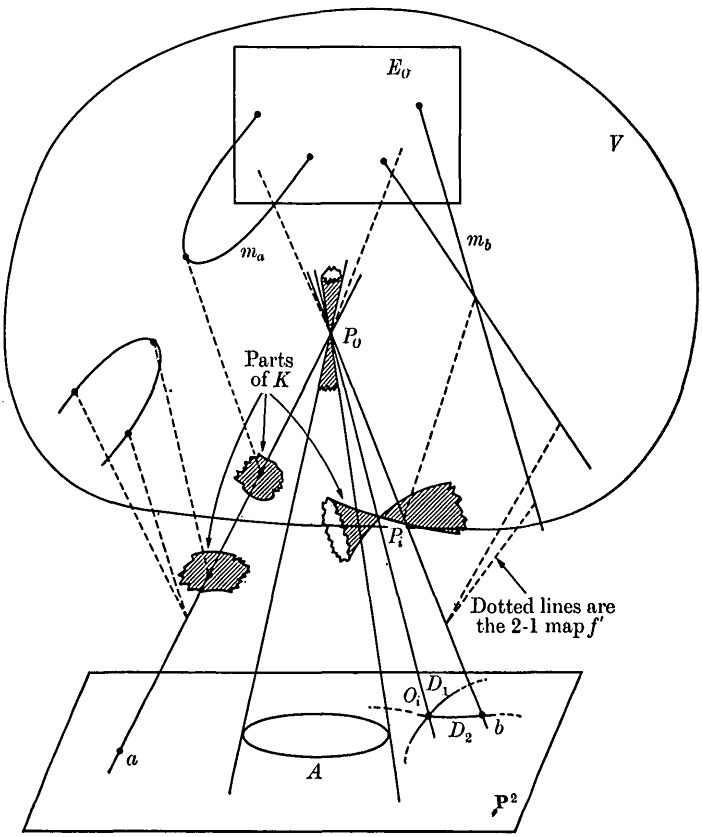


FIG. 2

This means that if we uniformize the elliptic curve D_i by the plane $\mathbf{C}$ modulo two periods ω_i', ω_i'', then for a suitable choice of periods, we get the situation:

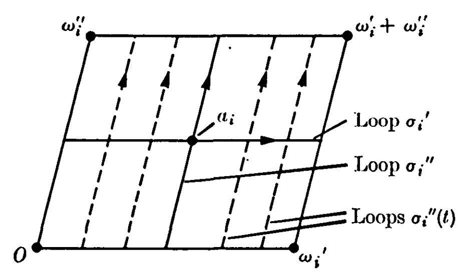


FIG. 3

(i) moving around σ_i', m_a' and m_a'' *are* interchanged,

(ii) moving around σ_i'', m_a' and m_a'' *are not* interchanged.

We are now in a position to prove that V has 2-torsion. We shall use the brutal procedure of constructing a 2-dimensional cycle α and a

3-dimensional cycle β such that

$$2\alpha = \partial\lambda,$$

$$2\beta = \partial\mu,$$

$$|\lambda| \cap |\beta| \text{ is one point } x \quad \text{and}$$

$$|\lambda|, |\beta| \text{ meet transversely at } x.$$

It follows that the cohomology classes $\bar{\alpha}, \bar{\beta}$ of α and β have order at most 2, and that their linking number is $\frac{1}{2}$; hence their order is exactly 2 ([**12**], § 77).

Construction of α. Fix a base point $a_i \in D_i$ as in Fig. 3. α is to be the *algebraic* cycle

$$\alpha = m'_{a_1} - m'_{a_2}.$$

In fact, moving around the loop σ'_i it follows that

$$m'_{a_i} \sim m''_{a_i}$$

Hence

$$2m'_{a_i} \sim m'_{a_i} + m''_{a_i} = m_{a_i}.$$

But for any $b, c \in \mathbf{P}^2 - D_1 \cap D_2$,

$$b \sim c \quad \text{on} \quad \mathbf{P}^2 - D_1 \cap D_2;$$

hence

$$m_b \sim m_c \quad \text{in} \quad V.$$

Therefore

$$\begin{aligned} 2\alpha &= 2m'_{a_1} - 2m'_{a_2} \\ &\sim m_{a_1} - m_{a_2} \\ &\sim 0. \end{aligned}$$

Construction of β. Moving the cycle m''_a around the loop σ''_1, it comes back to itself. Therefore

$$\bigcup_{a \in \sigma_1''} m''_a = \beta$$

is a 3-cycle. But moving the whole loop σ''_1 around the curve D_1 as indicated by the dotted lines $\sigma''_1(t)$ in Fig. 3,† β is transformed into

$$\beta^* = \bigcup_{a \in \sigma_1''} m'_a.$$

Thus $\beta \sim \beta^*$, and $2\beta \sim \beta + \beta^* = \bigcup_{a \in \sigma_1''} m_a$. But in $\mathbf{P}^2 - D_1 \cap D_2$, $\sigma''_1 \sim 0$; therefore in V, $\beta + \beta^* \sim 0$.

Finally λ, for instance, is easily seen to be made up of

(a) chains outside $f^{-1}(D_1)$,

(b) for each $a \in \sigma'_1$, one of the two components of m_a.

† Nine of these lines will pass through points O_i. Then the definition of β should be slightly modified to include the whole curve $m''_0 + n''_1$ in the fibre $f^{-1}(O_i)$ (see Fig. 1).

If our notation is chosen suitably, we may assume that λ contains m'_a, if $a = \sigma'_1 \cap \sigma''_1$, hence $|\lambda| \cap |\beta|$ is the one point $m'_a \cap m''_a$, where $a = \sigma'_1 \cap \sigma''_1$. It is clear that the intersection is transversal.

3. The Brauer group of a function field of two variables

Let S be a complete non-singular algebraic surface over an algebraically closed field k. We propose to compute the Brauer group of its function field K in terms of the étale cohomology of S. Since our results are valid only for the part of $\mathrm{Br}\, K$ prime to the characteristic of K, we work throughout this section 'modulo p-groups'. Cohomology will mean étale cohomology ([**2**]).

If S is simply connected, the computation is particularly simple.

THEOREM 1. *Suppose that $H^1(S, \mathbf{Q}/\mathbf{Z}) = 0$. There is a canonical exact sequence*

$$0 \longrightarrow \mathrm{Br}\, S \xrightarrow{\;i\;} \mathrm{Br}\, K \xrightarrow{\;a\;} \bigoplus_{\substack{\text{curves}\\ C}} H^1(K(C), \mathbf{Q}/\mathbf{Z})$$

$$\xrightarrow{\;r\;} \bigoplus_{\substack{\text{points}\\ p}} \mu^{-1} \xrightarrow{\;s\;} \mu^{-1} \longrightarrow 0,$$

where the groups and maps are explained below.

(i) μ_n denotes the group of nth roots of unity, $\mu = \bigcup_n \mu_n$, and $\mu^{-1} = \bigcup_n {\mu_n}^{-1} = \bigcup_n \mathrm{Hom}(\mu_n, \mathbf{Q}/\mathbf{Z})$. Thus μ and μ^{-1} are non-canonically isomorphic to $\mathbf{Q}/\mathbf{Z}$.

(ii) Br S denotes the Brauer group of Azumaya algebras on S, and the map i is the restriction to the general point. Since S is a smooth surface, we have $\mathrm{Br}\, S \approx H^2(S, \mathbf{G}_m)$ ([**9**]), and this group fits into an exact sequence

$$0 \to N \otimes \mathbf{Q}/\mathbf{Z} \to H^2(S, \mu) \to \mathrm{Br}\, S \to 0,$$

where N is the Neron–Severi group of S.

(iii) The sum in the third term is taken over all irreducible curves C on S, with function field $K(C)$. Thus $H^1(K(C), \mathbf{Q}/\mathbf{Z})$ is the group of cyclic extensions of $K(C)$, or the group of cyclic *ramified* coverings of the normalization $\bar{C}$ of C.

(iv) The local ring $\mathcal{O}_{S,C}$ of S at the generic point of C is a discrete valuation ring, and so the classical theory of maximal orders ([**7**]) associates to any finite central division ring D a cyclic extension L of the residue field $K(C)$. We recall that L is obtained from a maximum order of A for D over $\mathcal{O}_{S,C}$ as $A \otimes K(C)/(\text{radical})$. This yields the map a. The division ring D is usually said to be *ramified* along the curves C for which this cyclic extension is not trivial.

(v) In the fourth term, the sum is over all closed points of S. Given a cyclic extension of $K(C)$, one may measure its ramification at a point c of $\bar{C}$. This is canonically an element of μ^{-1} ([**2**], Exposé 18 and Exposé 19 (3.3)). The map r is defined as the sum of the ramification at all points of the various $\bar{C}$ lying over p.

(vi) The map s is the sum.

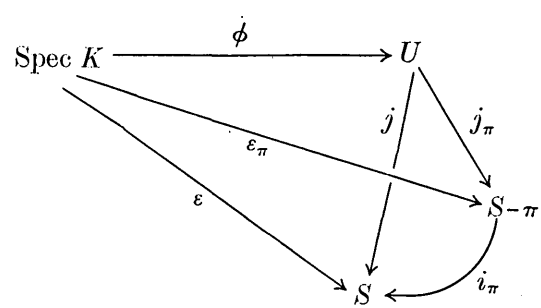
We will prove the analogous result for any irreducible, regular, excellent noetherian scheme S/k whose function field K is of transcendence degree 2 over k and such that $H^1(S, \mathbf{Q}/\mathbf{Z}) = 0$. For technical reasons, we do not assume K to be finitely generated. In order to do this, it is convenient to work formally with the complement of the points p of S of codimension 2 (the residue field at such a point is necessarily the field k). By this we mean the *pro-object* of schemes

$$U = \{S - \pi\}_{\pi \in I},$$

where I denotes the filtering system of finite sets π of points of codimension 2. The cohomology of U is by definition the direct limit

$$H^q(U, F_U) = \varinjlim_{\pi} H^q(S - \pi, F)$$

for any sheaf F on S. Thus computation with this cohomology is a substitute for an obvious limit argument. The relevant morphisms are



We have $R^3 i_{\pi *}\mathbf{G}_m = \bigoplus_{p \in \pi} \mu_p{}^{-1}$, where the subscript p denotes extension by zero outside p, and $R^q i_{\pi *}\mathbf{G}_m = 0$ if $q > 0$, $q \neq 3$. (To see this, note that the henselian local ring $\tilde{\mathcal{O}}_{S,p}$ of S at a point p of codimension 2 is necessarily the ring $k\{x,y\}$ of algebraic series in local parameters x,y. For, we have $k\{x,y\} \subset \tilde{\mathcal{O}}_{S,p} \subset k[[x,y]]$, the ring $k\{x,y\}$ is algebraically closed in $k[[x,y]]$, and $\tilde{\mathcal{O}}_{S,p}$ is algebraic over $k\{x,y\}$ since K has transcendence degree 2. Thus we may apply the results of [**2**] for algebraic schemes. The values of $R^q i_{\pi *}\mathbf{Z}/n\mathbf{Z}$ are given in [**2**] Exposé 16, Théorème (3.7) and the canonical twist by roots of unity is in Exposé 18, or in Exposé 19, Théorème (3.4). Then the values for $\mathbf{G}_m$ follow from Kummer theory (Exposé 9, Théorème (3.2)).) Passing to the limit over the spectral sequences for i_π, we obtain $\operatorname{Br} S = H^2(U, \mathbf{G}_m)$, and

$$(3.1) \qquad 0 \to H^3(S, \mathbf{G}_m) \to H^3(U, \mathbf{G}_m) \to \bigoplus_p \mu^{-1} \to H^4(S, \mathbf{G}_m) \to 0.$$

Next, we have $R^q \varepsilon_{\pi *} \mathbf{G}_m = 0$ if $q > 0$, $q \neq 2$; and this sheaf is concentrated at the points of $S - \pi$ of codimension 2, if $q = 2$ (cohomological dimension of K ([**2**], Exposé 10)). Thus $R^q \varphi_* \mathbf{G}_m = 0$ for all $q > 0$, i.e.

$$H^q(\operatorname{Spec} K, \mathbf{G}_m) \approx H^q(U, \varphi_* \mathbf{G}_m).$$

Moreover, we have an exact sequence

$$0 \to \mathbf{G}_m \to \varphi_* \mathbf{G}_m \to \bigoplus_C \mathbf{Z}_{K(C)} \to 0,$$

where C runs over irreducible closed sets of codimension 1, and where $\mathbf{Z}_{K(C)}$ denotes the extension by zero of the constant sheaf on $\operatorname{Spec} K(C) = C \cap U$. Clearly $H^q(U, \mathbf{Z}_{K(C)}) \approx H^q(K(C), \mathbf{Z})$. Since

$$H^1(K(C), \mathbf{Z}) = 0 \quad \text{and} \quad H^3(K, \mathbf{G}_m) = 0$$

([**2**], Exposé 9 (3.6), and 10) we obtain the exact cohomology sequence

$$0 \longrightarrow H^2(U, \mathbf{G}_m) \longrightarrow H^2(K, \mathbf{G}_m)$$

$$\xrightarrow{\ r\ } \bigoplus_C H^2(K(C), \mathbf{Z}) \longrightarrow H^3(U, \mathbf{G}_m) \longrightarrow 0$$

or

$$(3.2) \qquad 0 \to \operatorname{Br} S \to \operatorname{Br} K \to \bigoplus_C H^1(K(C), \mathbf{Q}/\mathbf{Z}) \to H^3(U, \mathbf{G}_m) \to 0.$$

We have $H^q(S, \mathbf{G}_m) \approx H^q(S, \mu)$ for $q > 2$. (Since S is regular, $H^q(S, \mathbf{G}_m)$ is torsion for $q \geqslant 2$ ([**9**], p. 71). Thus this follows from Kummer theory.) Moreover $H^4(S, \mu) \approx \mu^{-1}$ if S is complete and $H^3(S, \mu)$ is dual to $H^1(S, \mu) \approx H^1(S, \mathbf{Q}/\mathbf{Z})$. Thus the sequences (3.1) and (3.2) yield the exact sequence of the theorem. The fact that s is the sum is given by the canonical identification of the fundamental class on a complete surface ([**2**], Exposé 18), and it is standard that a is the correct map.

It remains to determine the map r, and for this purpose we may pass to the henselization at a given point $p \in S$. Since then S is acyclic, the sequences (3.1) (3.2) reduce to $H^3(U, \mathbf{G}_m) \approx \mu^{-1}$ and

$$(3.3) \quad 0 \longrightarrow \operatorname{Br} K \longrightarrow \bigoplus_C H^1(K(C), \mathbf{Q}/\mathbf{Z}) \xrightarrow{\ r\ } \mu^{-1} \longrightarrow 0,$$

where now $U = S - p$. Since S is henselian, so is each C, and so there is a canonical isomorphism $H^1(K(C), \mathbf{Q}/\mathbf{Z}) \approx \mu^{-1}$. We want to check that with this identification r becomes the identity map on each summand.

If C is non-singular, this is equivalent to the transitivity assertion of [2], Exposé 19 (3.4) for the inclusions $p \subset C \subset S$, as is easily seen. In order to prove it in general, it suffices to show that the map

$$H^1(K(C), \mathbf{Q}/\mathbf{Z}) \xrightarrow{r} \mu^{-1}$$

does not change if we blow up the point p in S and rehenselize at the closed point p' of the proper transform C' of C.

Let $\pi: S' \to S$ be this blowing-up. Choose a non-singular branch D in S, tangent to C, so that its proper transform D' passes through p'. Let $\alpha \in H^1(K(C), \mathbf{Q}/\mathbf{Z})$ have image $r(\alpha)$ in μ^{-1}, and choose $\beta \in H^1(K(D), \mathbf{Q}/\mathbf{Z})$ with $r(\beta) = r(\alpha)$. By the exact sequence (3.3), there is a unique class $d \in \operatorname{Br} K$ with $a(d) = \alpha - \beta$. Consider this class on the scheme S'. The irreducible closed sets of S' of codimension 1 are the proper transforms of branches in S, and the exceptional curve E. Therefore if we denote by a prime the replacement of S by S', we have

$$a'(d) = \alpha - \beta + \varepsilon,$$

where $\varepsilon \in H^1(K(E), \mathbf{Q}/\mathbf{Z})$. Since C', D' both pass through p', ε can ramify only at p'. But E is a rational curve, and so this implies that $\varepsilon = 0$ (we are ignoring p-groups!). Therefore $r'(\alpha) = r'(\beta)$. Since D is non-singular, $r'(\beta) = r(\beta)$. Thus $r'(\alpha) = r(\alpha)$ as required.

4. Conic bundles over surfaces

Let S be a non-singular complete simply connected surface over k as in § 3, but assume now that $\operatorname{char}(k) \neq 2$. We want to specialize the results of § 3 to quaternion algebras. (By a quaternion algebra, we mean simply a rank 4 Azumaya algebra.) It is a classical result that there is a one–one correspondence between:

(a) quaternion algebras A_η over the function field K of S, and

(b) curves V_η over K, isomorphic over the algebraic closure $\bar{K}$ of K to $\mathbf{P}^1_{\bar{K}}$.

Moreover each such curve V_η is isomorphic to a conic in $\mathbf{P}^2_K$, and this conic is unique up to a projective transformation. This correspondence has been extended by Grothendieck ([9]) to show, for instance, that for any Zariski-open set $U \subset S$, there is a one–one correspondence between:

(a′) quaternion algebras A over U, and

(b′) U-schemes $\pi: V \to U$, proper and flat over U, all of whose geometric fibres are isomorphic to $\mathbf{P}^1$. Moreover, such a V can be (essentially uniquely) embedded as a bundle of conics in a $\mathbf{P}^2$-bundle over U.

The correspondence is set up as follows. Given A, define V as a functor by

$$V(S') = \{\text{left ideals } L \text{ of } A \otimes \mathcal{O}_{S'} \text{ which are locally free of rank } 2\}.$$

This is clearly a closed subscheme of the Grassmannian of 2-dimensional submodules of A, and one sees easily that it is smooth over U with fibres isomorphic to $\mathbf{P}^1$.

Next, let A_η be any quaternion algebra over K: it represents an element $d \in \operatorname{Br} K$ of order 2. By Theorem 1 of § 3, there is a finite number of curves $C_1, \ldots, C_n$ on S at which $a(d)$ is not zero. The union $C = C_1 \cup \ldots \cup C_n$ is called the *ramification curve* of the algebra D and $S - C$ is the maximal Zariski open set U in S such that A_η extends to an Azumaya algebra over U. In fact the *maximal orders* A in A_η over U are precisely the Azumaya algebras extending A_η. What happens over C however? We want to analyse the case in which C is non-singular. Choose any maximal order A in A_η over the whole of S. Since S is a smooth surface, A will be locally free of rank 4. For further details, see [**3**].

PROPOSITION 2. *A maximal order A may be presented locally at a point $p \in C$ as the $\mathcal{O}_S$-algebra generated by elements x, y, with relations*

$$(4.1) \qquad \begin{cases} x^2 = a, \\ y^2 = bt, \\ xy = -yx, \end{cases}$$

where $t = 0$ is a local equation for C, and a, b are units in $\mathcal{O}_S$. Moreover, a is not congruent to a square (modulo t).

Conversely, when a is not congruent to a square, the algebra presented in this way is a maximal order in some (non-trivial) quaternion algebra.

Proof. We look first at a generic point of C. The local ring of X is a discrete valuation ring R with residue field $K(C)$, and we may apply the classical theory of maximal orders ([7]). It tells us that there is a unique prime ideal $p \subset A$ containing t, $A/p = L$ is a quadratic field extension of $K(C)$, and that $p^2 = At$. Choose $x \in A$ which reduces to a generator of L over $K(C)$, and has (reduced) trace zero, so that $x^2 = -\det x = a$ is a unit of R.

Next, note that if $y \in p$, then $\operatorname{tr} y \equiv 0 \pmod t$; for $y \to \bar{y} = \operatorname{tr} y - y$ is an anti-automorphism of A, hence maps p to p. Thus

$$\operatorname{tr} y = y + \bar{y} \in p \cap R = tR.$$

It follows that if y_0 is a non-zero element of p, then we can choose $\alpha, \beta \in Rt$ so that $y = \alpha + \beta x + y_0$ satisfies

$$\operatorname{tr} y = \operatorname{tr} xy = 0.$$

The required values are

$$(4.2)\qquad \begin{cases} \alpha = -\frac{1}{2}\operatorname{tr} y, \\ \beta = -\dfrac{1}{2a}\operatorname{tr} xy. \end{cases}$$

Then y, xy will form a basis for p/At over L, and by the Nakayama lemma, $\{1, x, y, xy\}$ is an R-basis for A. Note that $\overline{xy} = -xy$ since $\operatorname{tr} xy = 0$. Therefore

$$-xy = \overline{xy} = \bar{y}\bar{x} = (-y)(-x) = yx.$$

If we write $y^2 = bt$, then it is easily seen that b must be a unit of R since A is a maximal order. Thus the required presentation exists over R.

Now consider a closed point $p \in C$. Let $\mathcal{O}_X, \mathcal{O}_C$ denote the local rings of X and C there, and denote by p the kernel of the natural map $A \to A \otimes R \to L$. Then A/p is zero outside C, and hence is a free, rank 2 $\mathcal{O}_C$-module. We may choose an element $x \in A$ which reduces to a generator for A/p and has trace zero as above: $x^2 = a \in \mathcal{O}_X$. Choose $u, v \in p$ so that $\{1, x, u, v\}$ is an $\mathcal{O}_X$-basis for the free module A. By (4.2) we may adjust u, v to have trace zero.

The standard bilinear form $(\alpha, \beta) \to \operatorname{tr} \alpha\beta$ on A is non-degenerate wherever A is an Azumaya algebra, i.e. except on C. Thus its determinant δ (the discriminant of A) is of the form $\delta = \varepsilon t^r$, ε a unit. But calculation of the determinant with respect to the R-basis $\{1, x, y, xy\}$ yields $(4abt)^2$. Hence $r = 2$. Now calculate with respect to the basis $\{1, x, u, v\}$ using the fact that $\operatorname{tr} y \equiv 0 \,(\operatorname{mod} t)$ for all $y \in p$. This gives

$$\delta = \det\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2a & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}, \qquad * \equiv 0 \; (\operatorname{mod} t),$$

and hence

$$\varepsilon t^2 = 4a\xi t^2 + \eta t^3, \quad \text{with } \xi, \eta \in \mathcal{O}_X.$$

Therefore, a is a unit.

Since a is a unit, $A/p = \mathcal{O}_C[\sqrt{a}]$ is a semi-local dedekind domain. Therefore p/At is free of rank 1 as left A/p-module, and we choose a generator y of this module. By (4.2), we may assume that $\operatorname{tr} y = \operatorname{tr} xy = 0$, and then the above computation shows that $xy = -yx$, and $y^2 = bt$, $b \in \mathcal{O}_X$. The discriminant is $(4abt)^2 = \varepsilon t^2$. Thus b is a unit.

It remains to prove the converse, so let A have the above presentation. Then A is clearly an Azumaya algebra except on C, and if $\bar{A}$ is a maximal order containing A, then the determinant of the standard form on $\bar{A}$ is either $\equiv 0 \; (\operatorname{mod} t^2)$ as above, or is a unit (i.e. $\bar{A}$ is an Azumaya algebra).

Comparing determinants, we see that in the former case, $A = \bar{A}$, hence A is maximal, while in the latter, the cokernel $\bar{A}/A$ must have rank 1 on C. But since a is not a square, $(A/tA)_{\text{red}}$ is irreducible and quadratic over $\mathcal{O}_C$, and so that is impossible.

Using this proposition, we can extend the correspondence between orders and conic bundles as follows.

THEOREM 2. *There is a canonical one–one correspondence between*

(a″) *maximal orders A in quaternion algebras A_η over K, whose ramification curves are non-singular,*

(b″) *non-singular S-schemes $\pi: V \to S$ proper and flat over S, all of whose geometric fibres are isomorphic to $\mathbf{P}^1$ or to $\mathbf{P}^1 \vee \mathbf{P}^1$, such that the following condition holds: for every irreducible curve C_i along which the fibres of π are reducible, the two components of $\pi^{-1}(c_i)$ ($c_i \in C_i$, the generic point) are not rational over $K(C_i)$, but define a quadratic extension of this field.*

Moreover, the quadratic extensions so defined are just those given by $a(A_\eta)$, as in § 3.

We call V the *Brauer–Severi scheme* of A.

Proof. The correspondence is set up exactly as before. Given A, we let V represent the functor of left ideals of A. We need to check the structure of this scheme above points of C. First of all, it is immediately seen from the presentation (4.1) that at any point x of S, an element of A/m_xA generates a left ideal of dimension at least 2. Thus our left ideals L in $A \otimes \mathcal{O}_{S'}$ for any S' will all be principal. Next, L cannot lie entirely in the subspace spanned by $\{1, x, y\}$. Thus there is a non-zero element $u = p + qx + ry \in L$ which is unique up to scalar multiplication. Such an element u generates a left ideal of rank 2 if and only if u, xu, yu are linearly dependent. A small computation shows that this is equivalent to the equation

$$\varphi(t, p, q, r) = p^2 - aq^2 - btr^2 = 0,$$

homogeneous in p, q, r. Therefore $\varphi = 0$ defines V as a subscheme of $\mathbf{P}^2_s$. The locus defined by this equation is smooth over s except at the points $t = p = q = 0$, and at these points $\partial\varphi/\partial t \neq 0$. Thus V is non-singular. Moreover, if A is a maximal order, then a is not congruent to a square along C, hence the two components of $p^2 - aq^2$ are not rational.

Thus we have a functor from algebras A with local presentation (4.1) to non-singular S-schemes V as in the first part of (b″), and, by Proposition 2, A is a maximal order if and only if the final condition of (b″) holds.

Now the versal local deformation of $\mathbf{P}^1 \vee \mathbf{P}^1$ is the 1-parameter family

$$X_1X_2 - tX_0^2 = 0 \tag{4.3}$$

at $t = 0$, in the plane over $k[t]$. Thus any flat, proper deformation $V \to S$ of $\mathbf{P}^1 \vee \mathbf{P}^1$ is induced locally, say for the étale topology, by a map $S \to \operatorname{Spec} k[t]$. One sees immediately that if S is our surface and V is non-singular, the locus $t = 0$ on S must be a non-singular curve C, i.e. t must be a local parameter. Thus V can be put, locally for the étale topology, into the standard form (4.3). This is isomorphic to the *Brauer–Severi scheme* of the standard algebra having presentation (4.1) with $a = b = 1$. On the other hand, any algebra with presentation (4.1) is isomorphic to this standard one in the étale neighbourhood $\operatorname{Spec}(\mathcal{O}_S[\sqrt{a}, \sqrt{b}])$. Thus to show our correspondence one–one, it suffices by 'descent', to show that the map of étale sheaves on S:

$$\operatorname{Aut} A \xrightarrow{\ \varepsilon\ } \operatorname{Aut} V$$

is an isomorphism when A is the standard algebra above. This is certainly true outside C, where A is an Azumaya algebra. Hence the map is injective. Moreover, A admits the automorphism $x \to -x$, which interchanges the line pair $\pi^{-1}(c)$ ($c \in C$). Thus we need to consider only automorphisms fixing these pairs.

Consider the matrix representation

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}.$$

This identifies A with the subring of the matrix algebra $\bar{A} = M_2(\mathcal{O}_S)$ consisting of matrices

$$\begin{pmatrix} a & bt \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathcal{O}_S.$$

The inclusion $A \to \bar{A}$ induces a birational map

$$V \xrightarrow[f]{} \bar{V},$$

an isomorphism outside of C. Of course, $\bar{V} \cong \mathbf{P}^1_S$, and one sees easily that f is a morphism, and is in fact the contraction of one of the two families of lines making up $\pi^{-1}(C)$. This line family is mapped to the section v_1 of $\bar{V} \times_S C$ over C corresponding to the C-family L_1 of left ideals generated by $e_{11} = \frac{1}{2}(1+x)$.

Clearly, any automorphism σ of V not interchanging the lines will induce an automorphism $\bar{\sigma}$ of $\bar{V}$ leaving v_1 fixed, and $\bar{\sigma}$ comes from an automorphism $\bar{\varphi}$ of $\bar{A}$ such that $\bar{\varphi}$ carries L_1 to L_1 (modulo t). But L_1 is the ideal of matrices

$$\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \quad (\text{modulo } t),$$

and A is exactly the subring of $\bar{A}$ of right multipliers of L_1. Thus $\bar{\varphi}$ carries A to itself, i.e. induces an automorphism φ of A, which in turn induces σ. This proves the surjectivity of ε.

As an example of Brauer–Severi schemes, note that cubic hypersurfaces $H \subset \mathbf{P}^4$ with a sufficient generic line blown up are Severi–Brauer schemes V over $\mathbf{P}^2$. In fact, it is well known ([**6**]) that there is an irreducible 2-dimensional family l_α ($\alpha \in Z$) of lines on H, and it is easy to check that for almost all these lines l_α, there is *no* plane $L \supset l_\alpha$ tangent to H along l_α, or equivalently such that $L \cdot H = 2l_\alpha + l_{\alpha'}$ (some line $l_{\alpha'}$). Thus there is only a 1-dimensional set of lines l_α ($\alpha \in Z_0 \subsetneqq Z$) such that for some plane $L \supset l_\alpha$, $L \cdot H = l_\alpha +$ (double line). Pick any $\alpha \in Z - Z_0$. Let H^* be the blow up of H along l_α. Then the projection of $\mathbf{P}^4$ to $\mathbf{P}^2$ with centre l_α extends to a morphism:

$$\pi\colon H^* \to \mathbf{P}^2$$

and it is easily seen that all fibres $\pi^{-1}(p)$ are either irreducible conics or pairs of distinct lines. It can be checked that the ramification curve C is a non-singular quintic in $\mathbf{P}^2$ and that the generic line-pair over C does not split.

We would like to look, however, at cases where C is reducible (hence, since we are assuming C non-singular, C is also disconnected).

PROPOSITION 3. *With notation as in Theorem 2, assume that C is disconnected. Then the Brauer–Severi scheme V has 2-torsion in $H^4(V, \mathbf{Z}_2)$.*

Here the cohomology denotes the étale 2-adic cohomology. The reader who wishes to restrict to the case $k = \mathbf{C}$ can replace this by ordinary $\mathbf{Z}$-cohomology without changing the discussion below.

Proof. The idea here is the same as in § 2. We choose one of the lines l_i, making up $\pi^{-1}(p_i)$, where $p_i \in C_i$. Then $l_1 - l_2$ represents a class in H^4 of order 2, and we just need to check that this class is not itself zero. We do this by an analysis of the spectral sequence for the map π. If $p \in S$, then the cohomology of the fibre $\pi^{-1}(p)$ is just

$$H^0(\pi^{-1}(p), \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z},$$

$$H^1(\pi^{-1}(p), \mathbf{Z}/2\mathbf{Z}) = (0),$$

$$H^2(\pi^{-1}(p), \mathbf{Z}/2\mathbf{Z}) = \begin{cases} \mathbf{Z}/2\mathbf{Z}, & p \notin C, \\ \mathbf{Z}/2\mathbf{Z} + \mathbf{Z}/2\mathbf{Z}, & p \in C, \end{cases}$$

where the last isomorphism is as follows: given a class in $H^2(\pi^{-1}(p), \mathbf{Z}/2\mathbf{Z})$, for $p \in C$, restrict it to each component of $\pi^{-1}(p)$ and evaluate using the isomorphism $H^2(Y, \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$, if Y is an irreducible curve. Therefore by the proper base change theorem:

$$\pi_*(\mathbf{Z}/2\mathbf{Z}) = (\mathbf{Z}/2\mathbf{Z})_S,$$
$$R^1\pi_*(\mathbf{Z}/2\mathbf{Z}) = (0).$$

Now locally in the étale topology, π has sections through any smooth point. Taking the fundamental class of such sections, it follows that $R^2\pi_*(\mathbf{Z}/2\mathbf{Z})$ has sections locally which, evaluated point by point, are 0 or 1 on a component l of $\pi^{-1}(p)$ according as the section misses or hits l. In particular, if $p \in C$, and $\pi^{-1}(p) = l' \cup l''$, then taking the sum of the classes of sections through l' and l'', we get a section of $R^2\pi_*(\mathbf{Z}/2\mathbf{Z})$ which is 0 outside C, and has value $(1, 1)$ along C. Thus we get

$$(\mathbf{Z}/2\mathbf{Z})_C \to R^2\pi_*(\mathbf{Z}/2\mathbf{Z})$$

and it is easy to see that the cokernel is $(\mathbf{Z}/2\mathbf{Z})_S$ (we shall not need this however). To show that $\mathrm{cl}(l_1) - \mathrm{cl}(l_2) \in H^4(V, \hat{\mathbf{Z}}_2)$ is not zero, it suffices to produce an element $\zeta \in H^2(V, \mathbf{Z}/2\mathbf{Z})$ and prove that

$$[\mathrm{cl}(l_1) - \mathrm{cl}(l_2)] \cup \zeta \in H^6(V, \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$$

is not zero. Let i_1 and i_2 be the inclusion of l_1 and l_2 in V. Then

$$[\mathrm{cl}(l_1) - \mathrm{cl}(l_2)] \cup \zeta = i_1^*(\zeta) - i_2^*\zeta$$

(where $i_k^*(\zeta) \in H^2(l_k, \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$). The Leray spectral sequence for π gives

$$0 \to H^2(S, \mathbf{Z}/2\mathbf{Z}) \to H^2(V, \mathbf{Z}/2\mathbf{Z}) \to H^0(S, R^2\pi_*\mathbf{Z}/2\mathbf{Z}) \to H^3(S, \mathbf{Z}/2\mathbf{Z}).$$

But $H^3(S, \mathbf{Z}/2\mathbf{Z}) \cong (0)$ since S is simply connected. Let α_1 be a section of $(\mathbf{Z}/2\mathbf{Z})_C$ which is 1 on C_1 and 0 on the other components; let α_1' be its image in $H^0(S, R^2\pi_*\mathbf{Z}/2\mathbf{Z})$; and let $\alpha_1'' \in H^2(V, \mathbf{Z}/2\mathbf{Z})$ lift α_1'. Then

$$[\mathrm{cl}(l_1) - \mathrm{cl}(l_2)] \cup \alpha_1'' = i_1^*\alpha_1'' - i_2\alpha_1'' = 1 - 0.$$

We are now in a position to construct some examples. Choose a conic A in $\mathbf{P}^2$. If we assign arbitrarily points $p_1, \ldots, p_r$ on A, we can find non-singular curves C of degree r having a double intersection with A at each of the points $p_1, \ldots, p_r$. This is easy to see. Now choose $n \geqslant 2$ such curves $C_1, \ldots, C_n$ of degree $r_i \geqslant 3$. (The points need not be the same for the various curves.) The example of § 2 will be obtained by taking $n = 2$, $r_1 = r_2 = 3$. Let q be the rational function on $\mathbf{P}^2$ whose divisor is $A - 2L$, with L the line at infinity.

LEMMA. *If $r \geqslant 3$, the restriction of q to C is not a square in $K(C)$.*

Proof. Let $\bar{q}$ denote the restriction of q to C. Suppose that $\bar{q} = \bar{s}^2$, $\bar{s} \in K(C)$. Then $\bar{s} \in \Gamma(C, \mathcal{O}_C(C\cdot L))$. But the map

$$\Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(L)) \to \Gamma(C, \mathcal{O}_C(C\cdot L))$$

is surjective, so $\bar{s}$ lifts to a function s. Let $(s) = L' - L$, where L' is another line. Then

$$\begin{aligned} A \cap C &= \text{zeros of } q \text{ on } C \\ &= \text{zeros of } s \text{ on } C \\ &= L' \cap C; \end{aligned}$$

hence $A \cap C \subset A \cap L'$ which consists of at most 2 points. But $A \cap C$ consists of $r \geqslant 3$ points. Contradiction.

Applying the lemma to our curves C_i, we obtain a quadratic extension L_i of $K(C_i)$ by adjoining $\sqrt{q}$. This extension is *everywhere unramified* on C_i, since C_i has only double intersections with the zeros and poles of q.

Let S be the result of blowing up $\mathbf{P}^2$ at the intersection points of the C_i until their proper transforms become disjoint. Denote these proper transforms by C_i as well. Now S is a rational surface, and hence has trivial Brauer group (this follows easily from the exact sequence of Theorem 1 (ii)), and is simply connected. Therefore we may apply Theorem 1 to find a unique class $d \in \operatorname{Br} K$ ($K = K(S)$) such that $a(d)$ is zero except on $C_1, \ldots, C_n$, and is the class of L_i on C_i.

This class has order 2 since each extension is quadratic. Now ordinarily we do not know which classes of order 2 in $\operatorname{Br} K$ correspond to quaternion algebras. But in this case, we can split d in the quadratic extension $K(Y) = K[\sqrt{q}]$. In fact, going back to $\mathbf{P}^2$, our element d is unramified except on C_i. The double cover Y splits the assigned classes in $H^1(K(C_i), Q/\mathbf{Z})$ by construction. Thus the pull-back of d to $K(Y)$ is everywhere unramified, i.e. lies in $\operatorname{Br} Y$. But Y is a rational surface, in fact is isomorphic to a quadric in $\mathbf{P}^3$. Hence $\operatorname{Br} Y = 0$ and so d splits in $K(Y)$. By Brauer's construction ([**14**], p. 167), a class in $\operatorname{Br} K$ which splits in a Galois extension of degree n is represented by an algebra of rank n^2. Thus d is the class of a quaternion algebra D.

Let V be the Brauer–Severi variety of a maximal order of D over $\mathcal{O}_S$. Then V is a non-singular 3-fold by Theorem 2 and $H^4(V, \mathbf{Z}_2)$ contains 2-torsion by Proposition 3. Finally, V is unirational. For, since D splits in $K(Y)$, the generic fibre of V over S becomes isomorphic to $\mathbf{P}^1$ over $K(Y)$, which is a rational function field. Thus $K(V)[\sqrt{q}]$ is rational.

REFERENCES

1. S. Abhyankar, *Resolution of singularities of embedded algebraic surfaces* (Academic Press, New York, 1966).
2. M. Artin, A. Grothendieck and J.-L. Verdier, *Séminaire de géométrie algébrique: cohomologie étale des schémas* (Inst. Hautes Etudes Sci., 1963–64, mimeographed notes).

3. M. AUSLANDER and O. GOLDMAN, 'The Brauer group of a commutative ring', *Trans. Amer. Math. Soc.* 97 (1960) 367–409.
4. A. BOREL and A. HAEFLIGER, 'La classe d'homologie fondamentale d'un espace analytique', *Bull. Soc. Math. France* 89 (1961) 461–513.
5. N. BOURBAKI, *Algèbre commutative*, Chs. I, II (Hermann, Paris, 1961).
6. C. H. CLEMENS and P. A. GRIFFITHS, 'The intermediate jacobian of the cubic threefold', *Ann. of Math.*, to appear.
7. M. DEURING, *Algebren* (Springer, Berlin, 1935).
8. G. FANO, 'Sul sistema ∞^2 di rette contenuto in una varietà cubica', *Atti R. Accad. Sci. Torino* 39 (1904) 778–92.
9. A. GROTHENDIECK, 'Le groupe de Brauer' (*Dix exposés sur la cohomologie des schémas* (North Holland, Amsterdam, 1968)).
10. L. ROTH, *Algebraic threefolds* (Springer, Berlin, 1955).
11. B. SEGRE, 'Variazione continua ed omotopia in geometria algebrica', *Ann. Mat. Pura Appl.* 50 (1960) 149–86.
12. H. SEIFERT and W. THRELFALL, *Lehrbuch der Topologie* (B. G. Teubner, 1934, or Dover reprint).
13. J.-P. SERRE, 'On the fundamental group of a unirational variety', *J. London Math. Soc.* 34 (1959) 481–84.
14. —— *Corps locaux* (Hermann, Paris, 1962).
15. A. WEIL, 'On Picard varieties', *Amer. J. Math.* 74 (1952) 865–93.
16. V. A. ISKOVSKIKH and JU. I. MANIN, 'Three-dimensional quartics and counterexamples to the Lüroth problem', *Mat. Sb.* 86 (1971) 140–66.

Mathematics Department
Massachusetts Institute of Technology
Cambridge, Mass. 02139

Mathematics Department
Harvard University
2 Divinity Avenue
Cambridge, Mass. 02138